Course of "Automatic Control System"
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Laplace transform

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Laplace transform definition

- ***The Laplace transform*** of a function ***f(t)*** is defined as

$$f(t) \rightarrow F(s) = L(f(t)) = \int_{0}^{+\infty} f(t)e^{-st}dt$$

where $t \in R$ is a real variable, while $s = \alpha + j\omega \in C$ is a complex variable.

- Vice versa, given a function $F(s)$ in the Laplace domain, the original function in the time domain can be obtained using the ***Laplace anti-transformation***

$$F(s) \rightarrow f(t) = \lim_{\omega \rightarrow \infty} \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st}ds$$

- ***The Laplace transform is a bilateral only if the function f(t) is null for*** $t < 0$

Laplace transform main properties (1/2)

- ***Linearity***

$$L\left(af(t)+bg(t)\right)=aF(s)+bG(s)$$

- ***Translation in the Laplace domain***

$$L\left(e^{\alpha t}f(t)\right)=F(s-\alpha)$$

- ***Translation in the time domain***

$$L\left(f(t-T)\right)=F(s)e^{-sT}$$

Laplace transform main properties (2/2)

- ***Time domain derivation***

$$L\left(\frac{df(t)}{dt}\right)=sF(s)-f(0)$$

- ***Time domain integration***

$$L\left(\int_0^t f(\tau)\,d\tau\right)=\frac{1}{s}F(s)$$

- ***Time domain convolution***

$$L\big(f(t)*g(t)\big)=F(s)G(s)$$

Additional properties useful in control theory

- ***Initial value theorem***

$$f(0) = \lim_{s\to\infty} sF(s)$$

- ***Final value theorem***

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

- ***Initial value theorem of the derivate of the function***

$$\left.\frac{df(t)}{dt}\right|_{t=0} = \lim_{s\to\infty} s^2F(s) - sf(0)$$

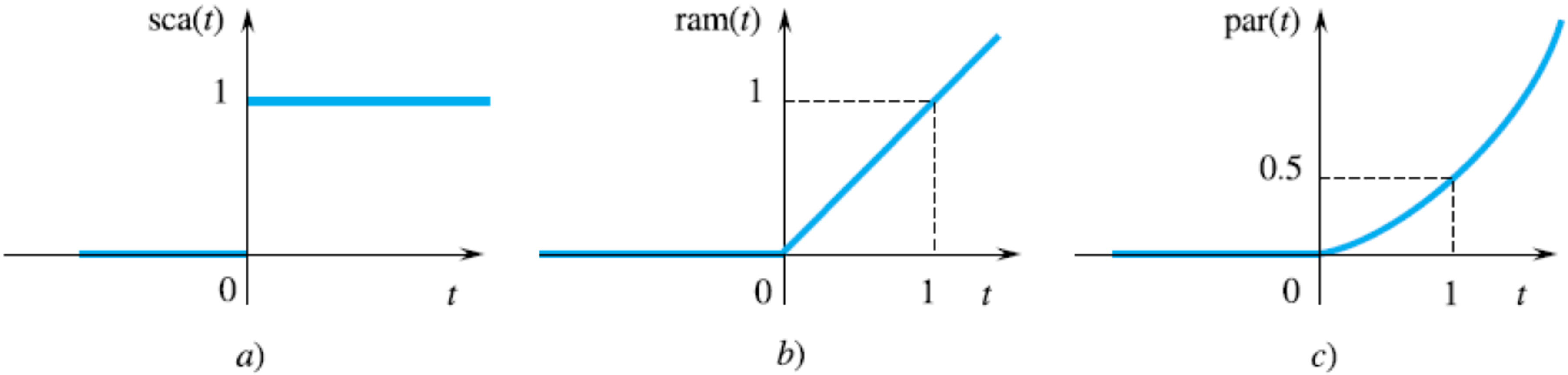
Selected Laplace transforms

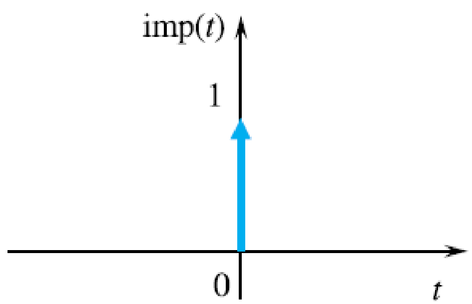
- In the system theory, we will mainly use the Laplace transform for the evaluation of the forced response of LTI systems to selected sets of input :

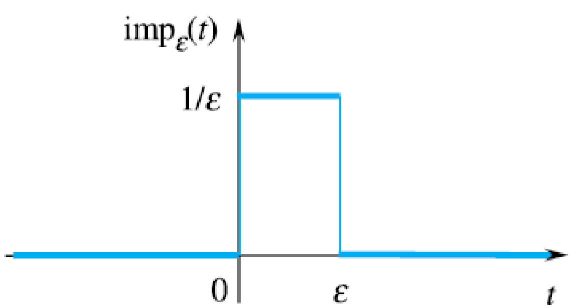
  - ***Polynomial inputs*** $\boldsymbol{u(t) = t^n 1(t)}$

  - ***Sinusoidal inputs*** $\boldsymbol{u(t) = sin(\omega t)\, 1(t)}$

    $\boldsymbol{u(t) = cos(\omega t) 1(t)}$


sca(t)
1
0
t
a)
ram(t)
1
0
1
t
b)
par(t)
0.5
0
1
t
c)


imp(t)
1
0
t


impε(t)
1/ε
0
ε
t

Selected Laplace transforms: polynomial signals

- In order to evaluate the Laplace transform of polynomial signals, let us firstly consider the Laplace transform of the impulse

  - ***Impulse $\delta(t)$*** $\longrightarrow$ $L(\delta(t)) = 1$ *(from the Laplace transform definition)*

- Then, using the ***time domain integration property***, we have

  - ***Step 1(t)*** $\longrightarrow$ $L(1(t)) = \dfrac{1}{s}$

  - ***Ramp $t \cdot 1(t)$*** $\longrightarrow$ $L(t \cdot 1(t)) = \dfrac{1}{s^2}$

  - ***Polinomial function $t^n \cdot 1(t)$*** $\longrightarrow$ $L(t^n \cdot 1(t)) = \dfrac{n!}{s^{n+1}}$

Selected Laplace transforms: sinusoidal signals

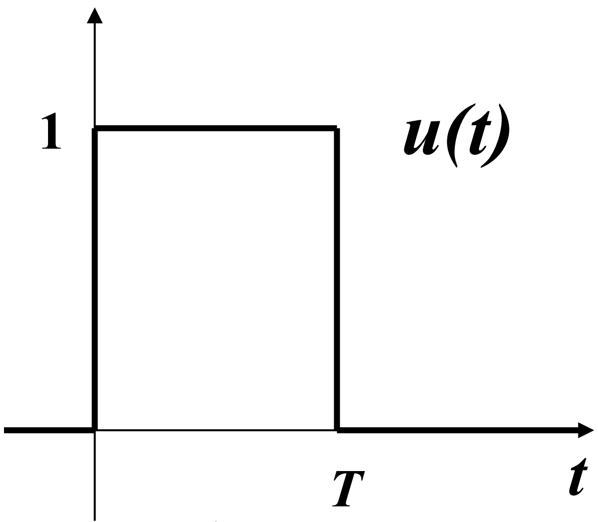
- The Laplace transform of sinusoidal functions
  - ***Sine*** $\boldsymbol{sin(\omega t)1(t)}$ $\longrightarrow$ $L(\sin(\omega t)\cdot 1(t)) = \dfrac{\omega}{s^2+\omega^2}$
  - ***Cosine*** $\boldsymbol{cos(\omega t)1(t)}$ $\longrightarrow$ $L(\cos(\omega t)\cdot 1(t)) = \dfrac{s}{s^2+\omega^2}$
- Finally, in the control theory the following transformations are of interest for the definition of the Laplace domain of the evolution modes of LTI systems

$$L(e^{\alpha t}1(t)) = \frac{1}{s-\alpha}$$

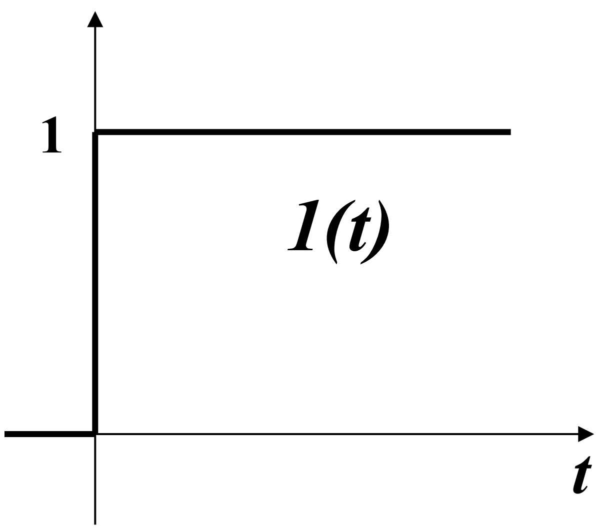
$$L\left(e^{\alpha t}\cos(\omega t)\cdot 1(t)\right) = \frac{s-\alpha}{(s-\alpha)^2+\omega^2}$$

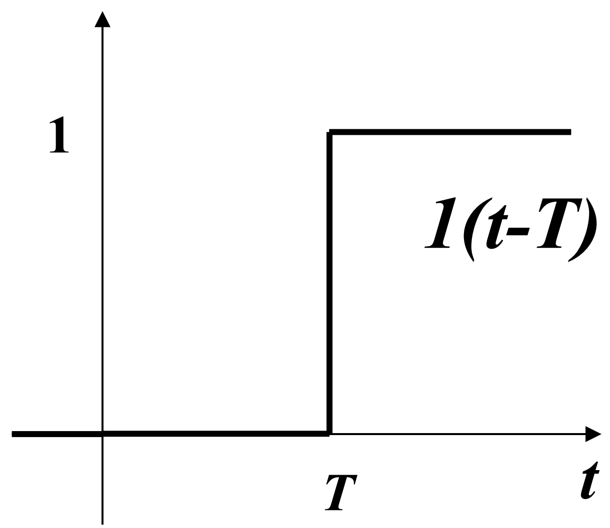
$$L\left(e^{\alpha t}sen(\omega t)\cdot 1(t)\right) = \frac{\omega}{(s-\alpha)^2+\omega^2}$$

Example: Laplace transform of a window signal



$$u(t)=1(t)-1(t-T)$$





Example: Laplace transform of a window signal

- The Laplace transform of a window signal can be evaluated from the Laplace transforms of two steps.

$$
\begin{aligned}
L\big(u(t)\big) &= L\big(1(t)-1(t-T)\big) \\
&= L\big(1(t)\big)-L\big(1(t-T)\big) \\
&= \frac{1}{s}-\frac{e^{-sT}}{s}=\frac{1-e^{-sT}}{s}
\end{aligned}
$$