Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

DiST — Dipartimento di Scienze e Tecnologie — Università Parthenope

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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Contents

- Linear independence.
- Basis of a space.

Example: why do 3 vectors in $\mathbb{R}^3$ generate a plane?

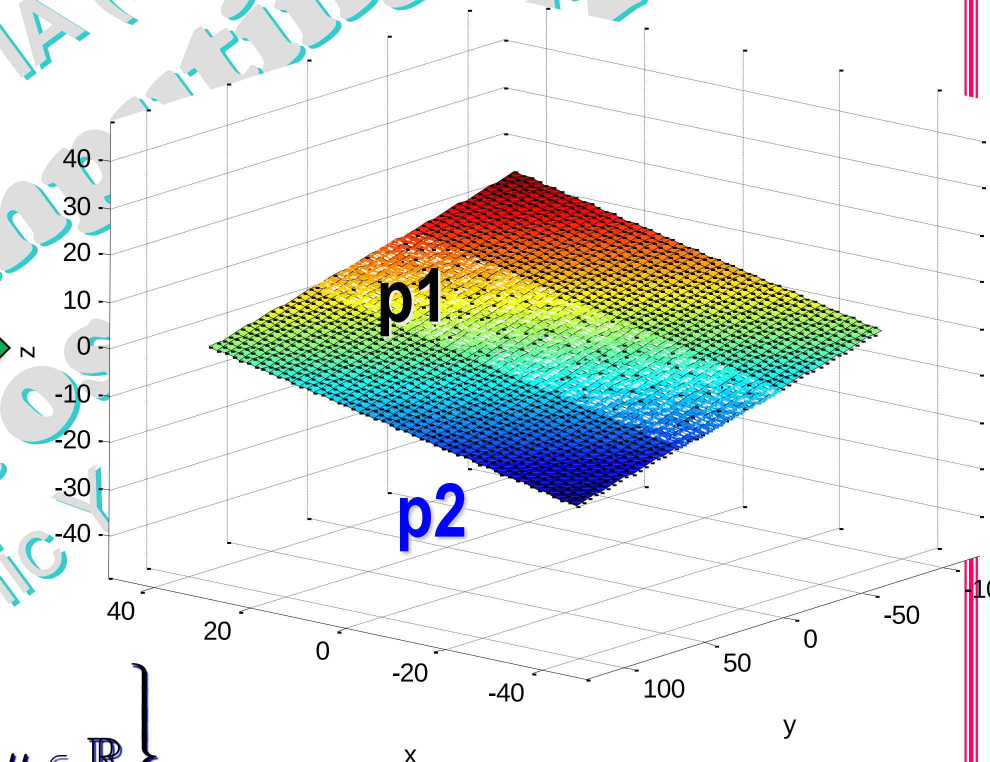
Display the range $\mathscr{R}(A)$ where: $A = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{pmatrix}$

Columns 1 and 3:

```
A=[1 3 3;2 6 9;-1 -3 3];
syms a b real
p1=A(:,[1 3])*[a b]';
ezmesh(p1(1),p1(2),p1(3))
```

Columns 2 and 3:

```
syms c d real
p2=A(:,2:3)*[c d]';
hold on
ezsurf(p2(1),p2(2),p2(3))
```



$$\mathscr{R}(A) = \left\{ \boldsymbol{w} = \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix},\ \lambda, \mu \in \mathbb{R} \right\}$$

The 3 vectors are linearly dependent!

Linear independence

k vectors in a linear space, $\{u_1, u_2, \dots, u_k\}$, are said to be **linearly independent** if the homogeneous linear system

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k = \underline{0} \quad \Leftrightarrow \quad \alpha_i = 0 \;\; \forall i$$

admits, as its solution $(\alpha_1, \alpha_2, \dots, \alpha_k)^\mathsf{T}$, only the zero vector. (trivial solution)

Otherwise they are said to be **linearly dependent**.

Examples

linearly independent $\leftarrow$ $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}\right\}$ $\rightarrow$ linearly dependent

linearly dependent $\leftarrow$ $\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}3\\6\\-3\end{pmatrix}, \begin{pmatrix}3\\9\\3\end{pmatrix}\right\}$

Example of linearly dependent vectors

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_k u_k = \underline{0} \quad \Leftrightarrow \quad \exists i : \alpha_i \neq 0$$

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{pmatrix}$$

```
A=[1 3 3;2 6 9;-1 -3 3]
A =
     1     3     3
     2     6     9
    -1    -3     3
N=null(A); N'
ans =
   -0.9487      0.3162      0.0000
N(1)*A(:,1)+N(2)*A(:,2)+N(3)*A(:,3)
ans =
  1.0e-014 *
             -0.0583
             -0.1082
              0.0749
```

solve the homogeneous system

linear combinations of the columns in *A*

```
A*N
ans =
  1.0e-014 *
             -0.0583
             -0.1082
              0.0749
```

$\Rightarrow \quad A\eta = \underline{0}, \ \forall \eta \in \mathcal{N}(A)$

1e-14 denotes a number so small that we consider it as zero!

Theorem

k vectors in a linear space are **linearly dependent** if, and only if, one of them is a **linear combination** of the remaining vectors.

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \underline{0}$$

$$\alpha_1 \neq 0 \Rightarrow$$

$$u_1 = -\frac{\alpha_2 u_2 + \alpha_3 u_3}{\alpha_1}$$

Consequences

2 vectors *u* and *v* of a linear space are **parallel** if, and only if, they are **linearly dependent**.

$$\exists \alpha, \beta \ : \ |\alpha| + |\beta| \neq 0 \ \wedge \ \alpha \underline{u} + \beta \underline{v} = \underline{0}$$

$$\Leftrightarrow \quad \beta \neq 0 \ \Rightarrow \ \underline{v} = -\frac{\alpha}{\beta} \underline{u}$$

scaled vectors: on the same line

3 vectors *u*, *v* and *w* of a linear space are **coplanar** if, and only if, they are **linearly dependent**.

$$\exists \alpha, \beta, \gamma \ : \ |\alpha| + |\beta| + |\gamma| \neq 0 \ \wedge \ \alpha \underline{u} + \beta \underline{v} + \gamma \underline{w} = \underline{0}$$

$$\Leftrightarrow \quad \gamma \neq 0 \ \Rightarrow \ \underline{w} = -\frac{\alpha}{\gamma} \underline{u} - \frac{\beta}{\gamma} \underline{v}$$

coplanar vectors: on the same plane

Matrix rank

Matrix **rank** denotes the number of rows and of columns which are **linearly independent**.

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{pmatrix}$$

```
A=[1 3 3;2 6 9;-1 -3 3]
A =
     1     3     3
     2     6     9
    -1    -3     3
```

```
rref(A)
ans =
     1     3     0
     0     0     1
     0     0     0
```

```
rank(A)
ans =
         2
```

rref: row-reduced echelon form

rref: row-reduced echelon form

A matrix is in **echelon form** if it has the shape resulting of a *Gaussian elimination*. **Row echelon form** means that *Gaussian elimination* has operated on the rows (row reduction).

```
A=[1 3 3;2 6 9;-1 -3 3];
[L,U,P]=lu(A); U
U =
     2     6     9
     0     0  -1.5
     0     0   7.5
```

A matrix is in **row-reduced echelon form** if it satisfies the following conditions:

- It is in row echelon form.
- Every leading coefficient is 1 and is the only nonzero entry in its column.

```
A=[1 3 3;2 6 9;-1 -3 3];
S=rref(A); S
S =
     1     3     0
     0     0     1
     0     0     0
```

The **row-reduced echelon form** of a matrix may be computed by *Gauss-Jordan elimination* (Gauss↓ + Gauss↑, respectively *forward* and *backward G-J*).

Unlike the *row echelon form*, the **row-reduced echelon form** of a matrix is **unique** and does not depend on the algorithm used to compute it.

Bases of a Linear Space

A sequence of vectors $\{u_1, u_2, \ldots, u_n\}$ of a linear space X is said a **basis of X**, if

- $X = \operatorname{span}\{u_1, u_2, \ldots, u_n\}$ (they span the space)
- the vectors are linearly independent.

EXAMPLE

a basis for $\mathbb{R}^3$ (standard basis)

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

another basis for $\mathbb{R}^3$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}$$

In a linear space there are infinitely many bases

Components of a vector

If $\{u^{(1)}, u^{(2)},\dots, u^{(n)}\}$ is a basis for X, then

$$\forall x \in X \quad \exists \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R} \;:\; x = \alpha_1 u^{(1)} + \alpha_2 u^{(2)} + \cdots + \alpha_n u^{(n)}$$

the scalars (α_i) are said **components** of x with respect to the basis $\{u^{(1)}, u^{(2)},\dots, u^{(n)}\}$.

EXAMPLE

$$x = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \quad \Rightarrow \quad x = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The components of x with respect to the standard basis are $(2, -1, 1)^T$.

Change of basis

w.r.t. the standard basis

To compute the components of $x = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ w.r.t. the following basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

we have to set $\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = x = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and solve the system:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 10/3 \\ -5/3 \\ 1/3 \end{pmatrix}$$

Theorem

If both sequences $\{u^{(1)}, u^{(2)}, \ldots, u^{(n)}\}$ and $\{v^{(1)}, v^{(2)}, \ldots, v^{(k)}\}$ are **bases** of a space X, then $k = n$.

That is, all bases of a Linear Space contain the same number of vectors.

$$x = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$$ with respect to the standard basis $$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = B_1$$

$$x = \begin{pmatrix} 10/3 \\ -5/3 \\ 1/3 \end{pmatrix}$$ with respect to the basis $$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} = B_2$$

$$x = \alpha_1 \vec{u_1} + \alpha_2 \vec{u_2} + \alpha_3 \vec{u_3} = \begin{pmatrix} \vec{u_1} & \vec{u_2} & \vec{u_3} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\vec{\beta} = B_1^{-1} B_2 \, \vec{\alpha}$$

$$\vec{\alpha} = B_2^{-1} B_1 \, \vec{\beta}$$

$$B_1 \vec{\beta} = x = B_2 \vec{\alpha}$$

$$x = \beta_1 \vec{v_1} + \beta_2 \vec{v_2} + \beta_3 \vec{v_3} = \begin{pmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$B_{new}^{-1} B_{old}$: change-of-basis matrix