Scuola Interdipartimentale delle **Scienze**, dell'**Ingegneria** e della **Salute**

DIPARTIMENTO DI SCIENZE E TECNOLOGIE

L. Magistrale in IA (ML&BD)

Scientific Computing (part 2 – 6 credits)

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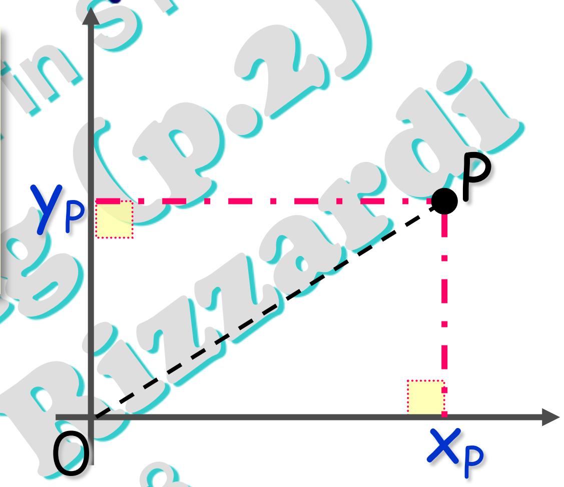
Contents

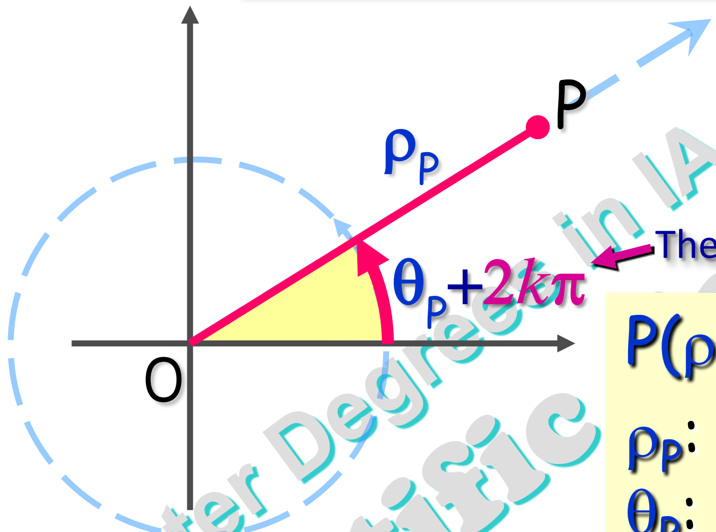
- **Brief notes on complex numbers.**
- **MATLAB 3D plots of a complex function with a complex argument.**

Coordinate systems in the plane

$P(x_P, y_P)$ cartesian coordinates:

x_P: abscissa of P
y_P: ordinate of P

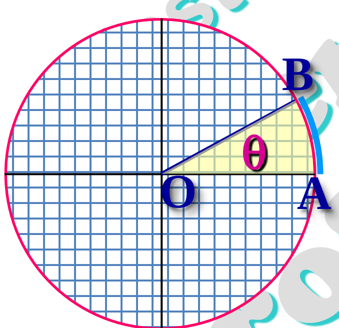




The angle is defined at less than multiples of 2π (a round angle)

$P(\rho_P, \theta_P)$ polar coordinates:

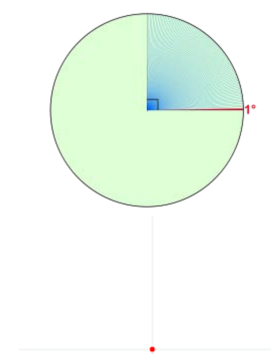
ρ_P: radial coordinate of P
θ_P: angular coordinate of P



An angle θ can be measured in: degrees or in radians.

1 degree: equals the 90th part of a right angle.

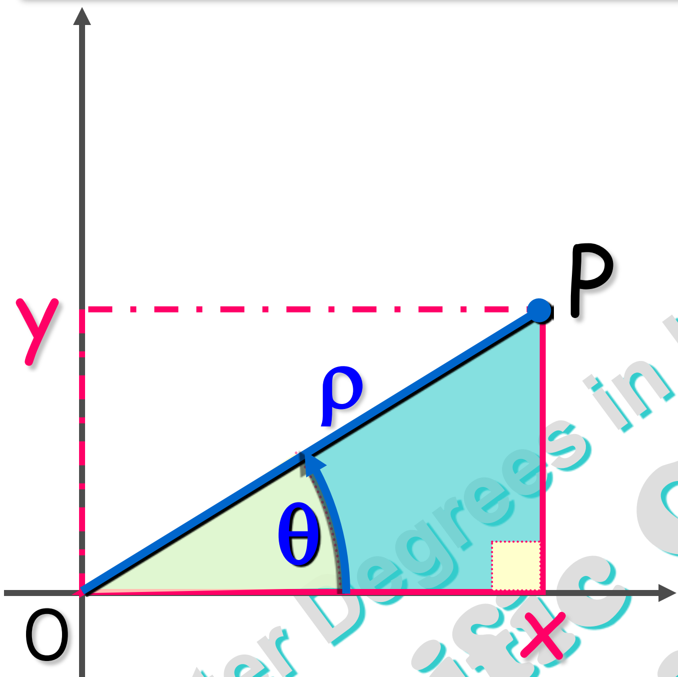
1 radian: it is the angle subtended by an arc equal in length to the radius of the circle. θ in radians is computed as the ratio between the **AB** arc length, and the **OA** radius of the circle.



to pass from deg() to rad() and conversely: $\deg(\theta) : 180 = \mathrm{rad}(\theta) : \pi$

Passing from a coordinate system to another

(x,y) **cartesian coordinates** and $[\rho,\theta]$ **polar coordinates** of P

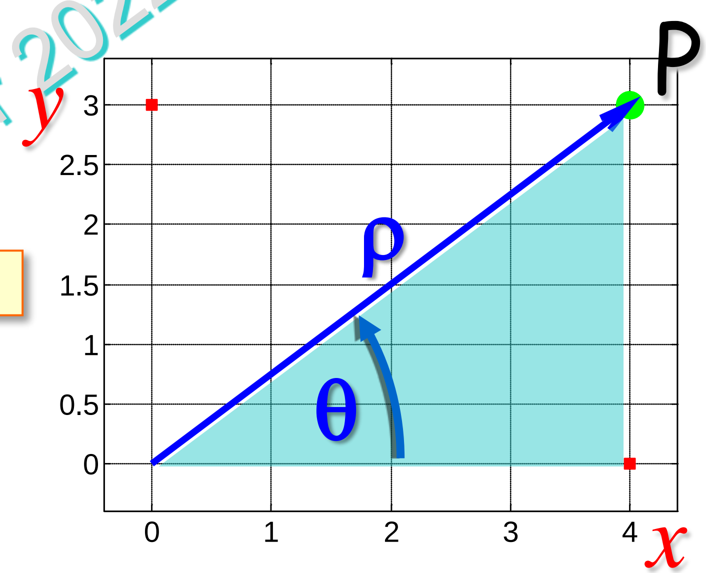


$$x = \rho\cos\theta \qquad \rho = \sqrt{x^2 + y^2}$$

$$y = \rho\sin\theta \qquad \theta = \arctan\frac{y}{x}$$

```
P=[4,3];   x=P(1);   y=P(2);
figure; plot(P(1),P(2),'og')
axis equal; hold on
rho=sqrt(x^2+y^2);  theta=atan(y/x);
plot([x 0],[0 y],'sr')
quiver(0,0,rho*cos(theta),rho*sin(theta),0)
```

```
theta=atan2(y,x);
```



Quiz: do you know the difference between **atan(…)** and **atan2(…)**?

Complex Numbers

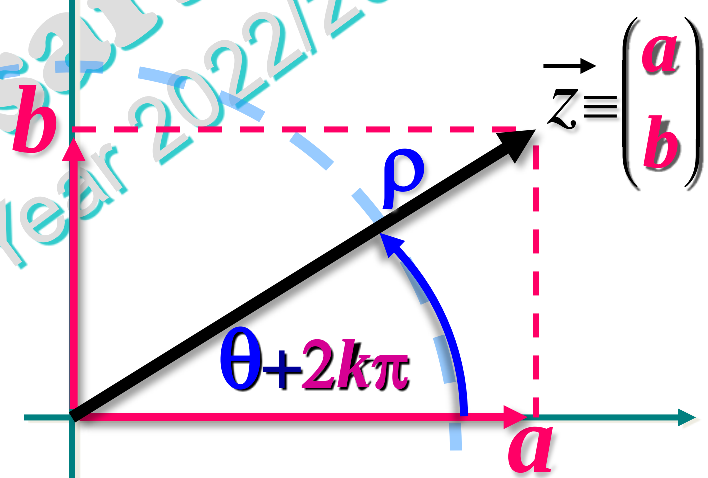
A complex number z is an ordered pair of real numbers (a,b)

$$\mathbb{C} = \{z=(a,b) : a,b \in \mathbb{R}\} \qquad \mathbb{C} \longleftarrow\!\bullet\!\longrightarrow \mathbb{R}^2$$

$\mathbb{C}$ denotes the set of complex numbers (or Complex Field)

a complex number z as a **point** of the plane $\mathbb{R}^2$ (euclidean affine space), with cartesian coordinates a and b.

a complex number z as a **vector** of the plane $\mathbb{R}^2$ (normed linear space), with components a and b along the axes.



cartesian coordinates $z \equiv (a,b)$
where
$a = \mathbf{Re}\, z$ (the real part of z)
$b = \mathbf{Im}\, z$ (the imaginary part of z)

polar coordinates $z \equiv [\rho,\theta]$
where
$\rho = |z|$ (the modulus of z)
$\theta = \mathbf{arg}\, z$ (the argument of z)

Representations of complex numbers

algebraic form $z = a + i\,b$

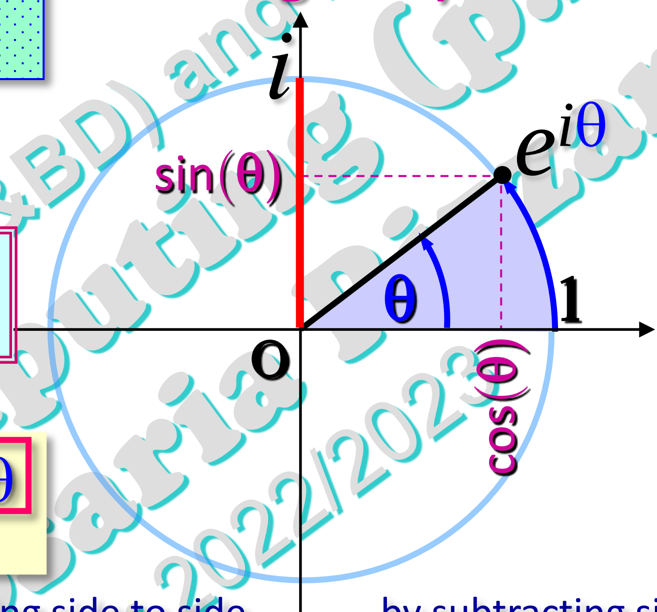
cartesian coordinates (a,b)

```
z=[3+2i;2;i;-3+2i;-3-2i];
a=real(z); b=imag(z);
disp([a  b])
     3     2
     2     0
     0     1
    -3     2
    -3    -2
```

i

```
syms a b real; z=a+i*b;
disp([real(z) imag(z)])
     a       b
```

imaginary unit $i = (0,1)$



...

$i^0 = 1$

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

...

Euler's formula* $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

* discovered around 1740

$$e^{+i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

by summing side to side

$$\cos\theta = \frac{1}{2}\left(e^{+i\theta} + e^{-i\theta}\right)$$

by subtracting side to side

$$\sin\theta = \frac{1}{2i}\left(e^{+i\theta} - e^{-i\theta}\right)$$

trigonometric form $z = \rho(\cos\theta + i\sin\theta)$

exponential form $z = \rho\, e^{i\theta}$

polar coordinates $[\rho,\theta]$

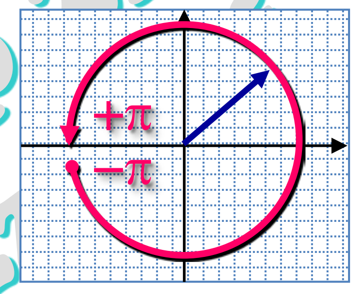
```
syms a b real; z=a+i*b;
disp([abs(z)  angle(z)])
[(a^2 + b^2)^(1/2), atan2(b, a)]
```

```
z=[3+2i;2;i;-3+2i;-3-2i];
r=abs(z); t=angle(z);
disp([r  t  t*180/pi])
    3.6056      0.588      33.69
         2          0          0
         1     1.5708         90
    3.6056     2.5536     146.31
    3.6056    -2.5536    -146.31
```

i

The **argument** of a complex number is defined at less than multiples of 2π. How does MATLAB handle arguments of complex numbers?

For a complex number z, MATLAB uses **Arg** z, the Principal Argument of z, i.e. it always takes the angles back to the interval $]-\pi, +\pi]$.

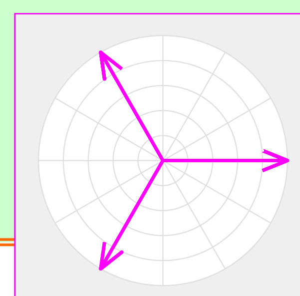


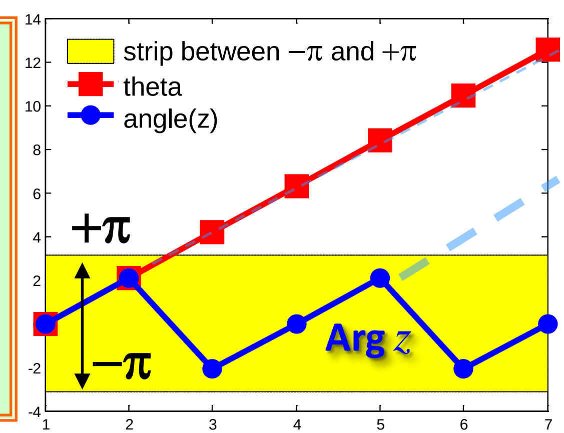

```
rho=2; theta1=pi/6; z1=rho*exp(i*theta1);
theta2=pi/6+2*pi;   z2=rho*exp(i*theta2);
theta3=pi/6-2*pi;   z3=rho*exp(i*theta3);
disp(rad2deg([angle(z1) angle(z2) angle(z3)]))
           30           30           30
```

rad2deg instead of *180/pi

How to go beyond the interval $]-\pi, +\pi]$?

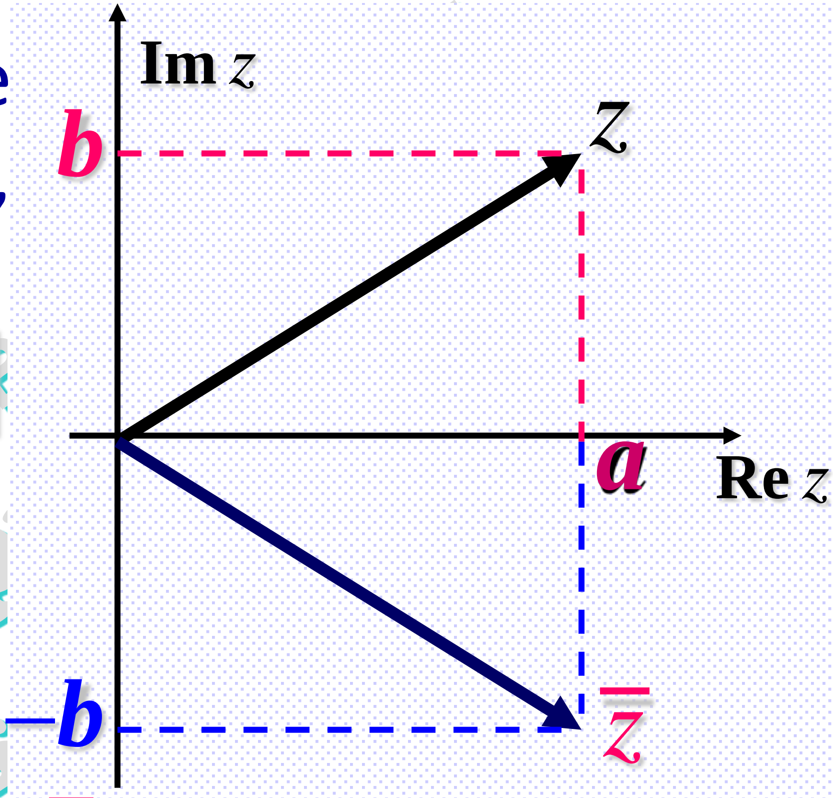
```
theta=(0:2*pi/3:4*pi)'; z=exp(i*theta); compass(z)
disp(rad2deg([theta  angle(z)  unwrap(angle(z))]))
1          0             0          0
2        120           120        120
3        240          -120        240
4        360   -1.4033e-14        360
5        480           120        480
6        600          -120        600
7        720   -2.8067e-14        720
```





If $z=(a,b)=[\rho,\theta]$ then the **complex conjugate** of z, denoted as $\overline{z}$, is

$$\overline{z}=(a,-b)=[\rho,-\theta]$$



$\overline{z}$ is the **symmetric** of z w.r.t. the real axis

$$\rho=\sqrt{z\cdot\overline{z}}$$

MATLAB `conj(z)`

```
z1=3+2i; z2=conj(z1); [z1;z2]
ans =
   3.0000 + 2.0000i
   3.0000 - 2.0000i
```

Operations on complex numbers

addition
(subtraction)

$$z_1 = a_1 + ib_1 \qquad z_2 = a_2 + ib_2 \;\Rightarrow\; z_3 = z_1 + z_2 = a_1 + a_2 + i(b_1 + b_2)$$

the same as summing two vectors in the real plane

product

cartesian coordinates: $\Rightarrow z_3 = z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) =$

$$= a_1 a_2 - b_1 b_2 + i[a_1 b_2 + a_2 b_1]$$

polar coordinates (exponential form):

$$z_1 = \rho_1 e^{i\theta_1} \qquad z_2 = \rho_2 e^{i\theta_2} \;\Rightarrow\; z_3 = z_1 z_2 = \rho_1 \rho_2 e^{i\theta_1} e^{i\theta_2} = \rho_1 \rho_2 \, e^{i(\theta_1 + \theta_2)}$$

$$\boxed{z_3 = z_1 z_2 = [\underbrace{\rho_1 \rho_2}_{\text{product of moduli}},\ \underbrace{\theta_1 + \theta_2 + 2k\pi}_{\text{sum of arguments}}]}$$

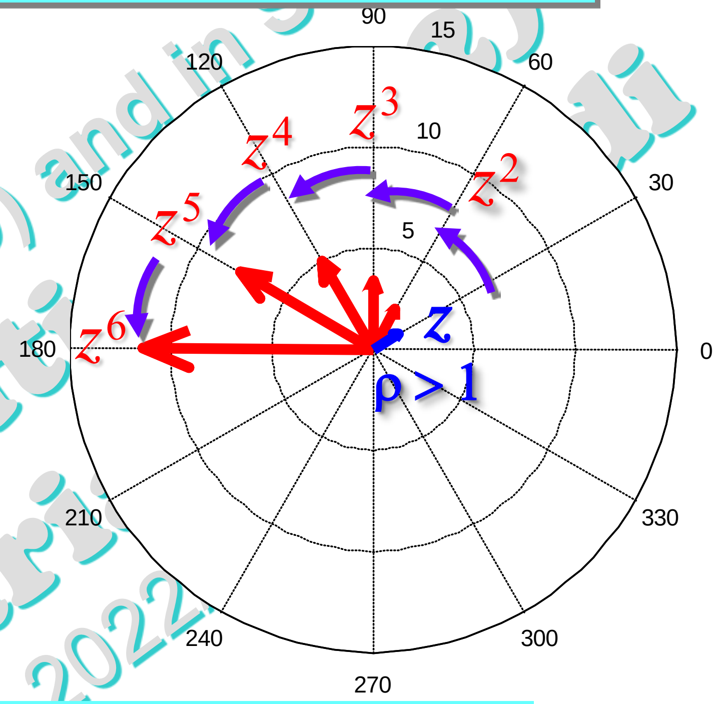
reciprocal

cartesian coordinates: $\Rightarrow z_3 = 1/z_1 = 1/(a_1 + ib_1) = (a_1 - ib_1)/|a_1 + ib_1|^2$

polar coordinates: $\Rightarrow z_3 = 1/z_1 = 1/(\rho e^{i\theta}) = (1/\rho)\, e^{-i\theta}$

Integer power of a complex number

$$z = [\rho, \theta+2k\pi]$$
$$z^2 = [\rho^2, 2\theta+2k\pi]$$
$$z^3 = [\rho^3, 3\theta+2k\pi]$$
$$z^4 = [\rho^4, 4\theta+2k\pi]$$
$$\dots$$
$$z^n = [\rho^n, n\theta+2k\pi]$$



$n^{\underline{th}}$ root of a complex number: $z : z^n = w$

```
z=[1.5+i;-3+2i]; w=z.^2;
disp([z     w.^(1/2)])
   1.5 + 1i   =   1.5 + 1i
    -3 + 2i   ≠     3 - 2i
```

OK!

???

```
z=[1+.6i;.6+i];  w=z.^5;
disp([z       w.^(1/5)])
    1 + 0.6i   =        1 +        0.6i
   .6 +    1i  ≠   1.1365 -     0.26162i
```

Fundamental Theorem of Algebra: a polynomial of degree $n \geq 1$ has exactly n complex roots, each counted with its multiplicity.

`w.^(1/n)` returns a single value not always equal to the original root

In order to compute all the n^{th} roots of $z\neq 0$, we have: from the eq. $w^n = z$

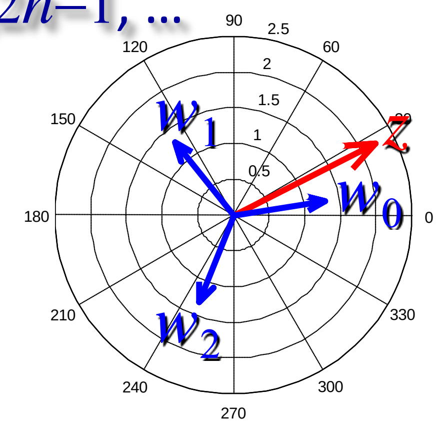
$\Longleftrightarrow$ $w^n = [r^n, n\varphi] = z = [\rho,\theta]$ $\Longleftrightarrow$ $r^n = \rho,\ n\varphi = \theta + 2k\pi$ solving w.r.t. r, φ

we obtain the following formula:

$$w_k = \sqrt[n]{z} = [r, \varphi_k] = \begin{cases} r = \sqrt[n]{\rho} \\ \varphi_k = \dfrac{\theta}{n} + \dfrac{2k\pi}{n}, \qquad k = 0, 1, \ldots, n-1 \end{cases}$$

where k must assume n **consecutive integer values**: for instance, $k=0,1,2,\ldots,n-1$, or $k=-2,-1,0,1,\ldots,n-3$, or $k=n,n+1,\ldots,2n-1$, ...

Example: cubic roots of $2+i$



$$w_k = \sqrt[3]{2+i} = [r, \varphi_k] = \begin{cases} r = \sqrt[3]{|2+i|} \\ \varphi_k = \dfrac{\arg(2+i)}{3} + \dfrac{2k\pi}{3}, \qquad k = 0, 1, 2 \end{cases}$$

```
z=2+i;   n=3;   k=(0:(n-1))';
[x,y]=pol2cart(angle(z)/n+2*pi/n*k, abs(z)^(1/n));
wk=complex(x,y); disp([wk wk.^n]) % check they give back z
    1.2921 + 0.20129i        2 + 1i
   -0.82036 + 1.0183i        2 + 1i
   -0.47171 - 1.2196i        2 + 1i
```

w_k^3 equals $z=2+i$

$$1w^3 + 0w^2 + 0w^1 - z\,w^0 = 0$$

```
rk=roots([1 0 0 -z])
rk =
   -0.82036 +   1.0183i
   -0.47171 -   1.2196i
     1.2921 + 0.20129i
```

polynomial coefficients

MATLAB **roots**

the roots are the same, but not in the same order

n^{th} roots of unity

$$w_k = \sqrt[n]{1} = \sqrt[n]{[1,0]} = \left[1, \frac{2k\pi}{n}\right], \qquad k = 0, 1, \ldots, n-1$$

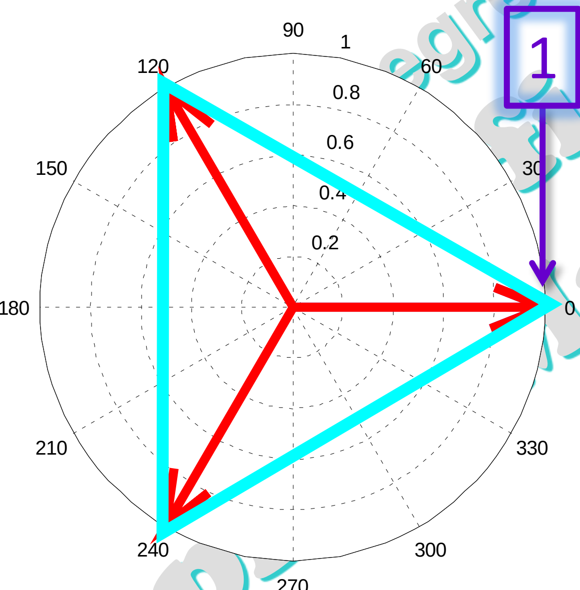
The n^{th} roots of the unit are placed on the vertices of the n-sided regular polygon inscribed in the unit circle. Among these roots there is always 1; when n is odd the others are complex and conjugate; when n is even, in addition to 1, there is also –1 as a real root.

Examples

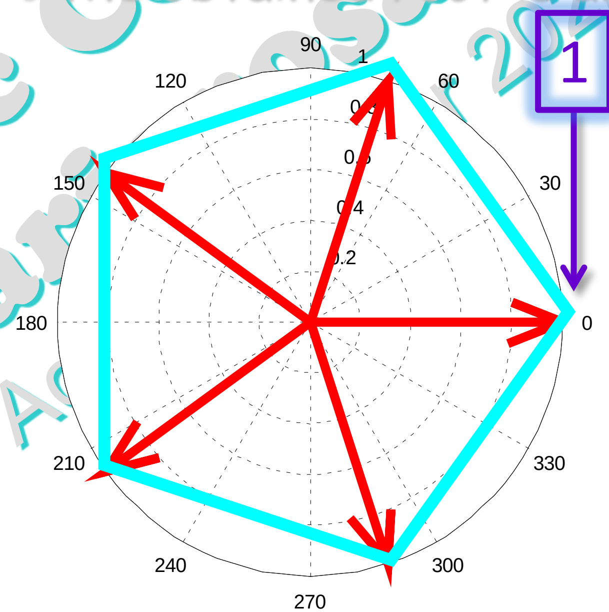
```
z=1;  n=…;  k=(0:(n-1))';  wk=exp(2i*pi*k/n);  compass(wk)
```

$w_k = \sqrt[3]{1}$ $\qquad$ $w_k = \sqrt[5]{1}$ $\qquad$ $w_k = \sqrt[6]{1}$

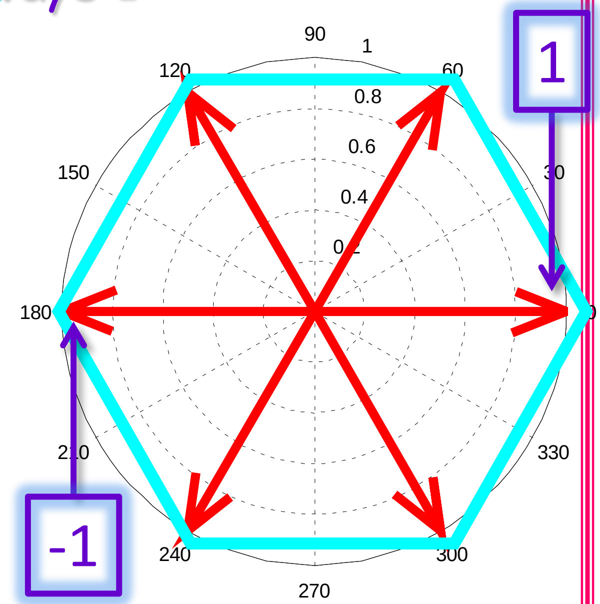
for $k=0$ the obtained root is always 1



equilateral triangle



pentagon



hexagon

Why the $n^{\underline{th}}$ roots of unity are important

Theor. The $n^{\underline{th}}$ roots of a complex number can be obtained by multiplying a particular root of the complex number by all the $n^{\underline{th}}$ of unity.

$$z_k = \sqrt[3]{2+5i} \quad \Rightarrow \quad z_k = z_p \cdot e_k$$

where

- z_p is a particular cubic root of $2+5i$
- $e_k = \sqrt[3]{1}$ are the cubic roots of 1

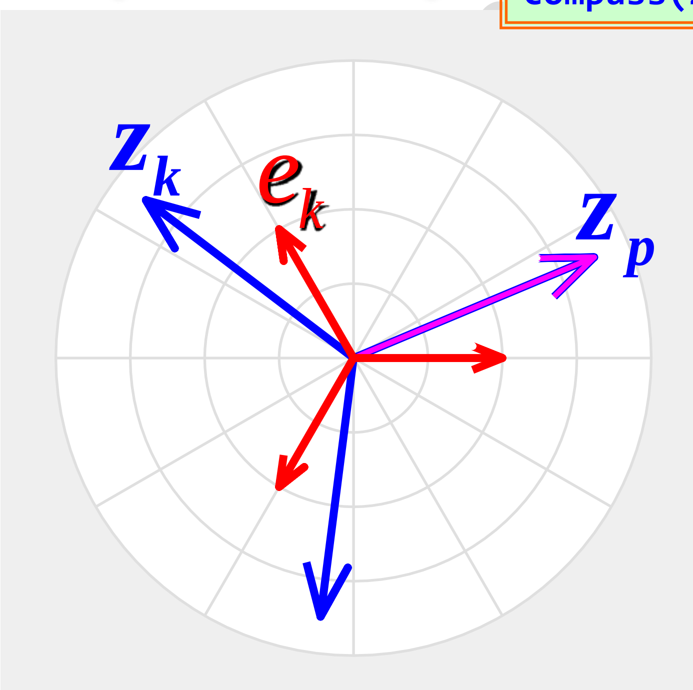
$$z_p\, e_k = |z_p| \exp(i \arg z_p)\, e_k$$

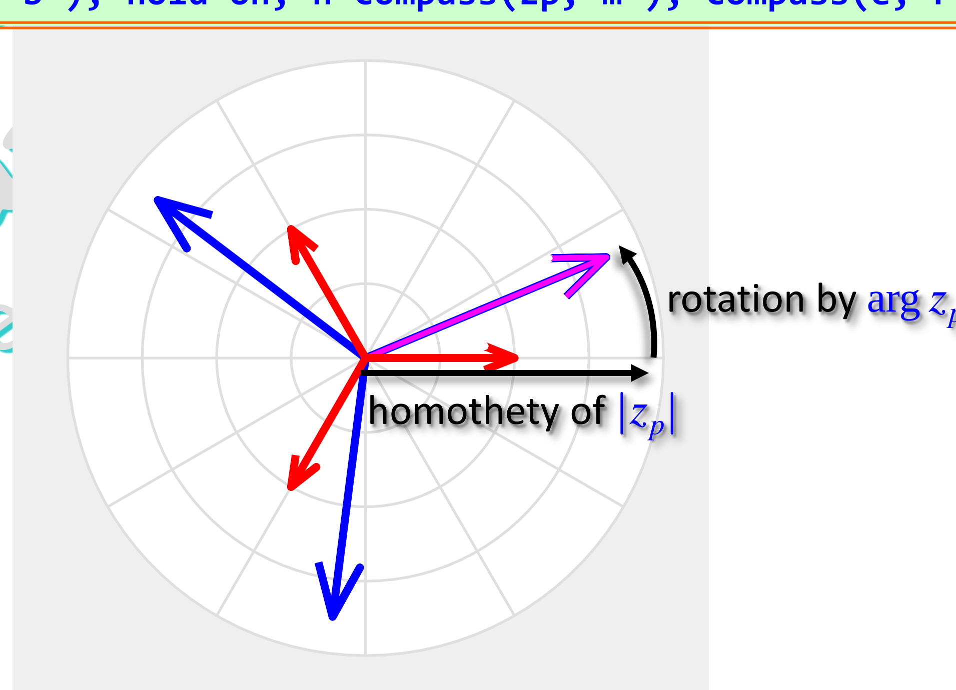
rotation

homothety

lengthen or shorten vector length

```
n=3; Z=2+5i; z=roots([1,zeros(1,n-1),-(2+5*i)]); % roots
zp=Z^(1/n); % a particular root
e=roots([1,0,0,-1]); % cubic roots of unity
compass(zp*e,'b'); hold on; h=compass(zp,'m'); compass(e,'r')
```





DEFINITION

A **primitive** n^{th} root of unity ζ is a particular n^{th} root of unity such that n is the smallest integer > 0 for which:

$$\boxed{\zeta^n = 1 \wedge \zeta^k \neq 1 \ \forall k : 0 < k < n, \text{ and they are all distinct}} \Rightarrow$$

(ζ^k is a n^{th} root of 1) $\boxed{(\zeta^k)^n = 1 \ \forall k = 0, 1, \ldots, n-1}$

Example: which roots are primitive?

```
n=4; k=(0:n-1)'; r=exp(2i*k*pi/n); disp(r.')
           1 + 0i    6.1232e-17 + 1i         -1 + 1.2246e-16i   -1.837e-16 - 1i
J=find(abs(real(r))<1e-10); r(J)=zeros(size(J)) + i*imag(r(J));
J=find(abs(imag(r))<1e-10); r(J)=real(r(J)) + i*zeros(size(J));
disp([r.'; (r.').^[k;n]])
^k         1 + 0i            0 + 1i            -1 + 0i            0 - 1i    roots
0          1 + 0i            1 + 0i             1 + 0i            1 + 0i
1          1 + 0i            0 + 1i            -1 + 0i            0 - 1i
2          1 + 0i           -1 + 0i             1 + 0i           -1 + 0i
3          1 + 0i            0 - 1i            -1 + 0i            0 + 1i
4=n        1 + 0i            1 + 0i             1 + 0i            1 + 0i
```

$r=1$: no $r=i$: yes $r=-1$: no $r=-i$: yes

Theor.

Among the $n^{\underline{\text{th}}}$ roots of unity

$$w_k = \sqrt[n]{1} = \left[1, \frac{2k\pi}{n}\right], \qquad k = 0, 1, \ldots, n-1$$

the primitive ones w_k are such that n and k are primes between them, i.e. $\mathbf{gcd}(n,k) = 1$.

gcd(): greater common divisor

```
disp([k gcd(k,n)])
     0       4
     1       1
     2       2
     3       1
```

$r=1$: no
$r=i$: yes
$r=-1$: no
$r=-i$: yes

(prof. M. Rizzardi)

Among the $n^{\underline{th}}$ roots of unity

$$w_k = \sqrt[n]{1} = \left[1, \frac{2k\pi}{n}\right], \qquad k = 0, 1, \dots, n-1$$

the roots for $k = 1$ and for $k = n-1$:

$$w_1 = \left[1, \frac{2\pi}{n}\right] = e^{i\frac{2\pi}{n}}, \qquad w_{n-1} = \left[1, \frac{2(n-1)\pi}{n}\right] = \left[1, -\frac{2\pi}{n}\right] = e^{-i\frac{2\pi}{n}}$$

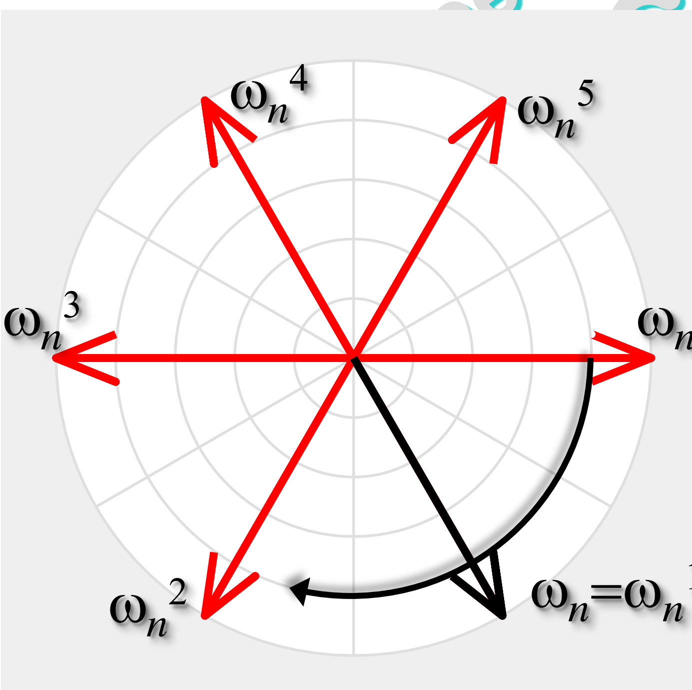
are always primitive. They are complex conjugate.

$\overline{\omega}_n$ in **IDFT**

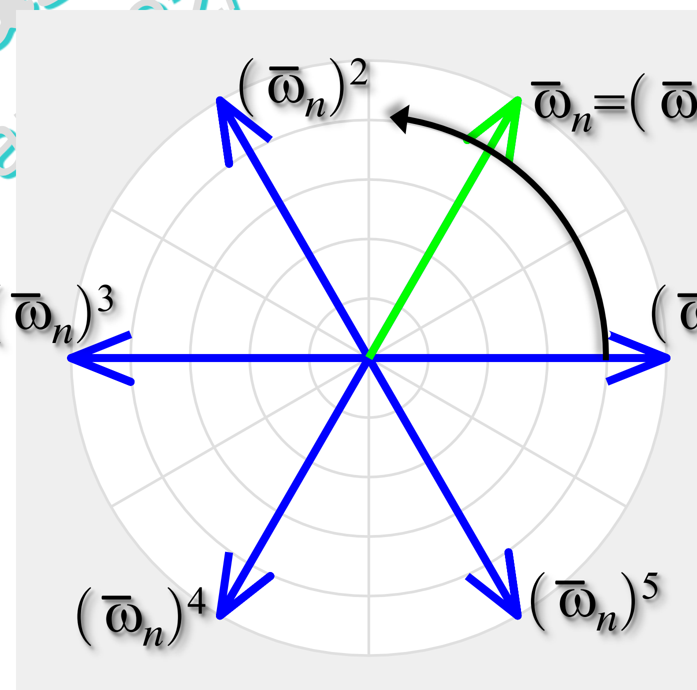
Inverse Discrete Fourier Transform

ω_n in **DFT**

Discrete Fourier Transform



$n=6$



Complex functions of a real variable

$$f : t \in \mathbb{R} \longrightarrow f(t) = x(t) + i y(t) \in \mathbb{C}$$

real part of f — imaginary part of f

Examples

$$f(t) = t + i2t$$
$$x(t) = t, \quad y(t) = 2t$$

$$f(t) = \cos t + i \sin t$$
$$x(t) = \cos t, \quad y(t) = \sin t$$

$$f(t) = t^2 + i\frac{1}{t}$$
$$x(t) = t^2, \quad y(t) = \frac{1}{t}$$

$$f(t) = e^{it}$$
$$x(t) = \cos t, \quad y(t) = \sin t$$

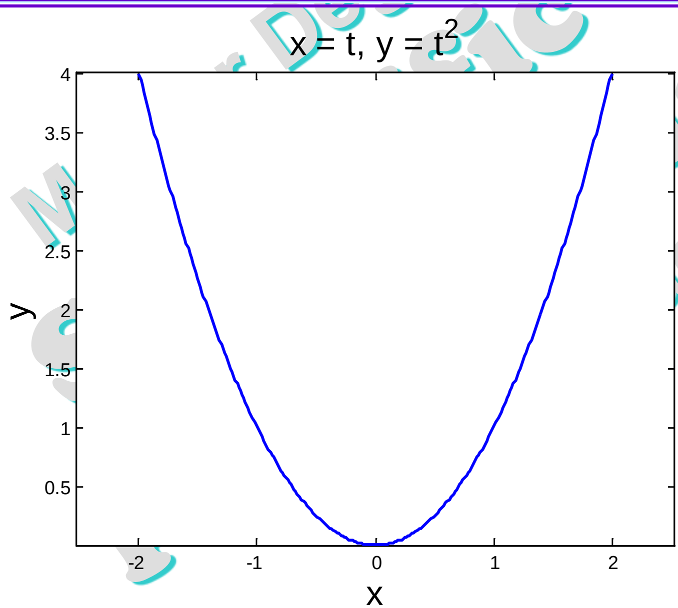
How to display a complex function $f(t)$ of a real variable t

Example: $f(t) = x(t) + iy(t) = t + it^2, \quad t \in \mathbb{R}$

1) plot of $(x(t), y(t)), \quad t \in \mathbb{R}$ **in** $\mathbb{R}^2$

```
syms t real;    f = t + 1i*t^2;
ezplot(real(f),imag(f),[-2 2]);
axis equal
```

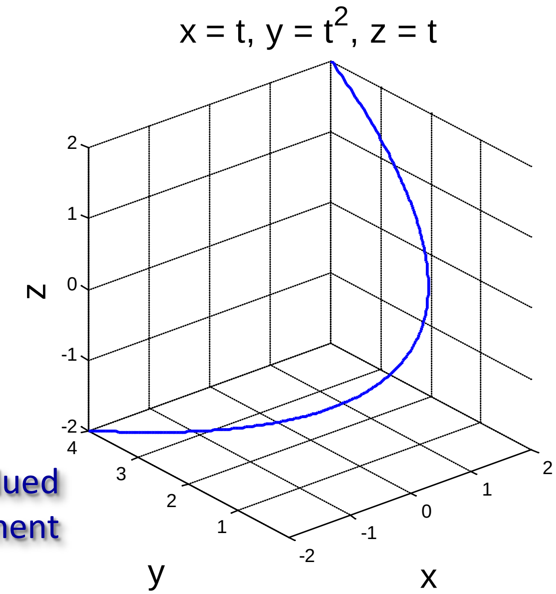
```
fplot(real(f),imag(f),[-2 2]);
axis equal
```



2) plot of $(f(t), t), \quad t \in \mathbb{R}$ **in** $\mathbb{R}^3$

```
syms t real;    f = t + 1i*t^2;
ezplot3(real(f),imag(f),t,[-2 2],'animate');
axis equal
```

```
fplot3(real(f),imag(f),t,[-2 2]);
axis equal
```



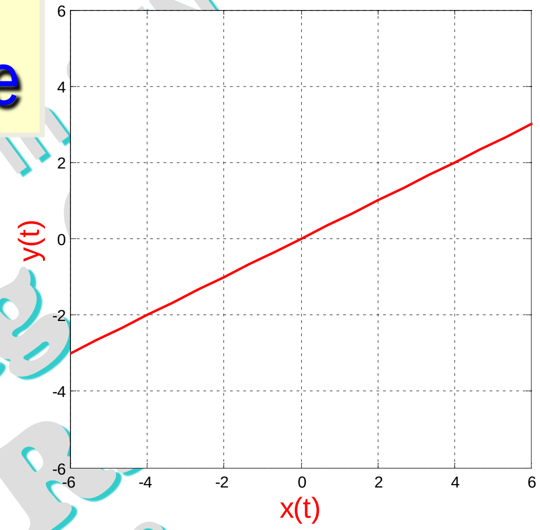
curves

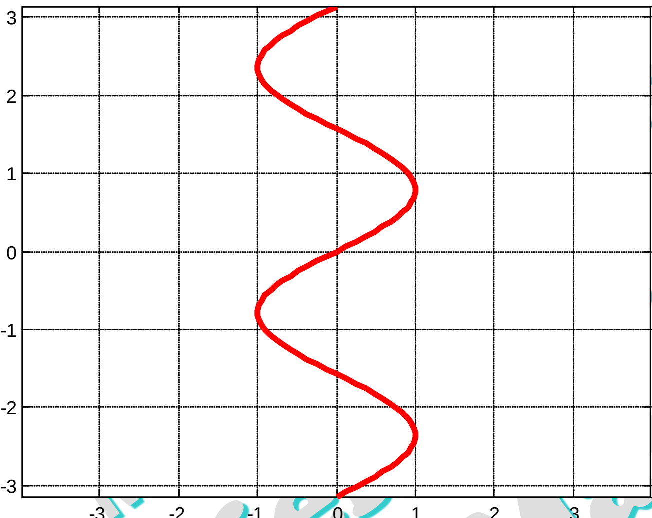
the plot of a complex-valued function of a real argument is a curve

Examples: curves in the plane as complex functions of a real variable

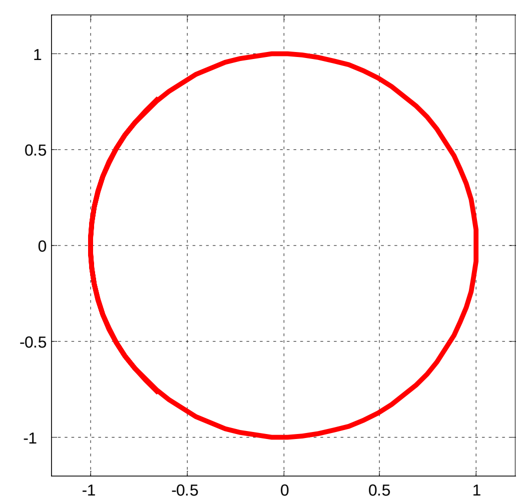
$$z = f(t) = x(t) + iy(t) = 2t + it$$

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$





$$f(t) = x(t) + iy(t) = \sin(2t) + it$$



$$f(t) = x(t) + iy(t) = \cos(t) + i\sin(t)$$

Complex functions of a complex variable

$$f : z = x + iy \in \mathbb{C} \longrightarrow f(z) = f(x,y) = \underbrace{u(x,y)}_{\mathrm{Re}\, f(z)} + i\underbrace{v(x,y)}_{\mathrm{Im}\, f(z)} \in \mathbb{C}$$

2 real variables

Examples

$$f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$$

$$f(z) = e^z = e^{x+iy} = \underbrace{e^x \cos y}_{u(x,y)} + i\,\underbrace{e^x \sin y}_{v(x,y)}$$

$$f(z) = |z| = \sqrt{x^2 + y^2} = \underbrace{\sqrt{x^2 + y^2}}_{u(x,y)} + i\,\underbrace{0}_{v(x,y)}$$

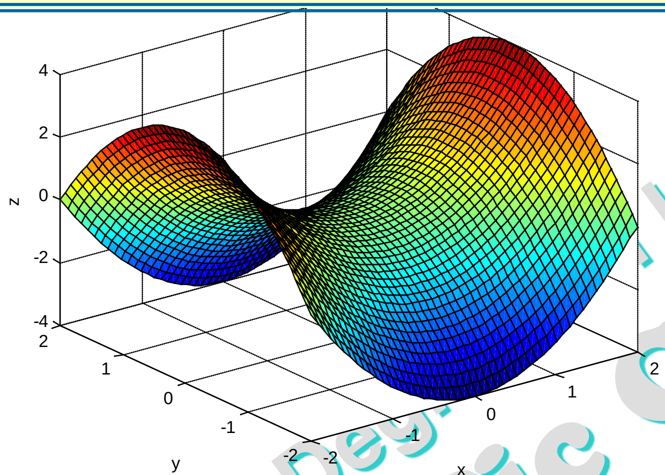
$$f(z) = \bar{z} = x - iy = \underbrace{x}_{u(x,y)} + i\,\underbrace{(-y)}_{v(x,y)}$$

How to display a complex function $f(z)$ of a complex variable z

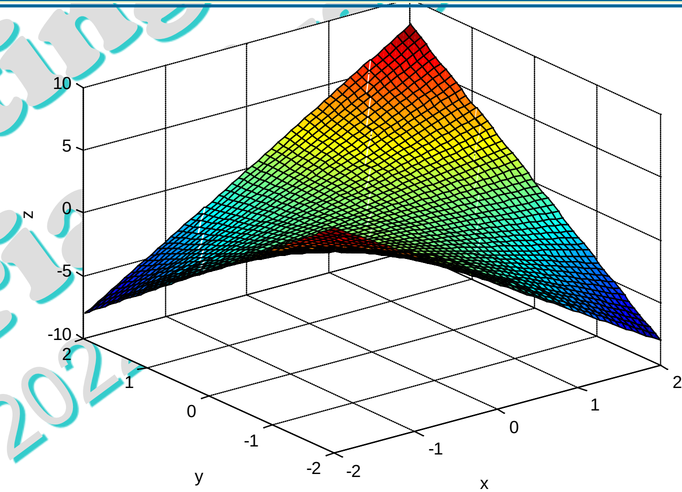
Example: $f(z) = z^2 = x^2 - y^2 + 2ixy$
on a square grid

```
syms x y real; z=x+i*y; f=z^2;
```

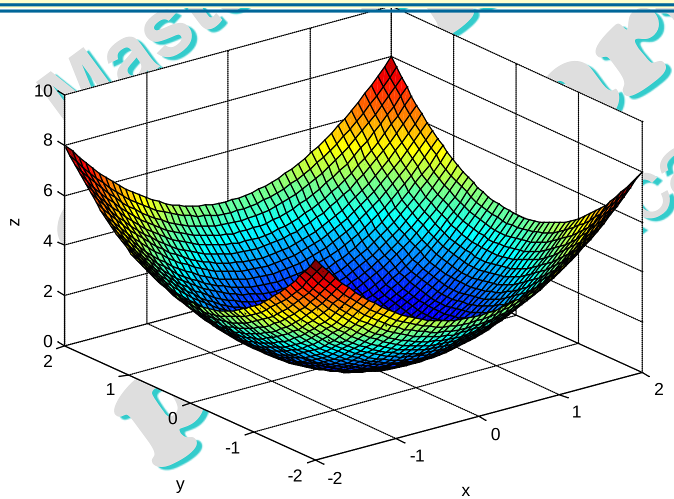
```
ezsurf(real(z),imag(z),real(f),[-2 2])
```



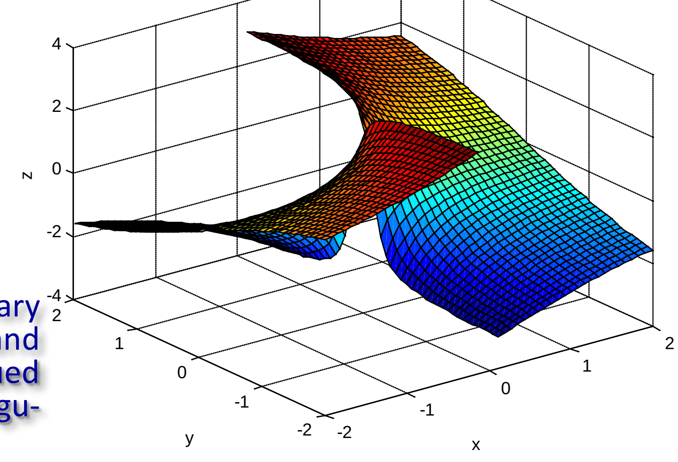

```
ezsurf(real(z),imag(z),imag(f),[-2 2])
```




```
ezsurf(real(z),imag(z),abs(f),[-2 2])
```




```
ezsurf(real(z),imag(z),angle(f),[-2 2])
```



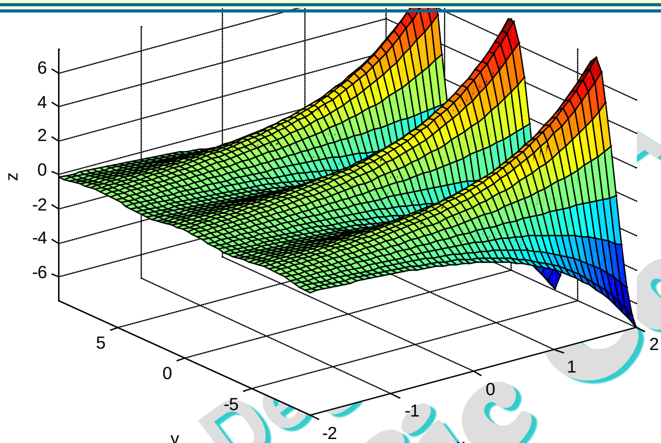
plots of the real and imaginary parts, of the modulus and argument of a complex-valued function of a complex argument are surfaces

How to display a complex function $f(z)$ of a complex variable z

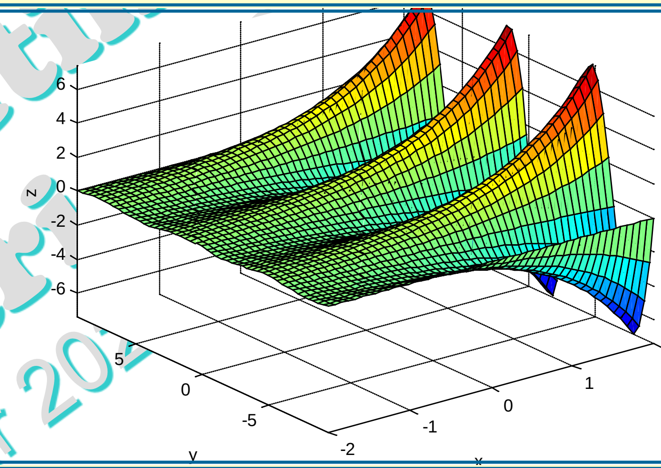
Example: $f(z) = e^z = e^{x+iy} = e^x\left[\cos y + i \sin y\right]$
on a rectangular grid

```
syms x y real; z=x+i*y; f=exp(z);
```

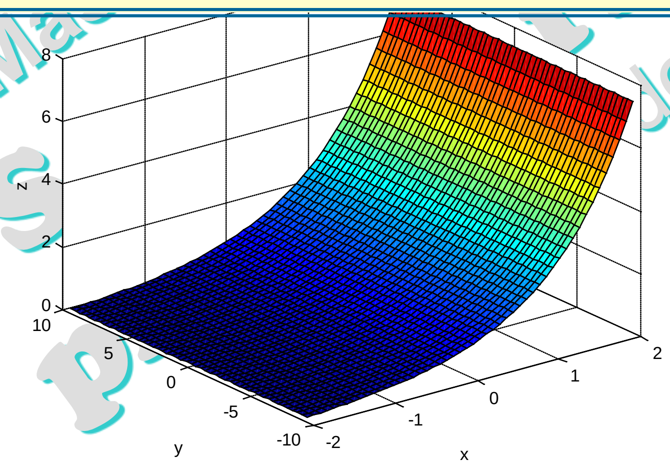
```
ezsurf(real(z),imag(z),real(f), ...
                     [-2 2 -3*pi 3*pi])
```



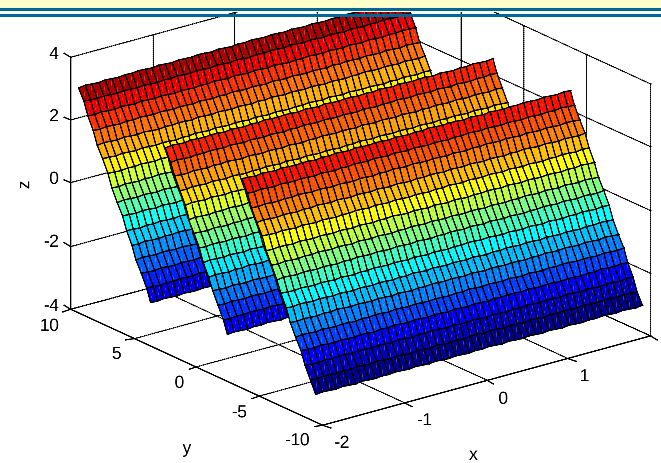

```
ezsurf(real(z),imag(z),imag(f), ...
                     [-2 2 -3*pi 3*pi])
```




```
ezsurf(real(z),imag(z),abs(f), ...
                     [-2 2 -3*pi 3*pi])
```




```
ezsurf(real(z),imag(z),angle(f), ...
                     [-2 2  -3*pi 3*pi])
```

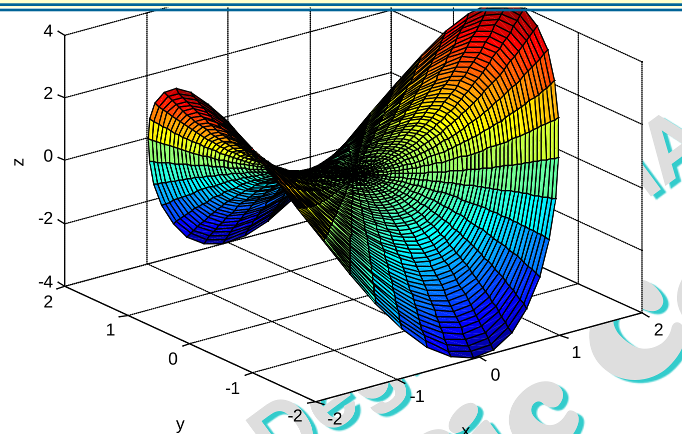


How to display a complex function $f(z)$ of a complex variable z

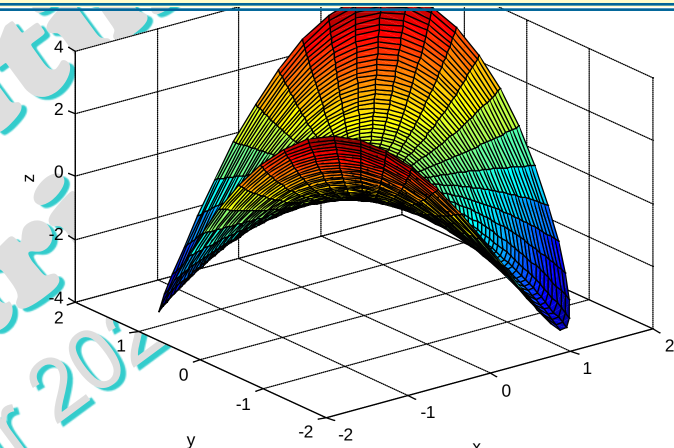
Example: $f(z) = z^2 = [\rho,\theta]^2 = \rho^2 e^{i2\theta}$
on a circular grid

```
syms rho positive;  syms th real;
z=rho*exp(i*th);  f = z^2;
```

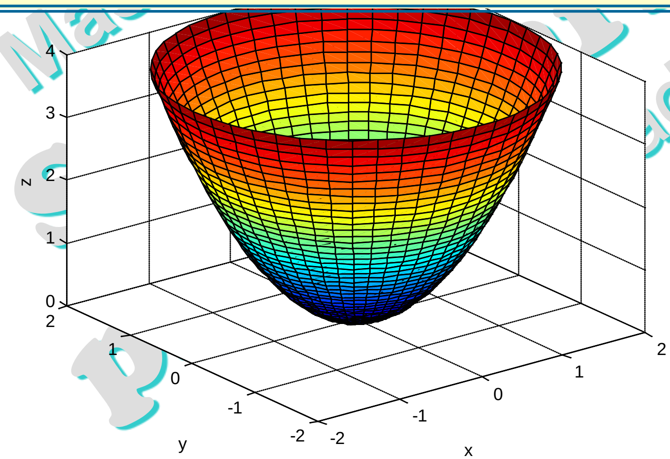
```
ezsurf(real(z),imag(z),real(f), ...
                        [0 2 -pi pi])
```




```
ezsurf(real(z),imag(z),imag(f), ...
                        [0 2 -pi pi])
```




```
ezsurf(real(z),imag(z),abs(f), ...
                        [0 2 -pi pi])
```




```
ezsurf(real(z),imag(z),angle(f), ...
                        [0 2 -pi pi])
```