Scuola Interdipartimentale delle Scienze, dell'Ingegneria e della Salute

DiST – Dipartimento di Scienze e Tecnologie

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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Contents

- Linear spaces and subspaces.

Basics of Algebra

Commutative group (or abelian group) $\langle G,\# \rangle$

It is a set G together with an operation # such that:

$$\forall \alpha,\beta \in G,\ \ \alpha \# \beta \in G$$

1. $\forall \alpha,\beta,\gamma \in G \ \ (\alpha\#\beta)\#\gamma = \alpha\#(\beta\#\gamma)$ (associativity of #)
2. $\exists \alpha_o \ \ \ \beta\#\alpha_o = \beta$ (∃ identity element)
3. $\forall \beta\ \exists\ \beta^* : \beta\#\beta^* = \alpha_o$ (∃ inverse element)
4. $\forall \alpha,\beta \ \ \ \alpha\#\beta = \beta\#\alpha$ (commutativity of #)

Field $\langle A,+,\times \rangle$

It is a set A that is a ***commutative group*** with respect to two compatible operations, addition (+) and multiplication (×).

In addition the following property holds:

$$\forall a,b,c \in A,\ \ a\times(b+c) = a\times b + a\times c$$

(distributivity of × over +)

Linear (or vector) spaces

Let X be a set (named as *support* of the space) and K a *field of numbers* (usually K is $\mathbb{R}$ or $\mathbb{C}$).

$\langle X, K, +, * \rangle$ is a **LINEAR SPACE** (or **VECTOR SPACE**) if X is a collection of objects called vectors, which may be added together (+) and multiplied (*) or "scaled" by numbers in K, called scalars in this context:

- **addition**

internal $\longrightarrow$ inside the space

$$+ \; : \; (u, v) \in X \times X \longrightarrow u + v \in X$$

- **multiplication by a scalar**

external $\longrightarrow$ inside the space

$$* \; : \; (\alpha, v) \in K \times X \longrightarrow \alpha * v = \alpha v \in X$$

addition and multiplication: properties

1 $\exists\, 0 \in X \quad : \quad \forall x \in X \quad x + 0 = x$ (existence and unicity of identity element or zero vector)

2 $\forall x \in X \;\; \exists -x \in X \quad : \quad x + (-x) = 0$ (existence and unicity of the opposite, or additive inverse, of x)

3 $\forall x, y \in X \quad x + y = y + x$ (commutativity law)

4 $\forall x, y, z \in X \quad (x + y) + z = x + (y + z)$ (associativity law)

5 $\forall x \in X,\ \forall \alpha, \beta \in K \quad \alpha(\beta x) = (\alpha\beta)x$ (compatibility of scalar multiplication with field multiplication)

6 $\forall x \in X,\ \forall \alpha, \beta \in K \quad (\alpha + \beta)x = \alpha x + \beta x$

7 $\forall x, y \in X,\ \forall \alpha \in K \quad \alpha(x + y) = \alpha x + \alpha y$

(distributivity law)

Usually the multiplication symbol is omitted

Linear combination of two vectors

Given two scalars α and β and two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then the $\boldsymbol{w}$ vector defined as

$$w = \alpha u + \beta v$$

is named as linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

A linear space contains all the linear combinations of its vectors.

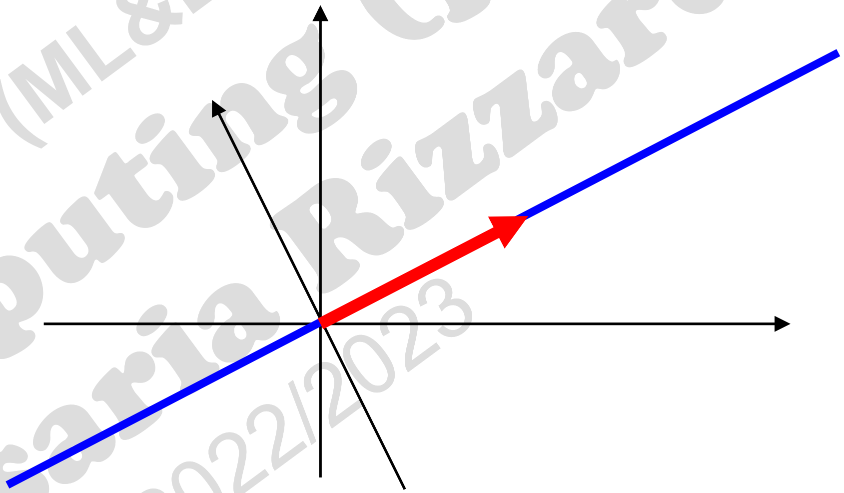
Examples of Linear spaces

- $\mathbb{R}^2$: set of vectors in the plane (a pair of real components).
- $\mathbb{C}^2$: set of vectors with two complex components.
- $\mathbb{R}^3$: set of vectors with three real components.
- $\mathbb{C}^3$: set of vectors with three complex components.
- $\mathbb{R}^n$: set of vectors with *n* real components (*n*-tuple).
- $\mathbb{C}^n$: set of vectors with *n* complex components.
- $M_{(m\times n)}(\mathbb{R})$: Set of real matrices of size m×n.
- $\mathcal{N}(A)$: Set of solutions of a homogeneous system $\boldsymbol{Ax}=\underline{0}$.
- $\boldsymbol{C(a,b)}$: Set of real functions $f(x)$ continuous on (a,b).

Examples

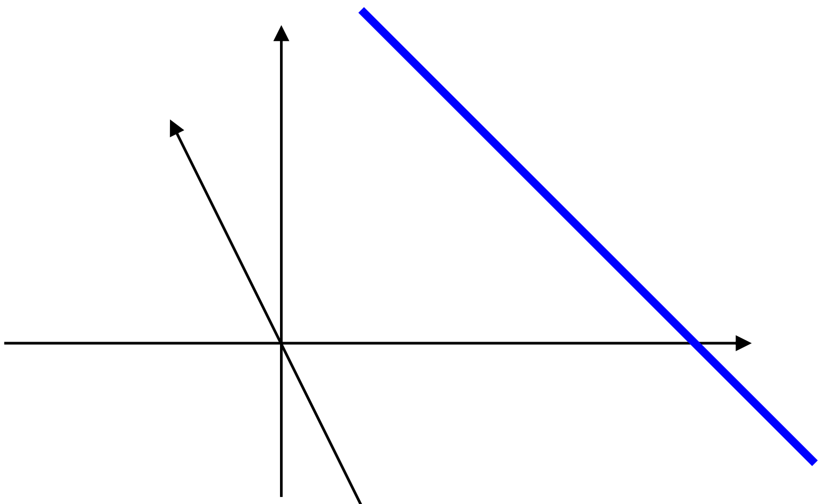
❖ Why a line in the plane and passing through the origin is a linear space?

This line contains all the vectors that lie on it, and this set verifies all the properties of a linear space.



❖ Why a line in the plane that does not pass through the origin **is not** a linear space?

Since this line does not pass through the origin, it does not contain the zero vector. Then it is not a linear space.



Subspaces of a linear space

W is a **subspace** of a linear space *X*, if:

- *W* is a non-empty subset of *X*;
- *W* is a linear space.

Theorem

W is a subspace of *X* if, and only if, all the linear combinations of its vectors stay in the subspace.

***W* is a subspace**

$\Updownarrow$

$$\forall u, v \in W, \forall \alpha, \beta \in K \quad \alpha u + \beta v \in W$$

(prof. M. Rizzardi)

Examples

❖ Why the set of solutions of a homogeneous system

$$Ax = \underline{0}$$

is a linear subspace?

Check if the previous Theor. holds

$$x, y \in \mathcal{N}\ A\ : \quad \begin{matrix} Ax = \underline{0} \\ Ay = \underline{0} \end{matrix} \quad \Rightarrow \quad A\ \ \alpha x + \beta y\ = \alpha Ax + \beta Ay = \underline{0}$$

It is named as Null Space of A and denoted as $\mathcal{N}$(A).

❖ Why the set of solutions of a non-homogeneous system

$$Ax = b, \qquad b \neq \underline{0}$$

is not a linear subspace?

$$\begin{matrix} Ax = b \\ Ay = b \end{matrix} \quad \Rightarrow \quad A\ \ \alpha x + \beta y\ = \alpha Ax + \beta Ay = \ \alpha + \beta\ \ b \neq b$$

The previous Theor. doesn't hold

Linear combination of n vectors

We can extend to n vectors the definition of a linear combination:

$$\forall v_1, v_2, \ldots, v_n \in X, \quad \forall \alpha_1, \alpha_2, \ldots, \alpha_n \in K$$

$$w = \underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n}_{\text{linear combination of } n \text{ vectors}} \Rightarrow \quad w \in X$$

The set W containing all combinations of n vectors $v_1, v_2, \ldots, v_n$ is a subspace of X, and it is denoted as

$$W = \textbf{span}\ \ v_1, v_2, \ldots, v_n$$

W is the subspace of X spanned by $v_1, v_2, \ldots, v_n$.
(or generated)

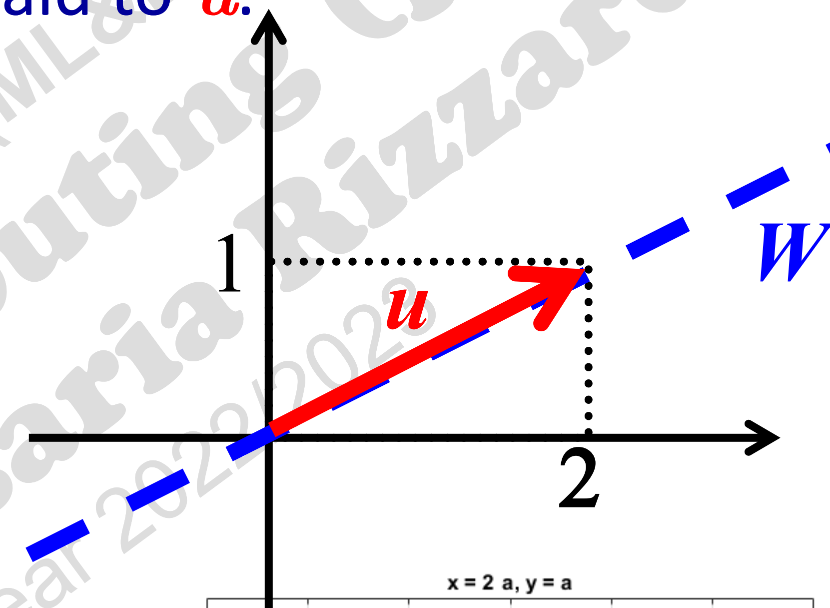
Example of span{…}

The subspace spanned by a single vector u of $\mathbb{R}^2$ is the line through the origin and overlaid to u.

$$u = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow W = \text{span } u$$

$$W = \left\{ \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \forall \alpha \in \mathbb{R} \right\}$$

director vector of W

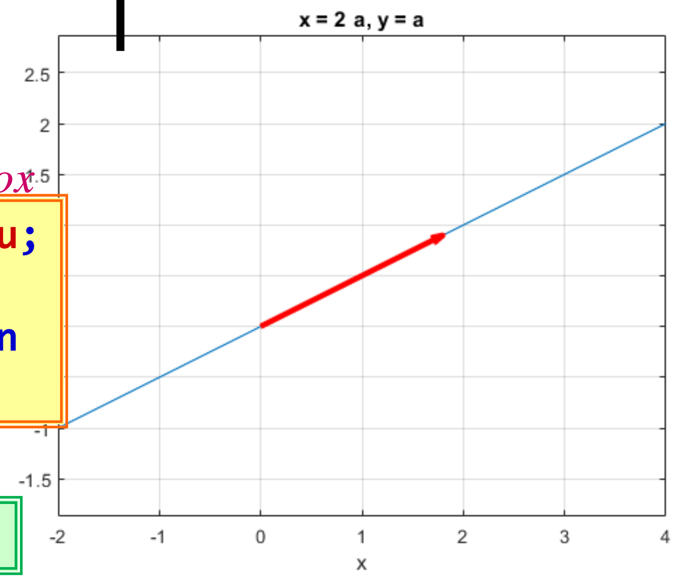


MATLAB *Symbolic Math Toolbox*

```
u=[2 1]'; syms a real; W=a*u;
ezplot(W(1),W(2),[-1 2])
hold on; axis equal; grid on
quiver(0,0,u(1),u(2),'r')
```

```
fplot(W(1),W(2),[-1 2])
```

```
compass(u(1),u(2),'r')
```

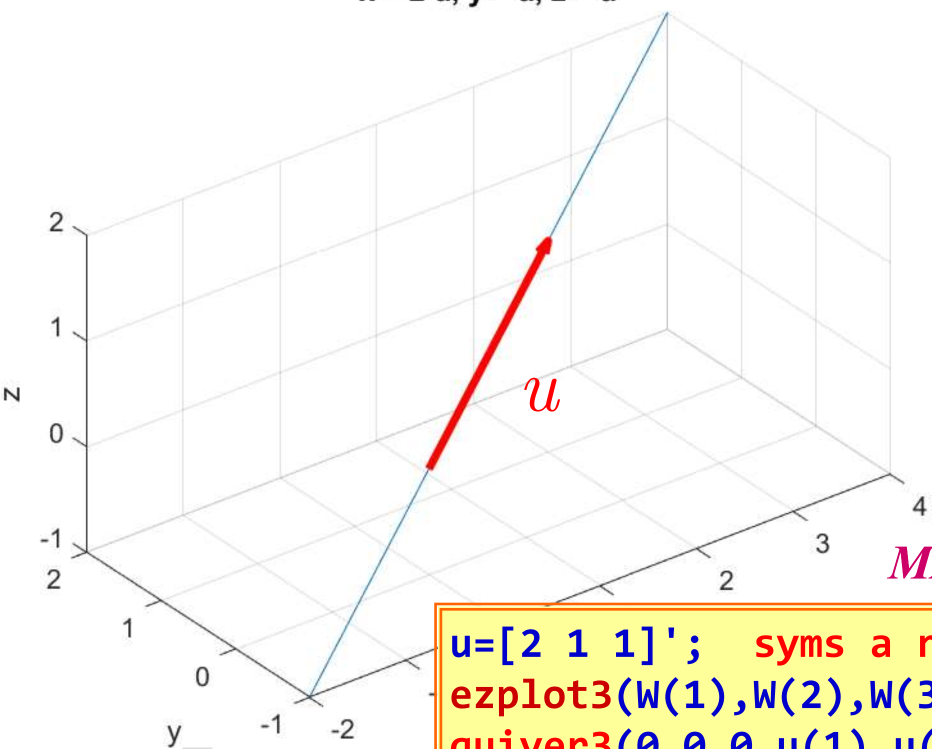


Example of span{…}

The subspace spanned by a single vector $\boldsymbol{u}$ of $\mathbb{R}^3$ is the line through the origin and overlaid to $\boldsymbol{u}$.

$$u = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \Rightarrow W = \text{span}\{u\} = \left\{ \alpha \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \forall \alpha \in \mathbb{R} \right\}$$

director vector of $\boldsymbol{W}$



MATLAB Symbolic Math Toolbox

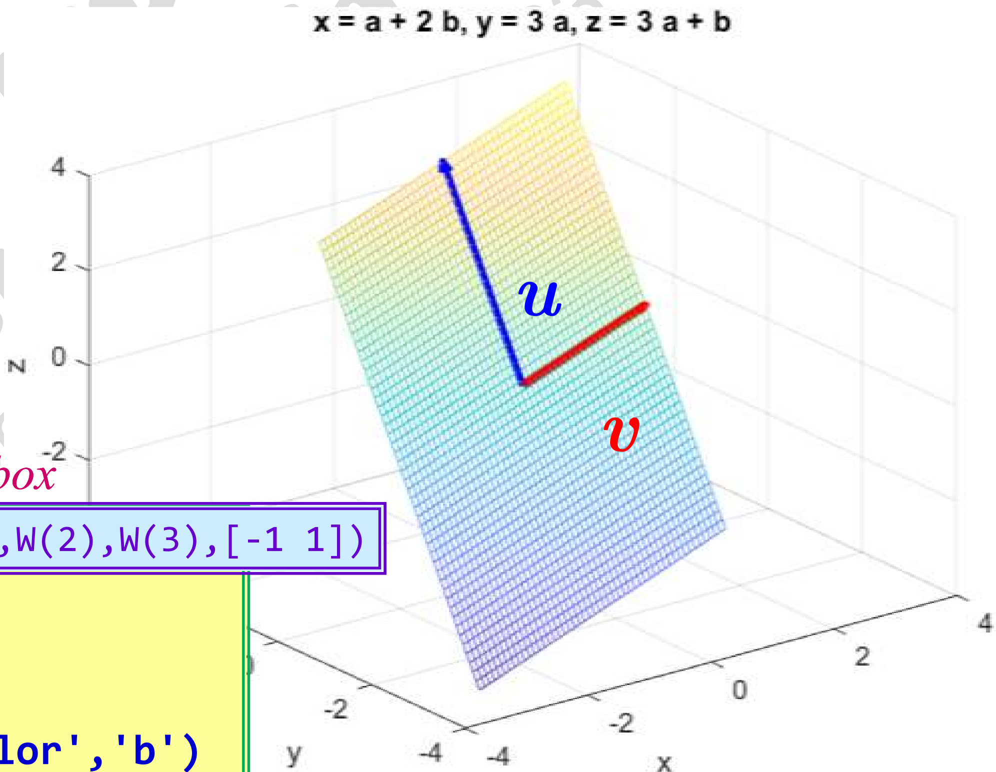
```
u=[2 1 1]';  syms a real;  W=a*u;
ezplot3(W(1),W(2),W(3),[-1 2]); hold on; axis equal
quiver3(0,0,0,u(1),u(2),u(3),1,'Color','r','LineWidth',3)
```

Example of span{…}

The subspace spanned by two vectors u, v of $\mathbb{R}^3$ is the plane through the origin that contains u and v.

$$u=\begin{pmatrix}1\\3\\3\end{pmatrix},\quad v=\begin{pmatrix}2\\0\\1\end{pmatrix}\quad\Rightarrow\quad W=\text{span}\{u,v\}$$

$$W=\left\{\alpha\begin{pmatrix}1\\3\\3\end{pmatrix}+\beta\begin{pmatrix}2\\0\\1\end{pmatrix},\forall\alpha,\beta\in\mathbb{R}\right\}$$



***MATLAB** Symbolic Math Toolbox*

```
u=[1 3 3]'; v=[2 0 1]';
syms a b real;  W=a*u+b*v;
ezmesh(W(1),W(2),W(3),[-1 1])
hold on
quiver3(0,0,0,u(1),u(2),u(3),0,'Color','b')
quiver3(0,0,0,v(1),v(2),v(3),0,'Color','r')
```

```
fmesh(W(1),W(2),W(3),[-1 1])
```