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Laplace domain analysis of LTI systems

Prof. Francesco Montefusco

Department of Economics, Law, Cybersecurity, and Sports Sciences

Università degli studi di Napoli Parthenope

francesco.montefusco@uniparthenope.it

Team code: **uxbsz19**

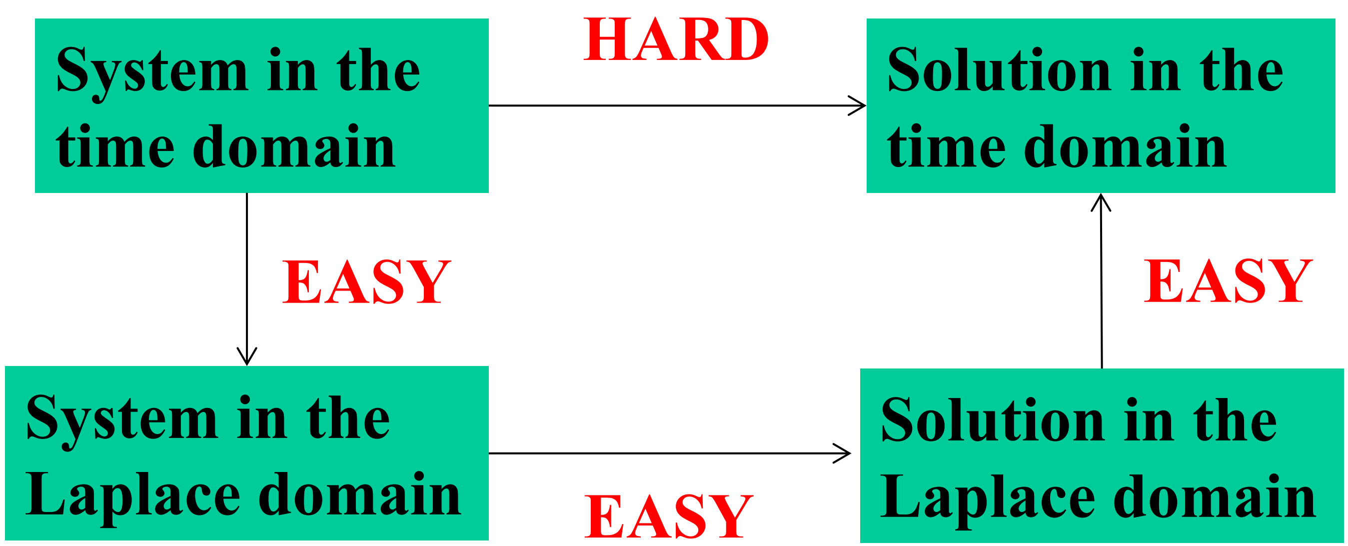
Evaluation of an LTI system response

- Let us consider a Linear Time Invariant (LTI) system in the state space form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \tag{1.a}$$

$$y(t) = Cx(t) + Du(t) \tag{1.b}$$

- The Evaluation of an LTI system response in a transformed domain is convenient only if



- Let us indicate with ***$X(s), U(s)$ and $Y(s)$ the Laplace transforms of the signals $x(t), u(t)$ and $y(t)$.***

- Transforming both the sides of the equation (1), we have

$$L(\dot{x}(t)) = L\big(Ax(t) + Bu(t)\big)$$

$$L(y(t)) = L\big(Cx(t) + Du(t)\big)$$

- Using the time domain derivation property of the Laplace transform, a linear system in the Laplace domain can be written has

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s) \qquad (2.1)$$

$$Y(s) = C(sI - A)^{-1}x_0 + C(sI - A)^{-1}BU(s) + DU(s) \qquad (2.2)$$

- ***Note that in the Laplace domain the dependency of the state variables $X(s)$ from the input $U(s)$ is expressed by a matrix product instead of a convolution***

LTI systems in the Laplace domain

- The matrix function $\Phi(s) = (sI - A)^{-1}$ is called ***Transition matrix*** whose dimension is given by the dimension of the A matrix.

- The matrix function $\mathrm{W}(s) = C(sI - A)^{-1}B + D$ is called ***Transfer function***.

$$X(s) = \Phi(s)x_0 + \Phi(s)BU(s) \qquad (3.1)$$

$$Y(s) = C\Phi(s)x_0 + W(s)U(s) \qquad (3.2)$$

- For ***Single Input Single Output (SISO)*** systems the transfer function $\mathrm{W}(s)$ is a scalar function;

- For ***Multiple Input Multiple Output (MIMO)*** systems the transfer function $\mathrm{W}(s)$ is a matrix whose element $\mathrm{W}(s)_{ij}$ will connect the output i with the input j.

Transfer function

- For SISO systems the scalar ***transfer function*** is given by the ratio of two polynomial functions

$$W(s) = \frac{N(s)}{D(s)} = \frac{a_m\, s^m + a_{m-1}\, s^{m-1} + \cdots + a_1\, s + a_0}{b_n\, s^n + a_{n-1}\, s^{n-1} + \cdots + b_1\, s + b_0}$$

where $m \leq n$.

- ***If $m < n$ the system is said strictly proper***. It happens when the *D* matrix of the LTI system in the state space is zero.

- ***If $m = n$ the system is said proper***. It happens when the *D* matrix of the LTI system in the state space is different from zero zero.

$$\mathrm{W}(s) = C(sI - A)^{-1}B + D$$

Transition matrix

- Given a ***transfer function***

$$W(s) = \frac{N(s)}{D(s)} = \frac{a_m\, s^m + a_{m-1}\, s^{m-1} + \cdots + a_1\, s + a_0}{b_n\, s^n + a_{n-1}\, s^{n-1} + \cdots + b_1\, s + b_0}$$

- The roots of the $\mathrm{N(s)}$ are said ***zeros.***
- The roots of the $\mathrm{D(s)}$ are said ***poles.***
- The polynomial $\mathrm{D(s)}$ is defined as $D(s) = \det(sI - A)$, hence
  - ***D(s) coincides with the characteristic polynomial of the system***
  - ***the poles coincide with the eigenvalues of the system*** except for possible pole-zero cancellation

Laplace antitransform

- For SISO systems ***the free evolution in the Laplace domain is given by the ratio of polynomial functions***

$$Y_{free}(s) = C\Phi(s)x_0$$

- ***This is also true for the forced evolution in case we restrict our attention to the case of polynomial and sinusoidal inputs***

$$Y_{forced}(s) = W(s)U(s)$$

- It is convenient to antitransform $Y(s)$ by reducing the ratio of high degree polynomial functions to the sum of selected signals transform such as

$$L\left(e^{\alpha t}\cos(\omega t)\cdot 1(t)\right) = \frac{s-\alpha}{(s-\alpha)^2+\omega^2} \qquad L\left(e^{\alpha t} sen(\omega t)\cdot 1(t)\right) = \frac{\omega}{(s-\alpha)^2+\omega^2}$$

$$L(e^{\alpha t}1(t)) = \frac{1}{s-\alpha}$$

- Different methods can be used to reduce the ratio of high degree polynomial functions to the sum polynomial functions of degree one or two, such as ***the residual method*** (see the book for details).

Residual method for real and distinct eigenvalues

(see the book for the other cases)

$$W(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_{i=1}^{n}(s-p_i)} \qquad p_i \neq p_j \text{ for } i \neq j$$

- In case of real and distinct eigenvalues, $W(s)$ can be also written as

$$W(s) = \sum_{i=1}^{n} \frac{A_i}{s - p_i}$$

where $A_k = \lim_{s \to p_k} (s - p_k)W(s)$. Hence

$$w(t) = \sum_{i=1}^{n} A_i \, e^{p_i t}$$

Laplace antitransform: example 1

CASE 1: real and distinct eigenvalues

$$Y_{free}(s) = C\Phi(s)x_0 = \frac{s-10}{(s+2)(s+5)}$$

- Appling the residual method we have

$$Y_{free}(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+5)}$$

with

$$A_1 = \lim_{s\to -2}(s+2)Y_{free}(s) = \lim_{s\to -2}\frac{s-10}{s+5} = -4$$

$$A_2 = \lim_{s\to -5}(s+5)Y_{free}(s) = \lim_{s\to -5}\frac{s-10}{s+2} = 5$$

Hence

$$y_{free}(t) = (-4e^{-2t} + 5e^{-5t}) \cdot 1(t)$$

CASE 2: real multiple eigenvalues

$$Y_{forced}(s) = W(s)U(s) = \frac{s+18}{(s+3)^2}U(s) \qquad \text{with } u(t) = 1(t)$$

- This function can be written as the sum of three terms

$$Y_{forced}(s) = \frac{s+18}{s(s+3)^2} = \frac{A_1}{s} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+3)^2}$$

- The residual method can be applied to evaluate A_1 and A_3, while A_2 can be evaluated by substitution

$$A_1 = \lim_{s\to 0} sY_{forced}(s) = 2 \qquad A_3 = \lim_{s\to -3} (s+3)^2\, Y_{forced}(s) = -5$$

while $A_2 = -2$. (By residual method for pole with multiplicity *r*, the l-th residual K_l, l=1,…,r, by $K_l = \frac{1}{(r-l)!}\frac{d^{r-l}}{ds^{r-l}}(s-p_i)^r Y(s)|_{s=p_i}$, then $A_2 = \frac{d(s+3)^2 Y_{forced}(s)}{ds}|_{s=-3}$)

Hence, $$y_{forced}(t) = (2 - 2e^{-3t} - 5te^{-3t}) \cdot 1(t)$$

CASE 3: complex conjugate eigenvalues

$$Y_{free}(s) = C\Phi(s)x_0 = \frac{100}{(s+1)(s^2+4s+13)}$$

- This function can be written as the sum of two terms

$$Y_{free}(s) = \frac{A_1}{(s+1)} + \frac{A_2 s + A_3}{(s^2+4s+13)}$$

- The residual method can be applied to evaluate A_1, while A_2 and A_3 can be evaluated by substitution. $A_1 = 10,\ A_2 = -10,\ \ A_3 = -30$.

- Hence, $Y_{free}(s) = \frac{10}{(s+1)} - 10\frac{s+2}{(s+2)^2+9} - \frac{10}{3}\frac{3}{(s+2)^2+9}$ and

$$y_{free}(t) = \left(10e^{-t} - 10e^{-2t}\cos(3t) - \frac{10}{3}e^{-2t}\sin(3t)\right)\cdot 1(t)$$

INVERSE OF A MATRIX $N \times N$

Inverse of a matrix

- Given a quadratic and invertible matrix

$$A = \begin{pmatrix} x_{1,1} & \dots & x_{1,j} \\ \vdots & \ddots & \vdots \\ x_{i,1} & \dots & x_{i,j} \end{pmatrix}$$

its inverse is defined as

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \mathrm{cof}(A, x_{1,1}) & \dots & \mathrm{cof}(A, x_{1,j}) \\ \vdots & \ddots & \vdots \\ \mathrm{cof}(A, x_{i,1}) & \dots & \mathrm{cof}(A, x_{i,j}) \end{pmatrix}^T$$

where the cofactor is

$$\mathrm{cof}(A, i, j) = (-1)^{i+j} \det(\mathrm{minor}(A, i, j))$$

and the minor (i, j) is the determinant of the matrix obtained excluding the row i and the column j.

Inverse of a 2×2 matrix

- Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

its inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse of a 3×3 matrix

- Given a matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

its inverse is

$$\frac{1}{\det(A)} \begin{pmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} & - \begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} & + \begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \\ - \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} & + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} & - \begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \\ + \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} & - \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} & + \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \end{pmatrix}$$

EIGENVALUES AND EIGENVECTORS

Eigenvalues and eigenvectors

- Given a matrix $A \in R^{n \times n}$, a scalar $\boldsymbol{\lambda \in C}$ is said ***eigenvalue*** of the matrix A if there exists a vector $\boldsymbol{v \in C^n}$, said ***eigenvector***, such that

$$\boldsymbol{Av = \lambda v}$$

- Taking into account account that eigenvalues and eigenvectors of a matrix verify the equation

$$(A - \lambda I)v = 0.$$

  The eigenvalues can be found evaluating the roots of the ***characteristic polynomial*** $\boldsymbol{p(\lambda)}$ defined as

$$p(\lambda) = det(A - \lambda I).$$

- The ***poles of*** $\boldsymbol{W(s)}$ coincide to the eigenvalues of the matrix A.

Examples

$$A=\begin{pmatrix}1 & 2\\ 3 & -4\end{pmatrix}$$

$$p(s)=s^2+3s-10$$

Eigenvalues

$$\lambda_1=2,\lambda_2=-5$$

$$A=\begin{pmatrix}1 & 2 & 3\\ 4 & -5 & 6\\ 7 & -8 & 9\end{pmatrix}$$

$$p(s)=s^3-5s^2-22s-24$$

Eigenvalues

$$\lambda_1=8.09$$

$$\lambda_2=\text{-}1.54+j0.765$$

$$\lambda_3=\text{-}1.54\text{-}\,j0.765$$