SIS Scuola Interdipartimentale delle Scienze, dell'Ingegneria e della Salute

DiST Dipartimento di Scienze e Tecnologie

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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- Properties of Linear Spaces and Subspaces.
- Dimension and a basis of the four fundamental subspaces of a matrix.

Dimension of a Linear Space

The **dimension** of a linear space X, denoted by **dim** X, *by definition*, is the number of vectors in any basis (its **cardinality**).

By definition, the subspace containing only the zero vector has

$$\mathbf{dim}\,\{\underline{0}\} = 0.$$

There are linear spaces with a finite dimension and spaces with an infinite dimension.

Examples

Π_n is the Linear Space of at most n-degree real polynomials. It contains vectors such as

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

A basis of Π_n is $\{1, x, x^2, x^3, \ldots, x^n\}$ so that

$$\dim \Pi_n = n+1$$

$\mathscr{A}$ is the Linear Space of real functions that are analytical* at 0. It contains vectors such as

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

A basis of $\mathscr{A}$ consists of all the power functions $\{x^k\}$ so that

$$\dim \mathscr{A} = \infty$$

* a function is said analytical at a point if locally it is the sum of a power series

Properties of Linear Subspaces

- If W is a subspace of X, then

$$\dim W \leq \dim X$$

- If W is a subspace of X and $\dim W = \dim X$ then $W = X$.

- If $\dim X = n$ and the vectors $u^{(1)}, u^{(2)}, \dots, u^{(n)}$ are linearly independent, then they form a **basis** for X.

Examples of basis and dimension

- The canonical basis of $\mathbb{R}^2$ contains the vectors

$$\{(1,0)^T, (0,1)^T\}$$

Then **dim $\mathbb{R}^2 = 2$**.

- The canonical basis of $\mathbb{R}^n$ contains the vectors

$$\{(1,0,\ldots,0)^T, (0,1,\ldots,0)^T, \ldots, (0,\ldots,0,1)^T\}$$

Then **dim $\mathbb{R}^n = n$**.

- A basis for Π_2 is the set of power functions

$$\{1, x, x^2\}$$

Then **dim $\Pi_2 = 3$**.

- If $M_{(2\times 2)} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \ \alpha,\beta,\gamma,\delta \in \mathbb{R} \right\}$ then a basis is ...
  and **dim $M_{(2\times 2)}$** = ... **?**

Fundamental subspaces associated with a matrix $A(m\times n)$: their dimension and a basis

The ***Null space of a matrix*** A: $\mathcal{N}(A)$ subspace of $\mathbb{R}^n$

$$\mathcal{N}(A)=\{x\in\mathbb{R}^n:\ Ax=\underline{0}\}\qquad \mathcal{N}(A)\subseteq\mathbb{R}^n$$

- **dim** $\mathcal{N}(A)=n-r$ where r is the rank of A.
- A basis is obtained by solving the system $Ax=0$.

The ***Left null space of a matrix*** A: $\mathcal{N}(A^{\mathsf{T}})$ subspace of $\mathbb{R}^m$

$$\mathcal{N}(A^{\mathsf{T}})=\{y\in\mathbb{R}^m:\ A^{\mathsf{T}}y=\underline{0}\}\qquad \mathcal{N}(A^{\mathsf{T}})\subseteq\mathbb{R}^m$$

- **dim** $\mathcal{N}(A^{\mathsf{T}})=m-r$ where r is the rank of A.
- A basis is obtained by solving the system $A^{\mathsf{T}}x=0$.

Fundamental subspaces associated with a matrix $A(m \times n)$: their dimension and a basis

The ***Column space of a matrix* A:** $\mathcal{R}(A)$ subspace of $\mathbb{R}^m$

$$\mathcal{R}(A) = \mathbf{span}\{A_{:,1}, A_{:,2}, \ldots, A_{:,n}\} \quad \mathcal{R}(A) \subseteq \mathbb{R}^m$$

- **dim** $\mathcal{R}(A) = r$ where r is the rank of A.
- A basis is obtained by the columns in A corresponding to pivots.

The ***Row space of a matrix* A:** $\mathcal{R}(A^\top)$ subspace of $\mathbb{R}^n$

$$\mathcal{R}(A^\top) = \mathbf{span}\{A_{1,:}, A_{2,:}, \ldots, A_{n,:}\} \quad \mathcal{R}(A^\top) \subseteq \mathbb{R}^n$$

- **dim** $\mathcal{R}(A^\top) = r$ where r is the rank of A.
- A basis is obtained by the rows in A corresponding to pivots.

Example: Null Space and Left Null Space

MATLAB

```
A=[1 0 1;5 4 9;2 4 6];
NA=null(A)
NA =
      0.57735
      0.57735
     -0.57735
disp(norm(NA))
        1
NAT=null(A')
NAT =
     -0.90453
      0.30151
     -0.30151
```

$\|\eta\|_2 = 1$

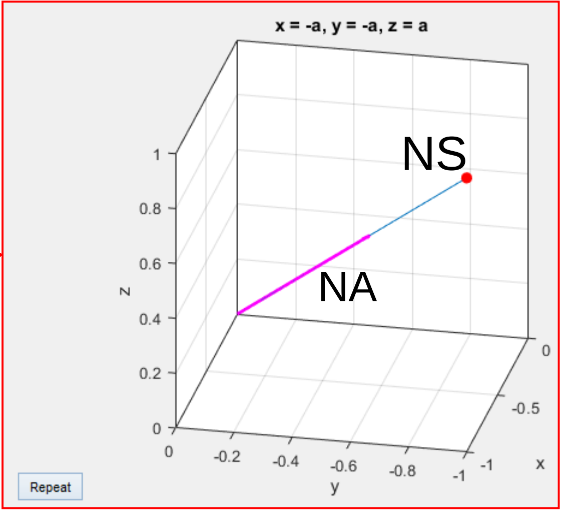
solution of the homogeneous system $Ax=0$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{pmatrix}$$

Symbolic Math Toolbox

```
S=sym([1 0 1;5 4 9;2 4 6]);
NS=null(S)
NS =
   -1
   -1
    1
disp(norm(NS))
3^(1/2)
NST=null(S')
NST =
   3
  -1
   1
```

```
syms a real;      r = a*NS;
ezplot3(r(1),r(2),r(3),[0 1],'animate')
axis tight; axis equal; box on; hold on
h=quiver3(0,0,0,NA(1),NA(2),NA(3),1);
set(h,'Color','m','LineWidth',2)
view(-78,23)
```



Example: Column Space and Row Space

MATLAB

```
A=[1 0 1;5 4 9;2 4 6];
disp(rank(A))
     2
RA = orth(A)
RA =
  -0.091519    0.41646
  -0.82791     0.47293
  -0.55335    -0.77646
RAT=orth(A')
RAT =
     -0.40119       0.71113
     -0.41527      -0.70301
     -0.81646       0.0081265
```

```
RA'*RA
ans =
              1    1.1102e-16
 1.1102e-16             1
norm(RA(:,1))
ans =
              1
```

orthonormal columns

```
disp(rref(A))
     1        0        1
     0        1        1
     0        0        0
```

```
disp(A(:,1:2))
     1        0
     5        4
     2        4
```

Symbolic Math Toolbox

```
S=sym([1 0 1;5 4 9;2 4 6]);
disp(rank(S))
       2
RS=colspace(S)
RS =
[   1, 0]
[   0, 1]
[ -3, 1]
RST=colspace(S')
RST =
[ 1, 0]
[ 0, 1]
[ 1, 1]
```

```
RS'*RS
ans =
[10, -3]
[-3,  2]
norm(RS(:,1))
ans =
10^(1/2)
```

```
disp(rank([A RA double(RS) double(RS1)]))
  2
```

The **orth** function can also be applied to a symbolic matrix, but **colspace** cannot be applied to a numeric matrix.

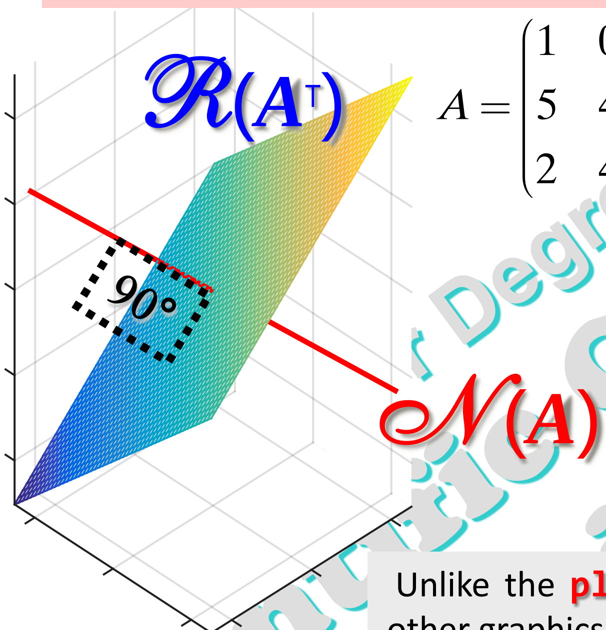
```
RS1=simplify(orth(S))
RS1 =
[30^(1/2)/30, -(7*330^(1/2))/330]
[ 30^(1/2)/6,       -330^(1/2)/66]
[30^(1/2)/15,  (8*330^(1/2))/165]
RS1'*RS1
ans =
[ 1, 0]
[ 0, 1]
```

```
norm(RS1(:,1))
ans =
1
```

Null Space and Row Space

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = \underline{0}\} \qquad \mathcal{N}(A) \subseteq \mathbb{R}^n$$

$$\mathcal{R}(A^T) = \text{span}\{A_{1,:}, A_{2,:}, \ldots, A_{m,:}\} \qquad \mathcal{R}(A^T) \subseteq \mathbb{R}^n$$



$$A = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{pmatrix}$$

```
A=[1 0 1;5 4 9;2 4 6];  S=rref(A)
S =
      1     0     1
      0     1     1
      0     0     0
syms a b real; p=A([1 2],:)'*[a b]';
fmesh(p(1),p(2),p(3),[-1 1]);
axis('equal')
N=null(A)*[-10 10]; % two points on 𝒩(A)
line(N(1,:),N(2,:),N(3,:),'Color','r')
```

linearly independent rows

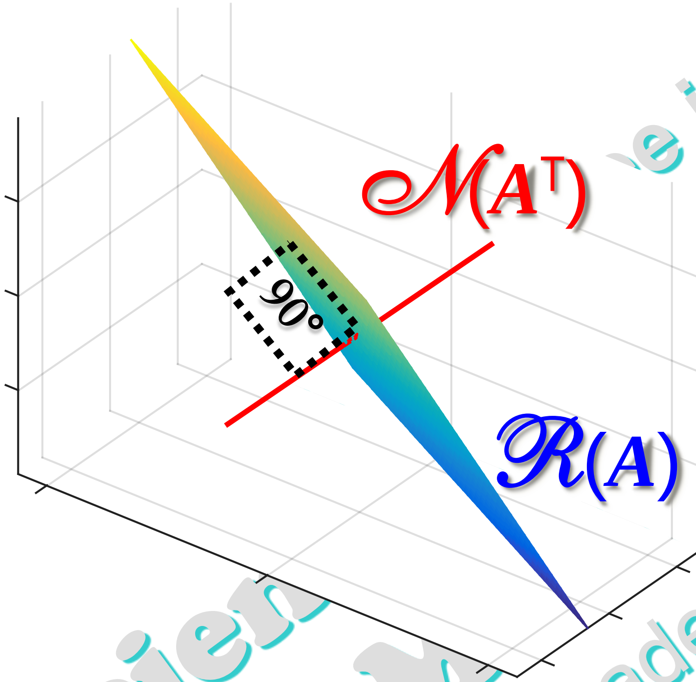
Unlike the **plot** function, **line** adds the line to the current axes without deleting other graphics objects: **hold on** doesn't need

```
p=A([1 2],:)'*[a b]';
q=S([1 2],:)'*[a b]';
fmesh(p(1),p(2),p(3),[-1 1])
axis('equal'); hold on
fsurf(q(1),q(2),q(3),[-5 5])
```

Left Null Space and Column Space

$$\mathcal{N}(A^{\mathsf{T}}) = \{y \in \mathbb{R}^m : A^{\mathsf{T}}y = \underline{0}\} \qquad \mathcal{N}(A^{\mathsf{T}}) \subseteq \mathbb{R}^m$$

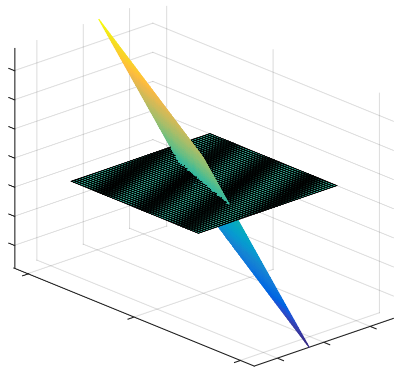
$$\mathcal{R}(A) = \text{span}\{A_{:,1}, A_{:,2}, \ldots, A_{:,m}\} \qquad \mathcal{R}(A) \subseteq \mathbb{R}^m$$




```
A=[1 0 1;5 4 9;2 4 6]; S=rref(A)
S =
     1     0     1
     0     1     1
     0     0     0
syms a b real; p=A(:,[1 2])*[a b]';
ezmesh(p(1),p(2),p(3)); axis('equal')
N=null(A')*[-10 10];% two points on 𝒩(Aᵀ)
line(N(1,:),N(2,:),N(3,:))
```

linearly independent cols

```
p=A(:,[1 2])*[a b]';
q=S(:,[1 2])*[a b]';
fmesh(p(1),p(2),p(3));
axis('equal')
hold on; fsurf(q(1),q(2),q(3))
```



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