



Course of "Automatic Control Systems"
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Stability of LTI systems

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Stability

- ✦ A linear system is said *stable* if no evolution mode is divergent (only convergent and constant evolution modes).
- ✦ It happens if all the eigenvalue of the matrix A (pole of the $W(s)$) have a negative or null real part and the eigenvalues with null real part have multiplicity 1.
- ✦ In a stable system
 - ✦ *the free evolution doesn't tend to infinity*
 - ✦ *the free evolution doesn't converge to zero* if the constant evolution mode is excited



Example 1

- ✦ Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- ✦ In order to evaluate the eigenvalues of the matrix A , we can calculate the roots of the characteristic polynomial
- ✦ In Matlab, it is possible to use the command *eig(A)*
- ✦ In this example we have $p_1 = p_2 = -1$.
- ✦ This system is *asymptotically stable* because it has all eigenvalues with negative real part



Example 2

✦ Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

✦ In this example we have $p_1 = p_2 = 1$.

✦ The system is *unstable* because it has two eigenvalues with positive real part



Example 3

✧ Let us consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

✧ In this example we have $p_1 = 0$, $p_2 = -1$.

✧ The system is *stable* because it has

✧ a null eigenvalue

✧ an eigenvalue with negative real part



Example 4

- ✦ Let us consider the transfer function of an LTI system

$$W(s) = \frac{s + 1}{s^2(s + 5)}$$

- ✦ This system is *unstable* because it has two null poles.



Routh-Hurwitz criterion

- ✦ Routh-Hurwitz criterion is used to study the sign of the real part of a polynomial roots.
- ✦ It is particularly useful in case of high order polynomials or polynomials with uncertain parameters where the evaluation of the roots can be difficult.
- ✦ Routh-Hurwitz criterion is of interest to study the stability of LTI systems both in the state-space form and in the Laplace domain

$$W(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

where the poles of $W(s)$ coincide with the eigenvalues of the matrix A



Necessary condition

✧ Let us consider a polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$$

and without loss of generality let us assume that

$$\spadesuit a_n > 0$$

$$\spadesuit a_0 \neq 0$$

✧ Stodola criterion (Necessary condition):

A necessary condition for the roots of the polynomial $D(s)$ to have negative real parts is that

$$\mathbf{sign(a_0) = sign(a_1) = \dots = sign(a_n).}$$

This condition is also sufficient for polynomials of degree $n = 1, n = 2$.



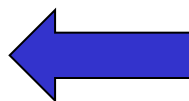
Routh table

✧ Let us consider the polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$$

✧ *The Routh table* is defined as follows

n	a_n	a_{n-2}	a_{n-4}	\dots	$b_{n-2} = -\frac{1}{a_{n-1}} \det \begin{pmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{pmatrix}$
$n-1$	a_{n-1}	a_{n-3}	a_{n-5}	\dots	
$n-2$	b_{n-2}	b_{n-4}	b_{n-6}	\dots	$b_{n-4} = -\frac{1}{a_{n-1}} \det \begin{pmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{pmatrix}$
$n-3$	c_{n-3}	c_{n-5}	\dots	\dots	
\dots	\dots	\dots	\dots	\dots	$c_{n-3} = -\frac{1}{b_{n-2}} \det \begin{pmatrix} a_{n-1} & a_{n-3} \\ b_{n-2} & b_{n-4} \end{pmatrix}$

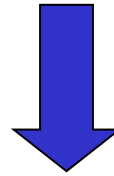




Routh table: Example

- ✦ Routh table, $n + 1$ rows and the last row has 1 element different from zero.
- ✦ Let us define the Routh table of the function

$$f(s) = s^4 + 2s^3 + 3s^2 + 5s + 10$$



4	1	3	10
3	2	5	0
2	0.5	10	0
1	-35	0	0
0	10	0	0



Routh criterion

- Let us consider the Routh table of the polynomial $D(s)$

n	a_n	a_{n-2}	a_{n-4}	\dots
$n-1$	a_{n-1}	a_{n-3}	a_{n-5}	\dots
$n-2$	b_{n-2}	b_{n-4}	b_{n-6}	\dots
$n-3$	c_{n-3}	c_{n-5}	\dots	\dots
\dots	\dots	\dots	\dots	

- The roots of the polynomial $D(s)$ have all negative real parts iff the elements of the first column of the Routh table are all positive.*
- Each sign variation of the element of the first column of the Routh table correspond to a root of $D(s)$ with a positive real part.*



Routh criterion: example

✧ Let us consider the polynomial

$$f(s) = s^4 + 2s^3 + 3s^2 + 5s + 10$$

✧ The Routh table of $f(s)$ is

4	1	3	10	0	<table border="1"><thead><tr><th>Roots of $f(s)$</th></tr></thead><tbody><tr><td>$0.544 + j1.60$</td></tr><tr><td>$0.544 - j1.60$</td></tr><tr><td>$-1.54 + j1.06$</td></tr><tr><td>$-1.54 - j1.06$</td></tr></tbody></table>	Roots of $f(s)$	$0.544 + j1.60$	$0.544 - j1.60$	$-1.54 + j1.06$	$-1.54 - j1.06$
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2	0.5	10	0	0						
1	-35	0	0	0						
0	10	0	0	0						



Routh Criterion: uncertain parameters

- Let us consider a transfer function $W(s)$ of an LTI system where the poles of $W(s)$ depends on an uncertain parameter p ,

$$W(s) = \frac{s+1}{2s^3 + 5ps^2 + (3+p)s + 1}$$

- From the Routh table we have that

3	2	3+p	→	{	5p > 0
2	5p	1			→
1	5p² + 15p - 2	0			↓
0	1	0		{	p > 0
					→
				{	p < -3.13 ∧ p > 0.128
					→
					p > 0.128



Routh criterion: Singular Cases

- ✦ In the design of the Routh table two singular cases can be found
 - a) *The first term of a row is null*
 - b) *All the terms of a row are null*

- ✦ In these cases, some mathematical manipulations of the Routh table can be adopted. However, it is not of interest for this course.