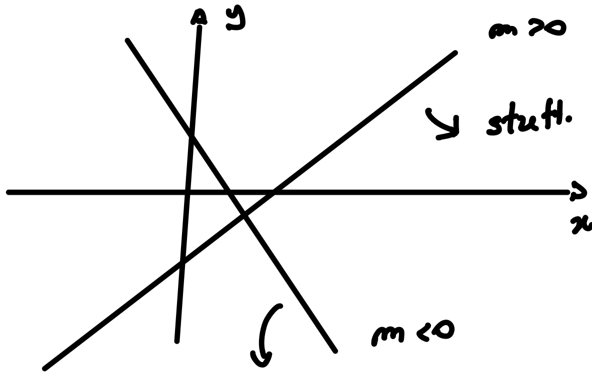
Le funzioni elementari

1) Funzioni lineari:

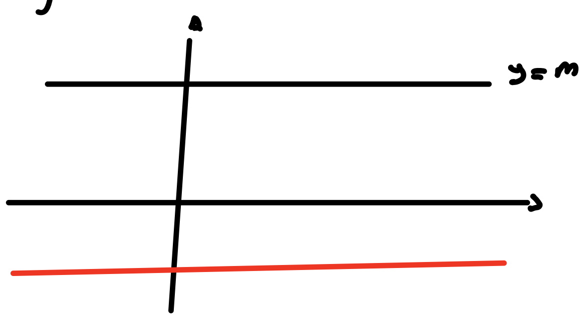
$$f(x) = mx + n \quad \forall x \in \mathbb{R}$$
$$m, n \in \mathbb{R}$$

$y = mx + n$ retta

$m \neq 0$



<u>$m = 0$</u> $\quad f(x) = n$ funzione costante



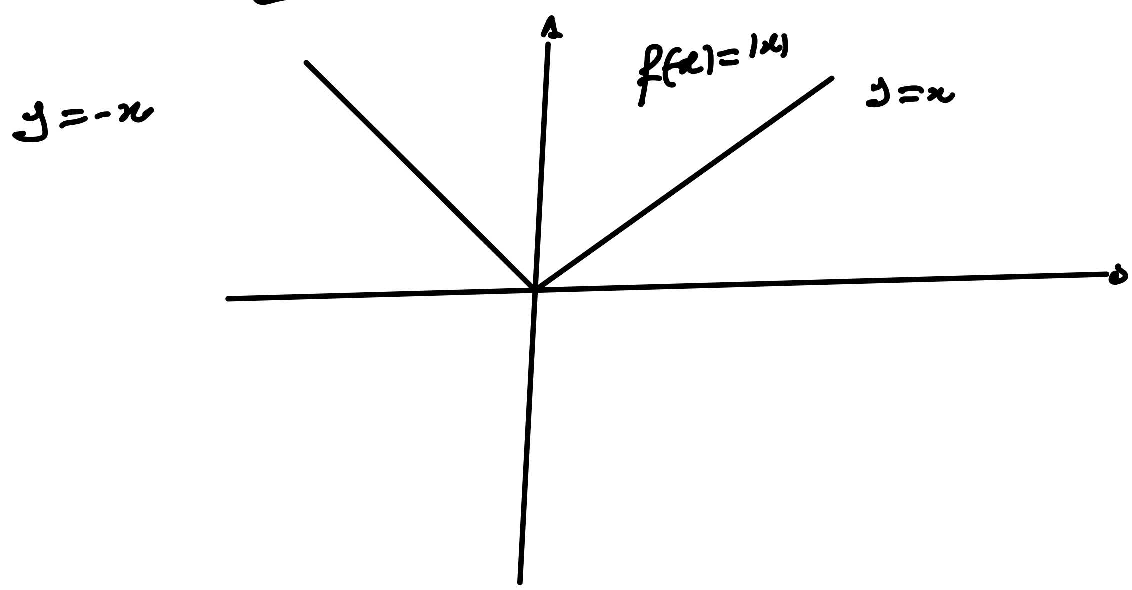
$y = 2x + 1, \quad y = -\frac{x}{2} - 4, \ldots$

Funzione valore assoluto

$f(x) = |x| \quad \forall x \in \mathbb{R} \quad : D_f = \mathbb{R}$

$$f(x) = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x \leq 0 \end{cases}$$

$y = x$
$x \geq 0$

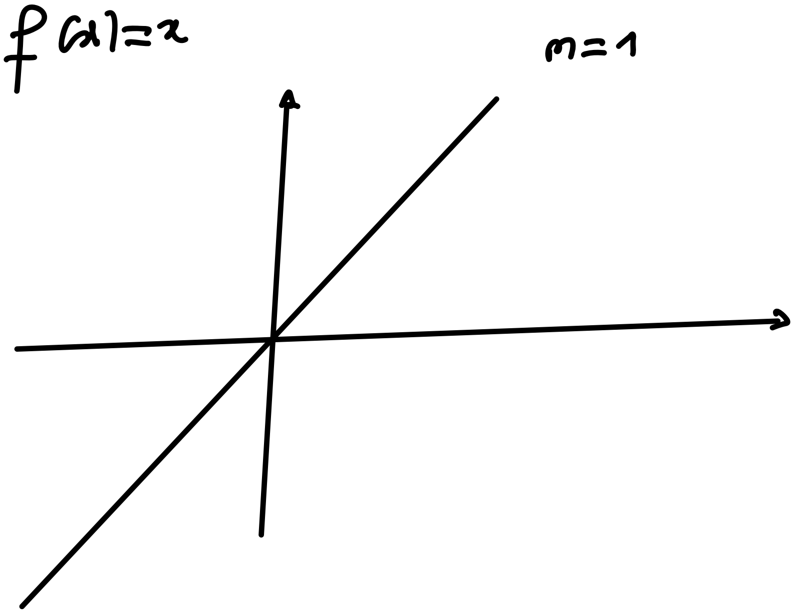


Funzione potenza ad esponente intero

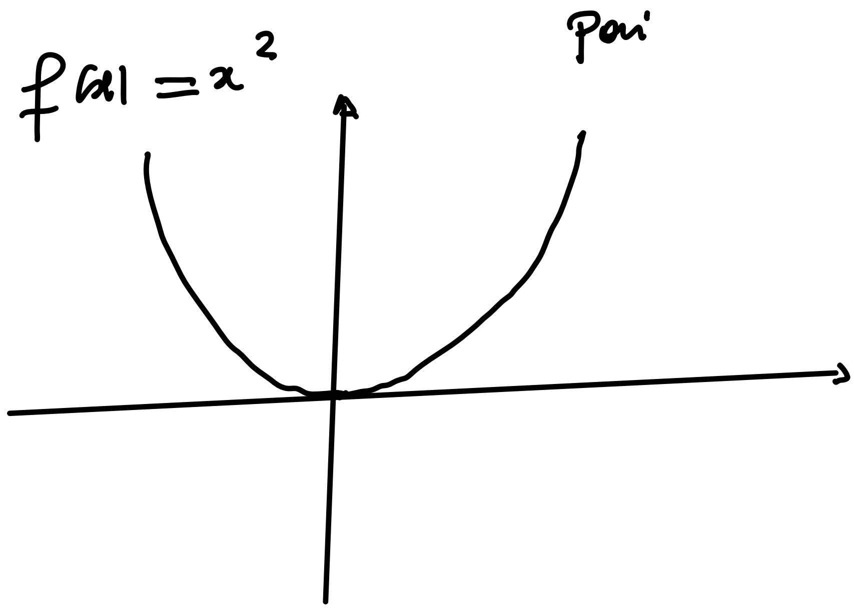
$f(x) = x^n \quad , n \in \mathbb{N} = \{1, 2, 3, 4, \dots\} \quad , \forall x \in \mathbb{R}$

$f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4, \dots$

$\underline{n=1}$: $y = f(x) = x$

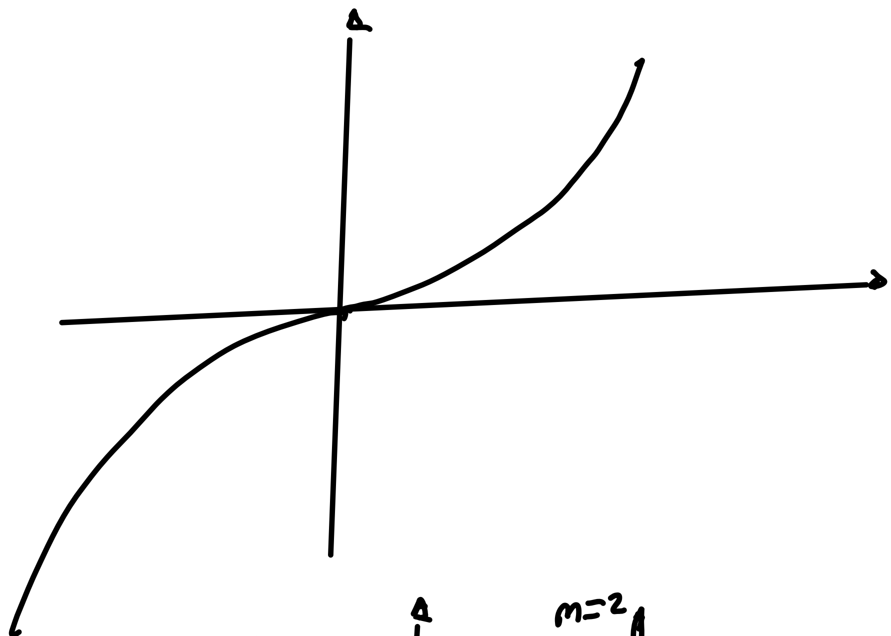


$n=2$ $y = f(x) = x^2$

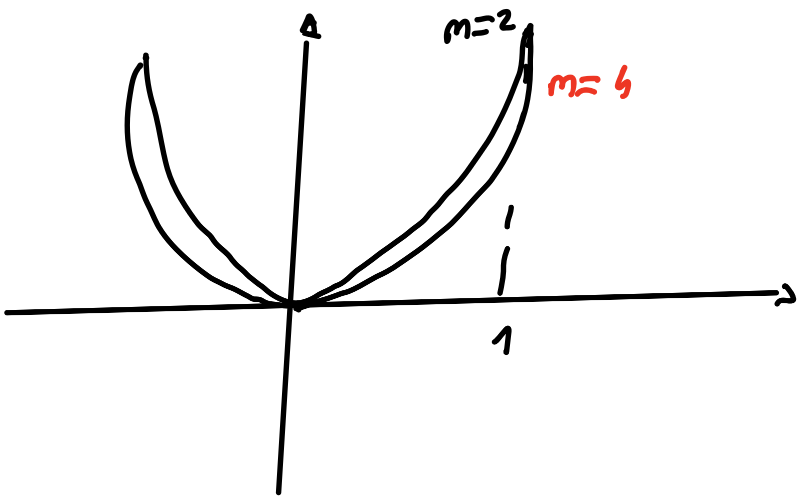


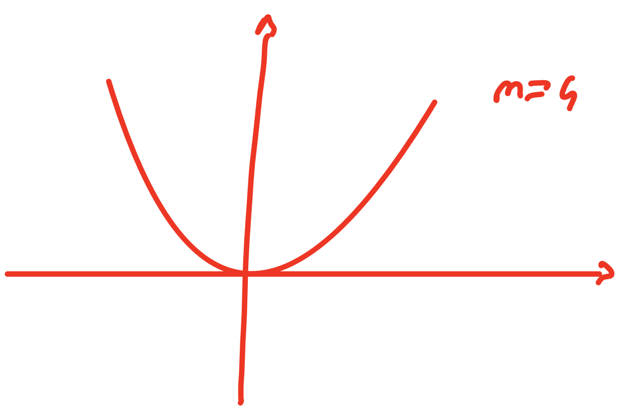
$f(-x) = f(x)$

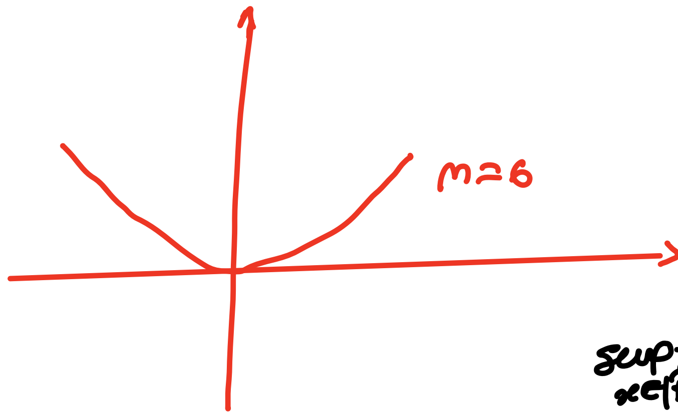
$n=3$ $\quad y = f(x) = x^3$ $\quad f(-x) = -f(x)$



<u>n pari</u>







$\sup_{x\in\mathbb{R}} f(x) = +\infty$

$\inf_{x\in\mathbb{R}} f(x) = \min_{x\in\mathbb{R}} f(x) = 0$

$f(x)=x^n$ $\qquad D_f = \mathbb{R}$

n pari

Codominio $= [0,+\infty[=$

$= \{ y\in\mathbb{R} : y\geq 0\}$

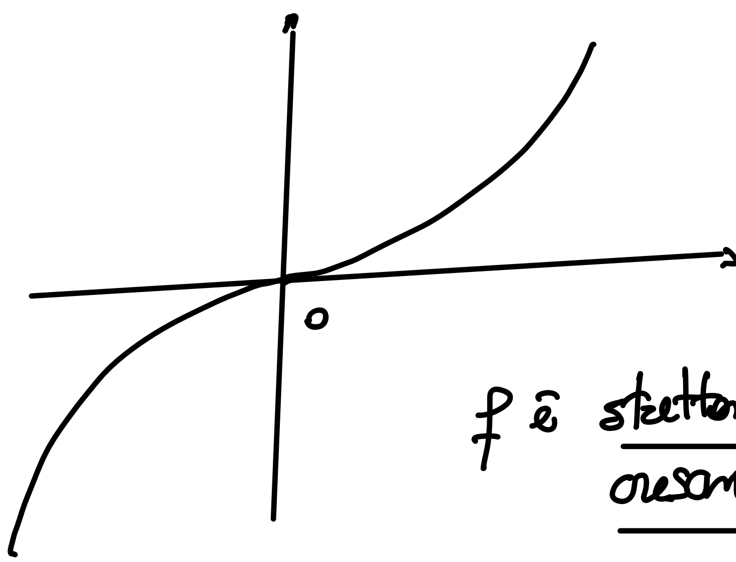
$\downarrow$

funzione decrescente per $x\leq 0$, crescente per $x\geq 0$

$f(x)= x^n$ $\qquad n$ dispari

$D_f = \mathbb{R}$

Codominio $=\mathbb{R}$



f è strettamente crescente

$$\inf_{x \in \mathbb{R}} f(x) = -\infty \quad , \quad \sup_{x \in \mathbb{R}} f(x) = +\infty$$

Funzione composta di due funzioni

$f = f(x) \qquad f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

I = intervallo

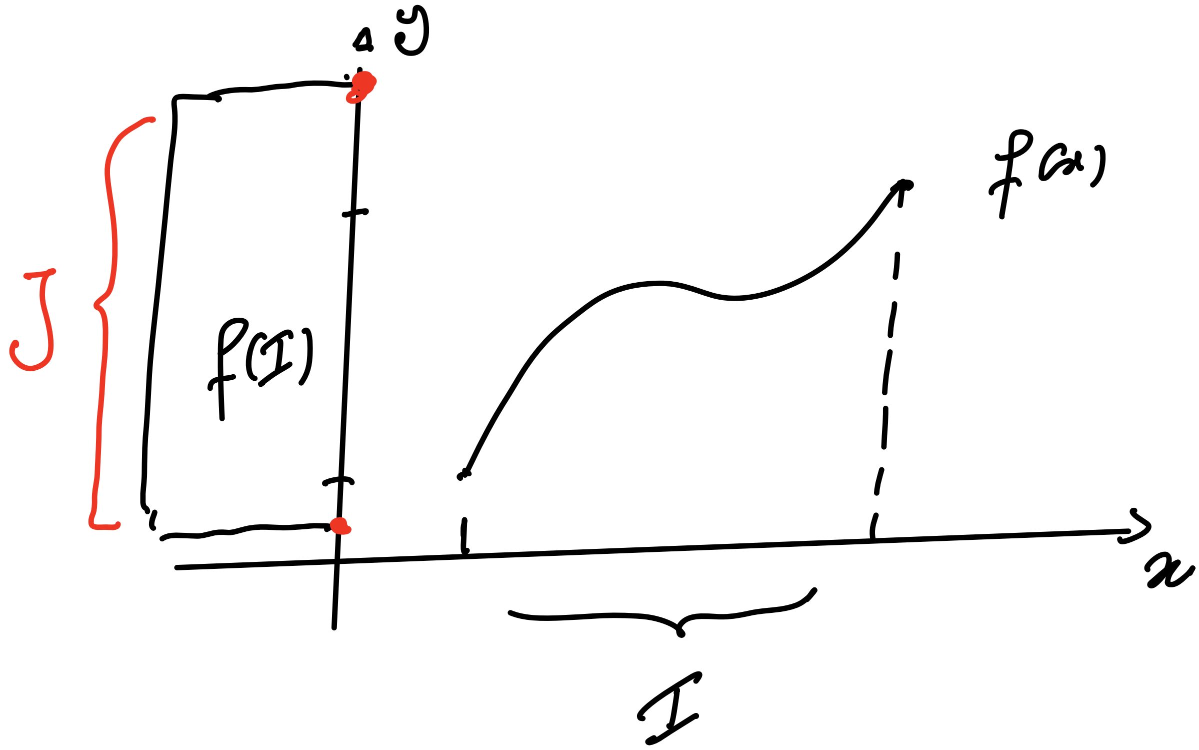
$g = g(y) \qquad g : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

J = intervallo

Supponiamo che il codominio di f sia contenuto in quello di g, ossia

$$f(I) \subseteq J$$

codominio di f



Funzione composta da f e g

$$g \circ f = \text{" } g \text{ composto } f \text{ "}$$

$$(g \circ f)(x) \underset{\text{def.}}{=} g(f(x))$$

$$f : I \longrightarrow J$$

$$g : J \longrightarrow \mathbb{R}$$

$$g(f(I)) = \{ g(y) : y \in f(I) \} = \{ g(f(x)) : x \in I \}$$

$$\begin{tikzcd}
x \arrow[r, "f"] & f(x) \arrow[r, "g"] & g(f(x)) \\
I \arrow[rr, bend right, "g \circ f"'] & J & \mathbb{R}
\end{tikzcd}$$

$$\begin{tikzcd}
 & f(I) \arrow[dr, "g"] & \\
I \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & g(f(I))
\end{tikzcd}$$

ES.

$$f(x) = x \qquad g(y) = y^2$$

$$(g \circ f)(x) = g(f(x)) = g(x) = x^2$$

$$D_f = \mathbb{R}$$

$$f(x) = x, \qquad g(y) = 1/y, \quad D_g = \mathbb{R} \setminus \{0\}$$

$$(g \circ f)(x) \qquad f(x) \neq 0 \iff$$

$$\iff x \neq 0$$

$$(g\circ f)(x) = g(f(x)) = g(x) = \frac{1}{x}$$

$f(x) = x^2$ $\qquad$ $g(y) = 1/y$

$$x^2 \neq 0 \iff x \neq 0$$

$D_f = \mathbb{R}\setminus\{0\}$

$$(g\circ f)(x) = g(f(x)) = g(x^2)$$

$$= \frac{1}{x^2}$$

$f(x) = x-1 \quad , \forall x \in \mathbb{R}$

$g(y) = \sqrt{y} \qquad \forall y \geq 0 \quad , D_g = [0, +\infty[$

$$f(x) \geq 0 \iff x-1 \geq 0 \iff \underline{x \geq 1}$$

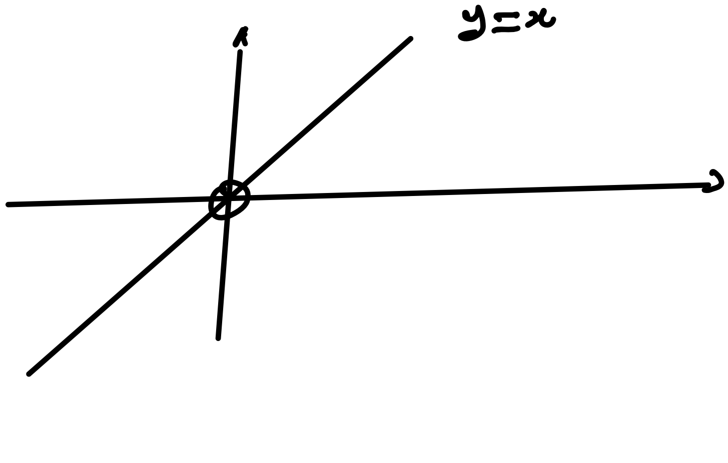
Quando $x \geq 1 \iff x \in [1, +\infty[$:

$$(g \circ f)(x) = g(f(x)) = g(x-1)$$
$$= \sqrt{x-1} \quad \forall x \geq 1$$

$f(x) = 1/x \quad \forall x \neq 0 \quad , D_f = \mathbb{R} \setminus \{0\}$

$g(y) = 1/y \quad \forall y \neq 0 \quad , D_g = \mathbb{R} \setminus \{0\}$

$$(g \circ f)(x) = g(f(x)) = g(1/x) = \frac{1}{1/x}$$
$$= x \quad \forall x \neq 0$$



N.B. $g \circ f \neq f \circ g$

Funzioni invertibili ; funzioni inverse

$f : A \longrightarrow B \qquad A, B \neq \emptyset$

f biettiva (ossia iniettiva e suriettiva)

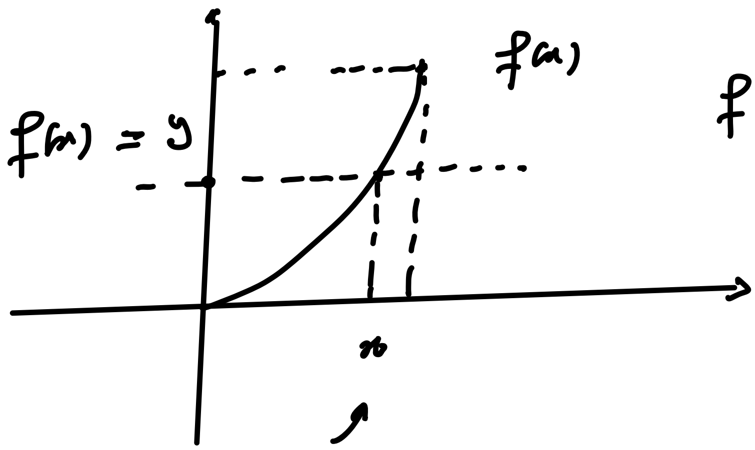
$\Leftrightarrow \forall y \in B \ \exists!\ x \in A : f(x) = y$

Si definisce la funzione inversa di f

$$f^{-1} : B \longrightarrow A$$

$\forall y \in B$, $f^{-1}(y) = x$ dove $x \in A$ è tale che

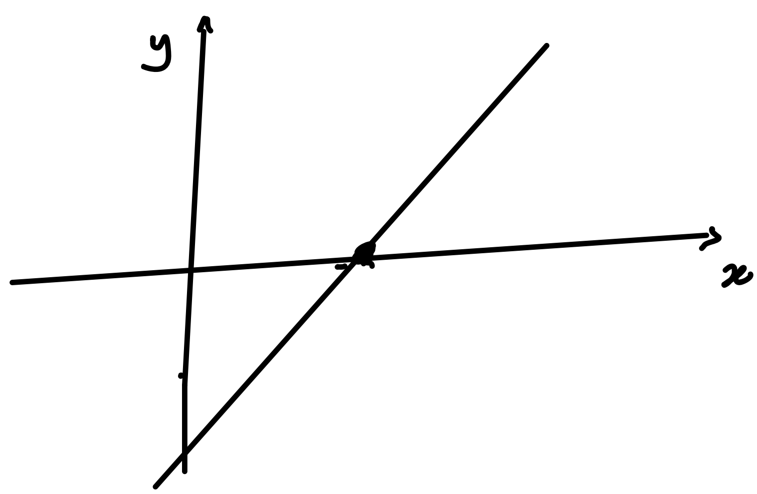
$f(x) = y$



$f^{-1}(y)$

$f(x) = y \iff x = f^{-1}(y)$

ES. $y = 2x - 3 = f(x)$ $\quad f$ invertibile



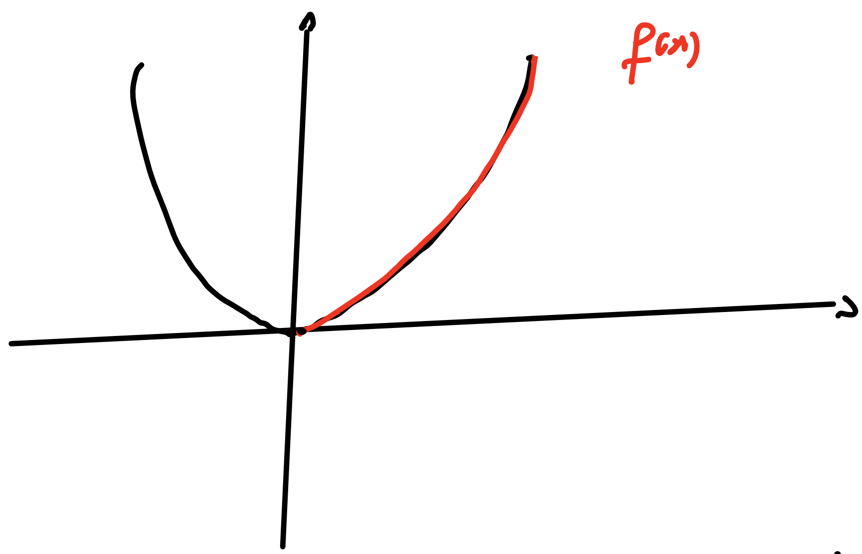
$y \in \mathbb{R}$ vogliamo trovare $f^{-1}(y) = x$, dove

x tale che $\quad f(x)=y \iff \underbrace{2x-3}=y$

$\iff 2x = 3+y \iff x = \dfrac{3+y}{2}$

<u>Quindi</u> $\quad f^{-1}(y) = \dfrac{3+y}{2}$

<u>ES.</u> $\quad f(x) = x^2 \qquad x \geq 0$



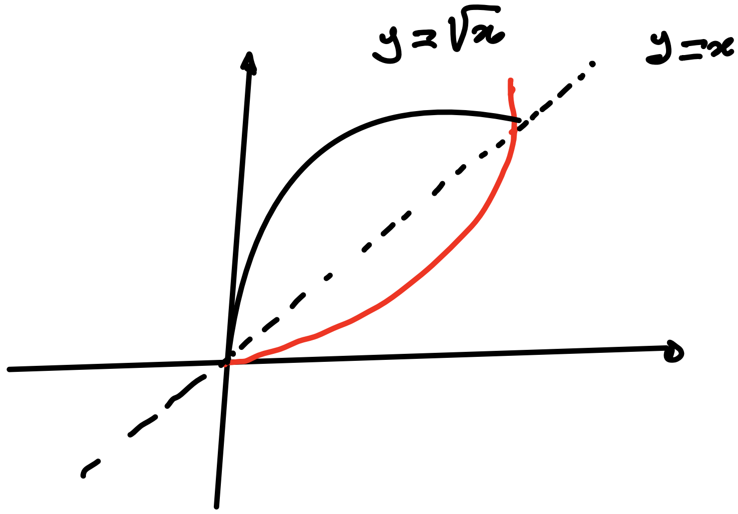
$f : [0,+\infty[\longrightarrow [0,+\infty[\quad$ è invertibile

$f^{-1}(y) = x \quad$ t.c. $\quad f(x) = y \iff x^2 = y$

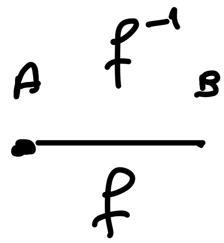
$\iff x = \sqrt{y}$

$$f^{-1}(y) = \sqrt{y}$$

$$f^{-1}(x) = \sqrt{x}$$



Proprietà delle funzioni inverse



$f : A \to B$ invertibile

$f^{-1} : B \to A$

$$f^{-1}(f(x)) = x \quad \forall x \in A$$

$$f(f^{-1}(y)) = y \quad \forall y \in B$$

NOTA Se f è strettamente crescente o decrescente, è certamente invertibile

Grafico di f^{-1}

$$f : A \to f(A)$$

Supponiamo che $f : A \longrightarrow \mathbb{R}$ sia invertibile

$$f^{-1} : f(A) \longrightarrow A$$

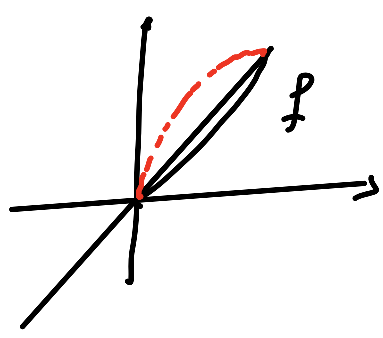
$f^{-1}(y) = x$ t.c. $f(x) = y$

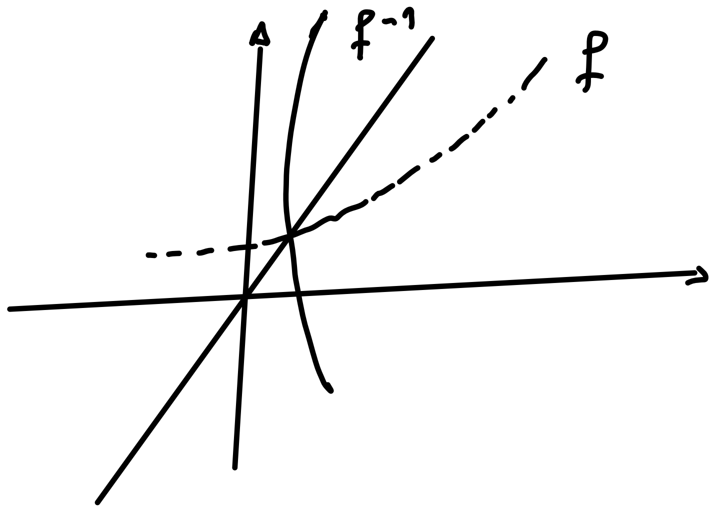
Se (x_0, y_0) appartiene al grafico di f,

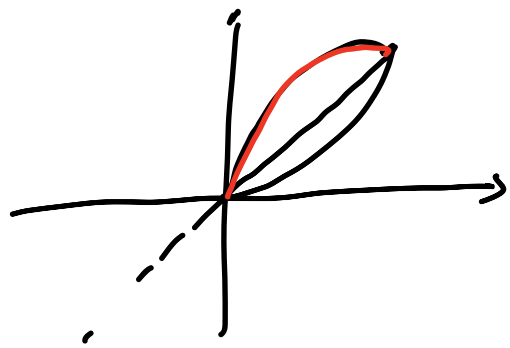
$$y_0 = f(x_0) \Leftrightarrow x_0 = f^{-1}(y_0)$$

$(y_0, x_0) = (y_0, f^{-1}(y_0)) \in$ grafico di f^{-1}

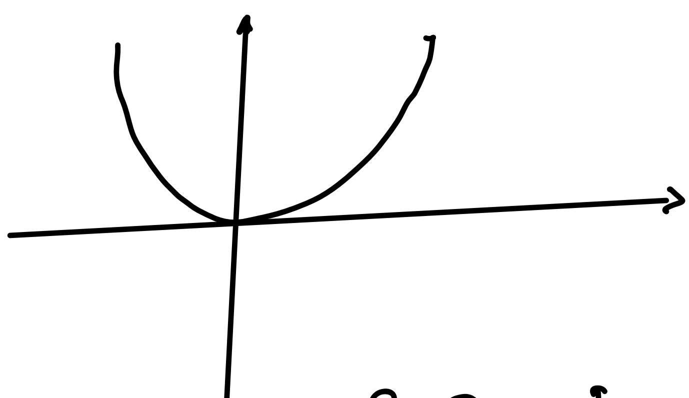
$$(x,y) \longleftrightarrow (y,x)$$







ES. $f(x) = x^2 \quad \forall x \in \mathbb{R}$

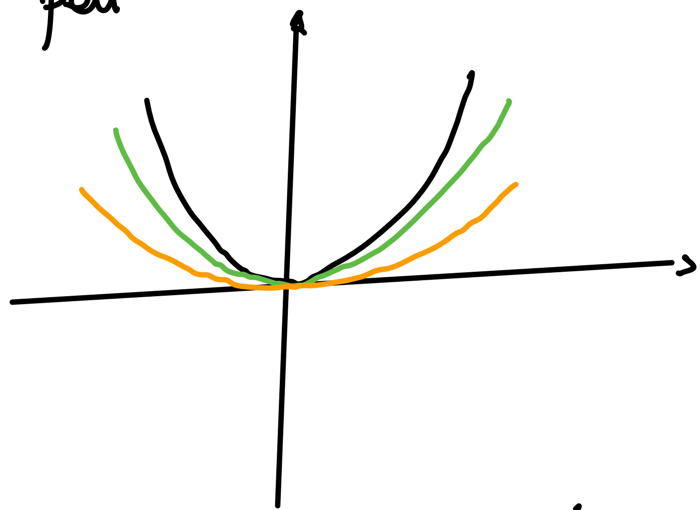


$f(x) = x^2 \ , \ \forall x \geq 0$

$f : [0, +\infty[\longrightarrow [0, +\infty[$ é invertibile

$f^{-1}(y) = \sqrt{y}$

$f(x) = x^n$ n pari



$f(x) = x^n \quad (x^2, x^4, x^6, \dots) \quad \forall x \geq 0$

$f : [0, +\infty[\longrightarrow [0, +\infty[$ è invertibile

$f^{-1} : y \in [0, +\infty[\longrightarrow f^{-1}(y) \in [0, +\infty[$

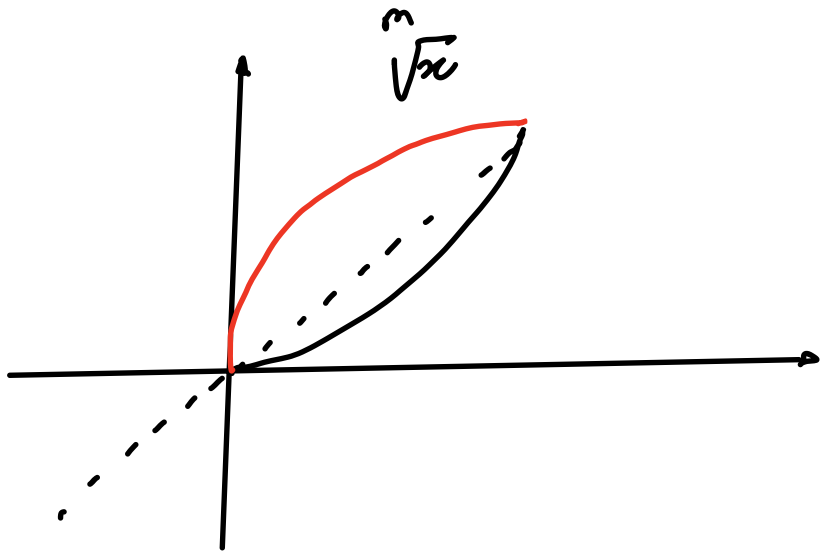
$f^{-1}(y) = x$ t.c. $f(x) = y$

$\iff x^n = y$

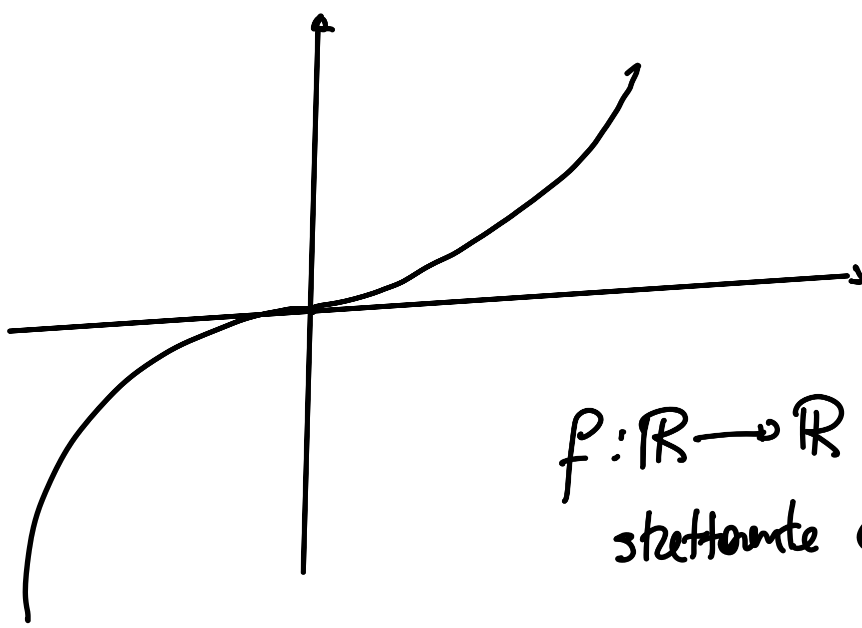
$\iff x = \sqrt[n]{y}$

$f^{-1}(y) = \sqrt[n]{y}, \quad \forall y \geq 0$

$$f^{-1}(x) = \sqrt[n]{x} = x^{1/n}$$



$f(x) = x^n$ n dispari $n = 3, 5, 7, 9, \ldots$



$f : \mathbb{R} \to \mathbb{R}$

strettamente crescente

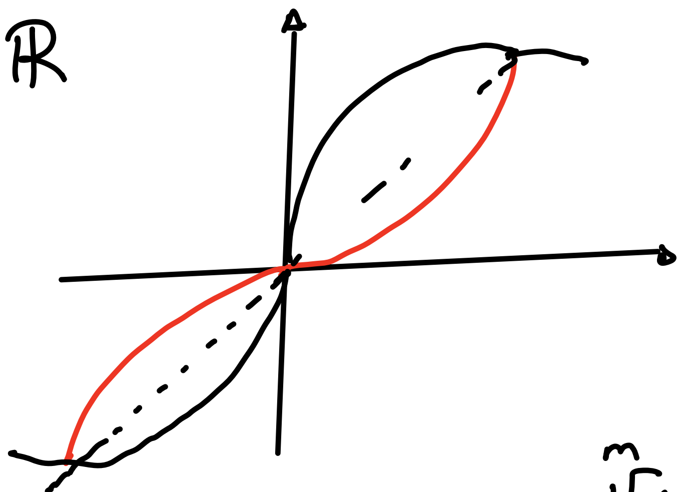
$y \geq 0$ $f^{-1}(y) = x$: $x^n = y \iff x = \sqrt[n]{y}$

$$f^{-1}(y) = \begin{cases} \sqrt[n]{y} & \text{se } y \geq 0 \\ -\sqrt[n]{-y} & y \leq 0 \end{cases}$$

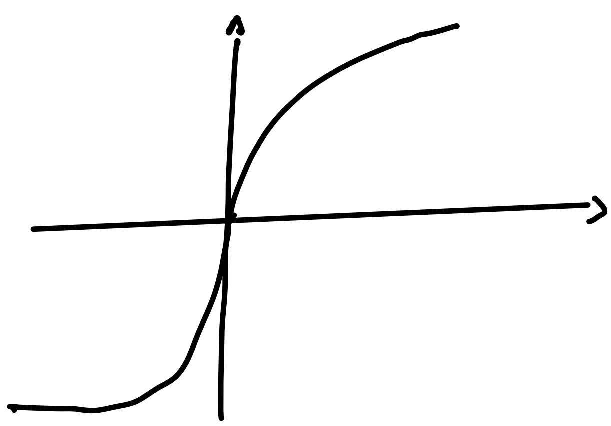
$$\sqrt[3]{y} = -\sqrt[3]{-y} \qquad y < 0$$

$$\sqrt[3]{-8} = -\sqrt[3]{8} = -2$$

$$f^{-1} : \mathbb{R} \to \mathbb{R}$$



$\sqrt[n]{x}$, n dispari



Funzioni ad esponente reale, funzione esponenziale

$a, b > 0, \qquad c, d \in \mathbb{R}$

1) $a^0 = 1 \quad \forall a \neq 0 \quad , \quad 1^c = 1 \quad \forall c$

2) $a^c > 0, \quad \forall c \in \mathbb{R}$

$$\begin{cases} a^c < 1 & \text{se } a < 1 \\ a^c > 1 & \text{se } a > 1 \end{cases} \qquad e \quad c > 0$$

3) $a^{c+d} = a^c \cdot a^d$

4) $(ab)^c = a^c \cdot b^c$

5) $(a^b)^c = a^{bc}$

6) $c < d \Rightarrow \quad a^c < a^d \quad a > 1$

$\qquad\qquad\quad a^c > a^d \quad 0 < a < 1$

7) $0 < a \leq b \Rightarrow a^c \leq b^c, \ \forall c > 0$

$\downarrow$

$x < y \Rightarrow x^b < y^b, \ \forall b > 0$

8) $0 < a < b \Rightarrow a^c > b^c \quad \forall c < 0$

$a^b \quad a > 0 \quad b \in \mathbb{R}$

$f(x) = x^b, \quad \forall x > 0 \quad \forall b \in \mathbb{R}$

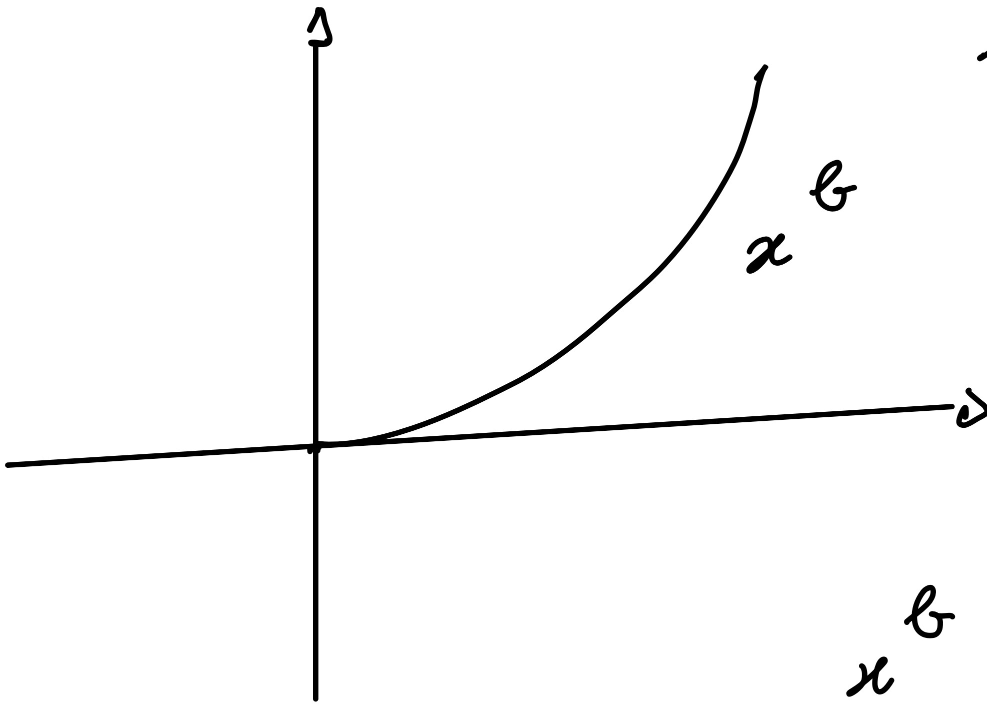
$f(x) = x^{\pi} \qquad f(x) = x^{\sqrt{2}}$

$f(x) = x^{1/2} = \sqrt{x}$

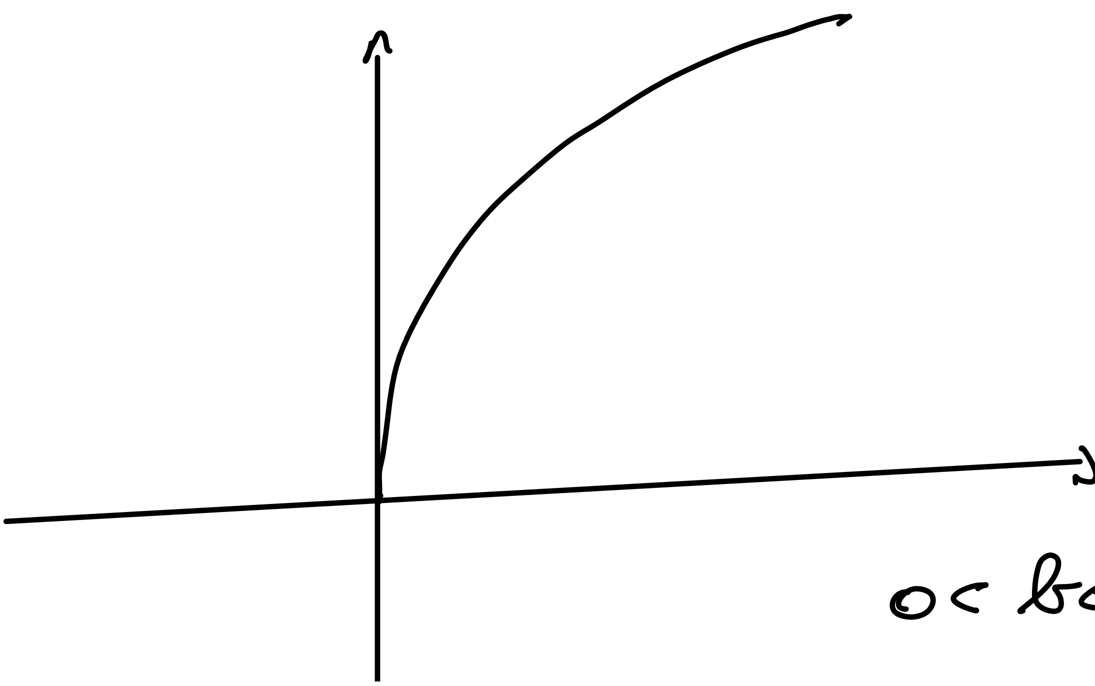
Funzione potenza ad esponente reale

Dalla (7), se $b>0$, la funzione è strettamente crescente

$b>1$



x^b

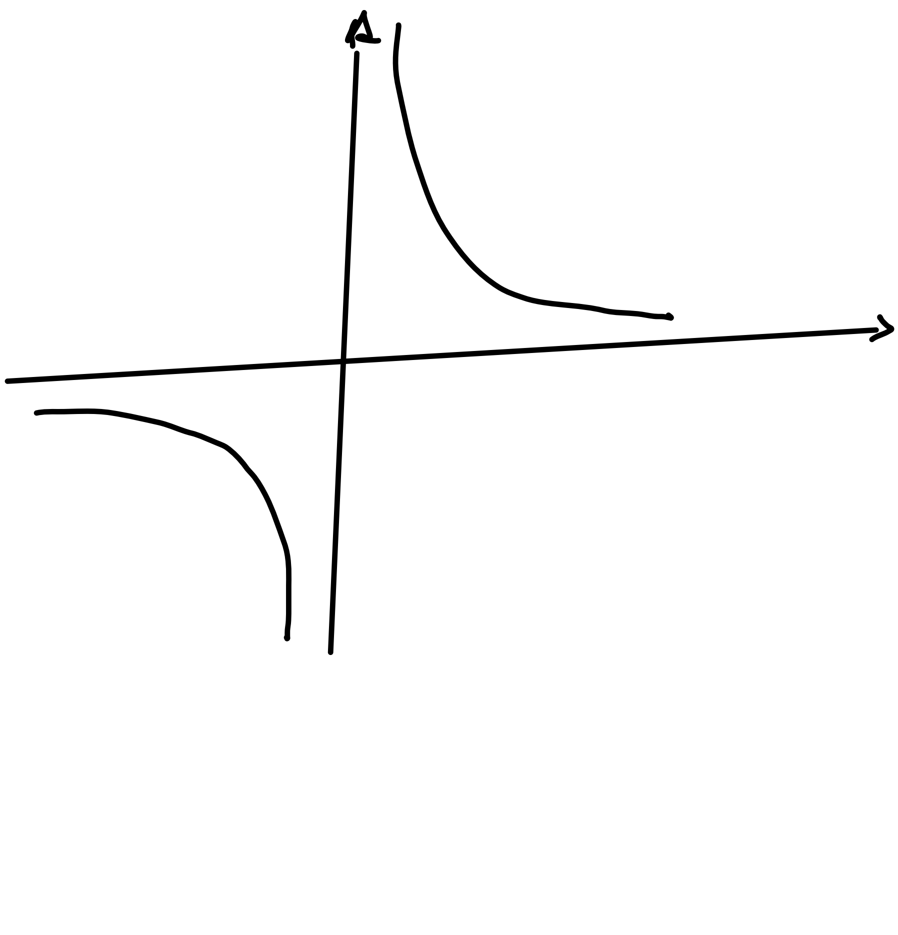


$0<b<1$

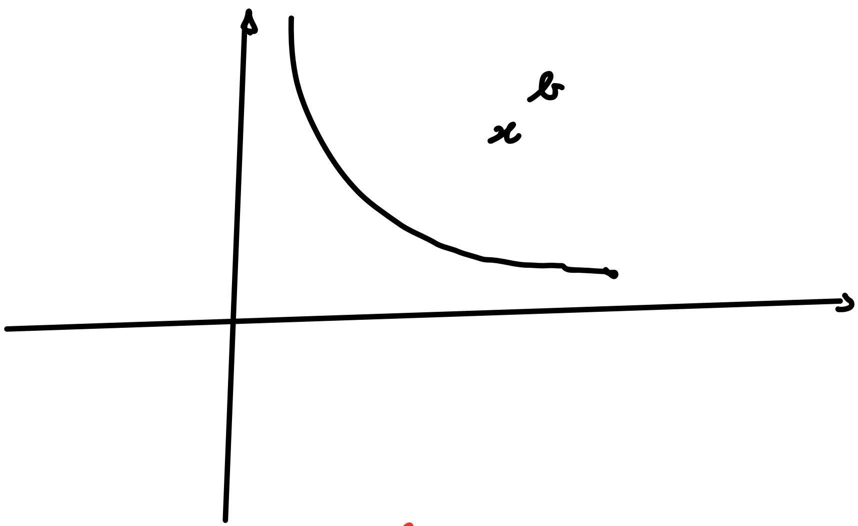
$\underline{b<0}$? (ES. $f(x) = x^{-1} = \frac{1}{x}$

$x > 0$

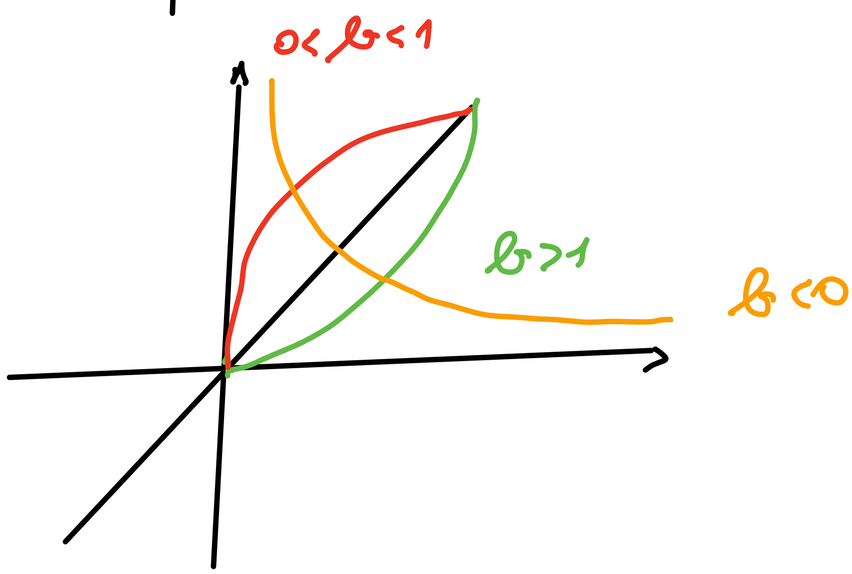
$y = \frac{1}{x}$ iperbole equilatera



$b < 0$



$f(x) = x^b$



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a^b $\quad a > 0 \quad , b \in \mathbb{R}$

$f(x) = a^x \quad \forall a > 0 \quad , \forall x \in \mathbb{R}$

funzione esponenziale

$\underline{a=1}$ $\quad f(x) = 1^x = 1$

$a \neq 1$

$f(x) = a^x > 0 \quad \forall x \in \mathbb{R}$

$0 < a < 1$ $\quad$ ES. $\; a = \frac{1}{2}$

$f(x) = \left(\frac{1}{2}\right)^x$

$a > 1$ $\quad f(x) = 2^x$

$a = e$ " Numero di Nepero "

$c < d \Rightarrow \quad a^c < a^d \quad a > 1 \quad (\Leftarrow)$

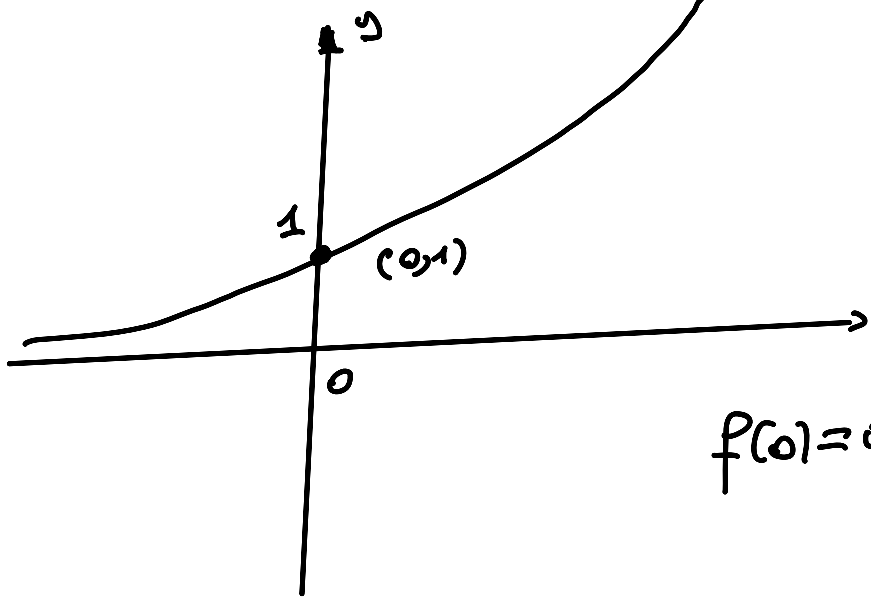
$\qquad\qquad\qquad a^c > a^d \quad 0 < a < 1 \quad (\Leftarrow)$

Se $a > 1$, da $\Leftarrow$ si ha che a^x è

strettamente crescente

$\sup\limits_{x \in \mathbb{R}} a^x = +\infty$

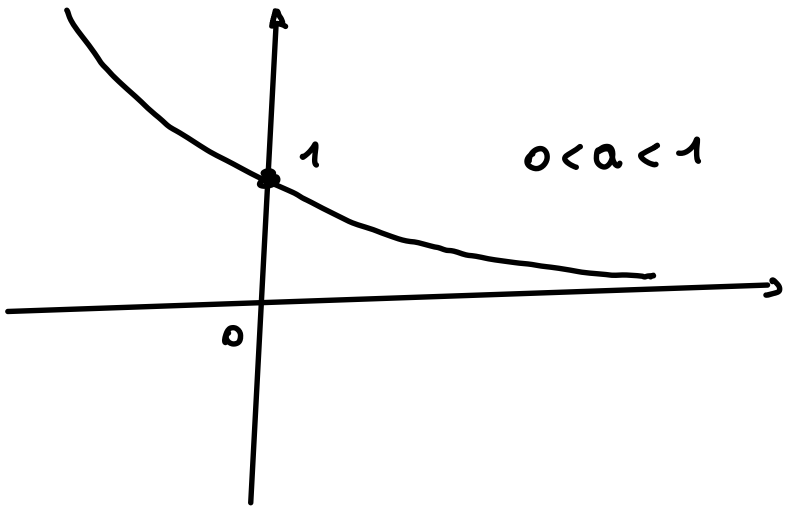
$\inf\limits_{x \in \mathbb{R}} a^x = 0$



$f(0) = a^0 = 1$

$D_f = \mathbb{R}$, Codominio di $f = f(\mathbb{R}) =]0, +\infty[$

Se $0 < a < 1$ da $(\Leftarrow)$, a^x è strettamente decrescente



$0 < a < 1$

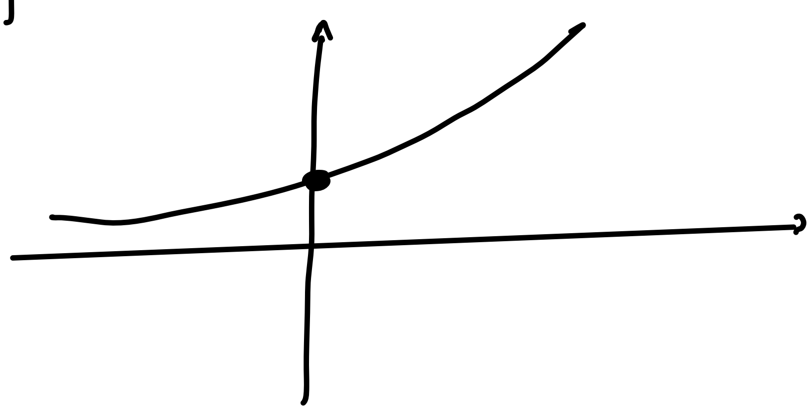
$\sup_{x \in \mathbb{R}} a^x = +\infty$

$\inf_{x \in \mathbb{R}} a^x = 0$

$\left(\frac{1}{2}\right)^x = 2^{-x}$

$a = e$ $\qquad e = 2{,}7182818\ldots$

$f(x) = e^x$ $\qquad 2 < e < 3$



Incasa? $\qquad a \neq 1, \quad a > 0, \quad f(x) = a^x$ è

invertibile: $\qquad f : \mathbb{R} \longrightarrow]0, +\infty[$

$\underline{a>1}$ $\qquad f^{-1} : \,]0,+\infty[\, \longrightarrow \mathbb{R}$

$\forall y>0 \qquad f^{-1}(y) = x \;:\; f(x) = y \iff \underbrace{a^x = y}$

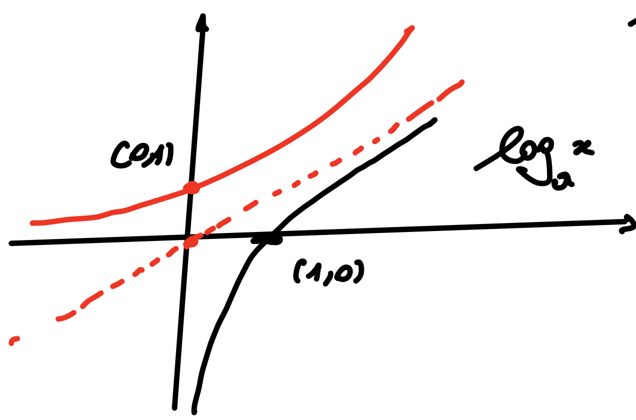
$$x = \log_a y \qquad \forall y>0$$

$$\log_2 \tfrac{1}{2} = -1 \qquad 2^{-1} = \frac{1}{2}$$

$$f^{-1}(y) = \log_a y \qquad \forall y>0$$

$\log_a 1 = 0$

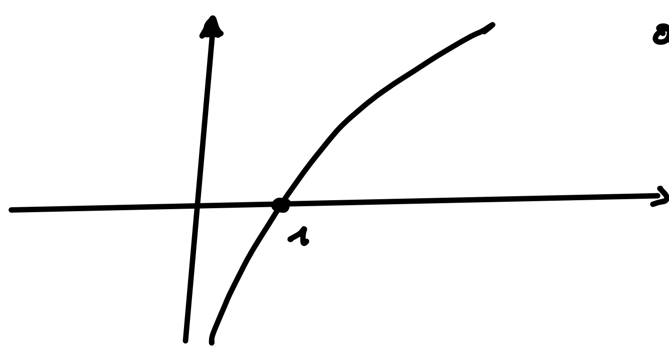
$a>1$



$\log_a x$

$a>1$

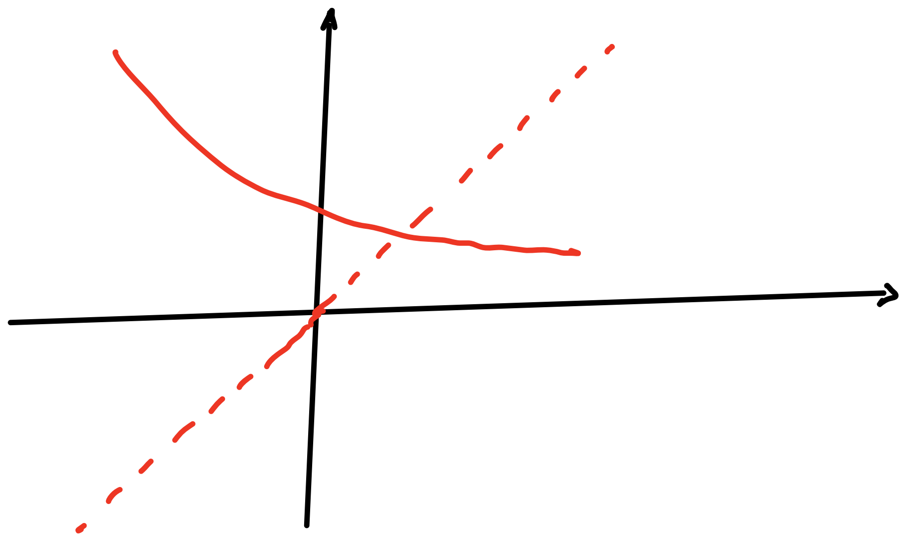
$\sup\limits_{x>0} \log_a x = +\infty$



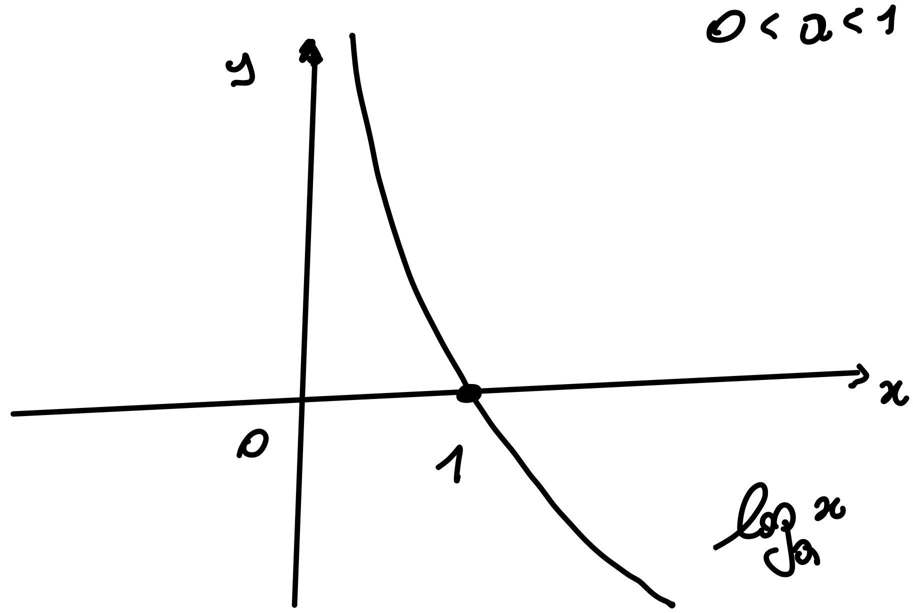
$\log_a x$

$\inf\limits_{x>0} \log_a x = -\infty$

$0 < a < 1$



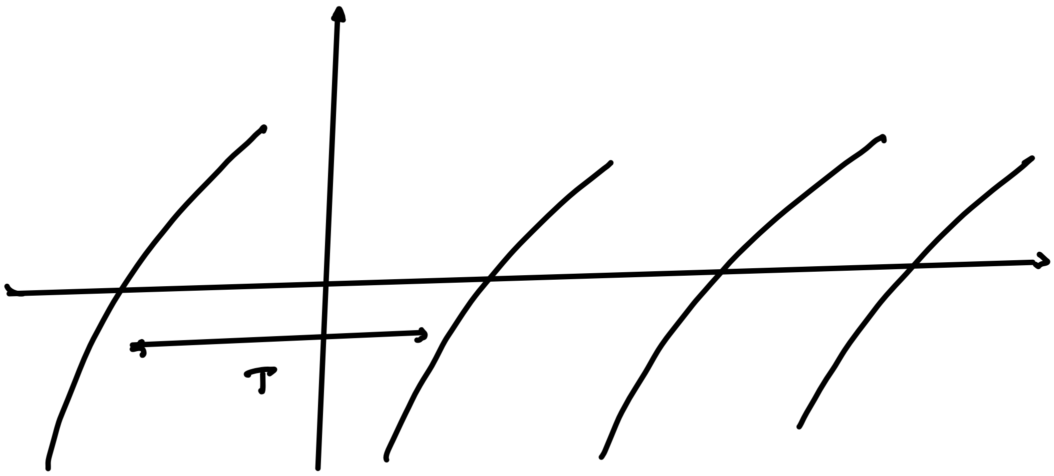
$0 < a < 1$



Funzioni trigonometriche

Def. $f: \mathbb{R} \to \mathbb{R}$ si dice periodica di periodo $T > 0$

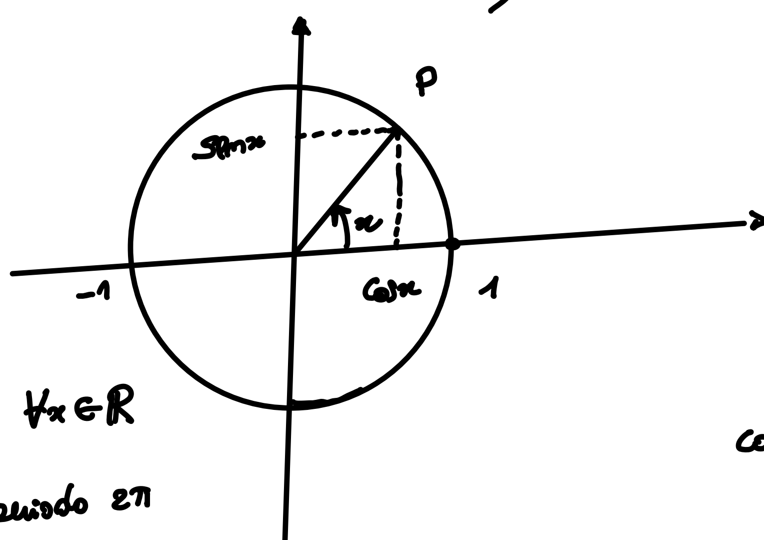
se accade che $f(x+T) = f(x)$, $\forall x \in \mathbb{R}$



Def Funzione coseno : $f(x) = \cos x \quad \forall x \in \mathbb{R}$

$-1 \leq \cos x \leq 1$

$|\cos x| \leq 1$

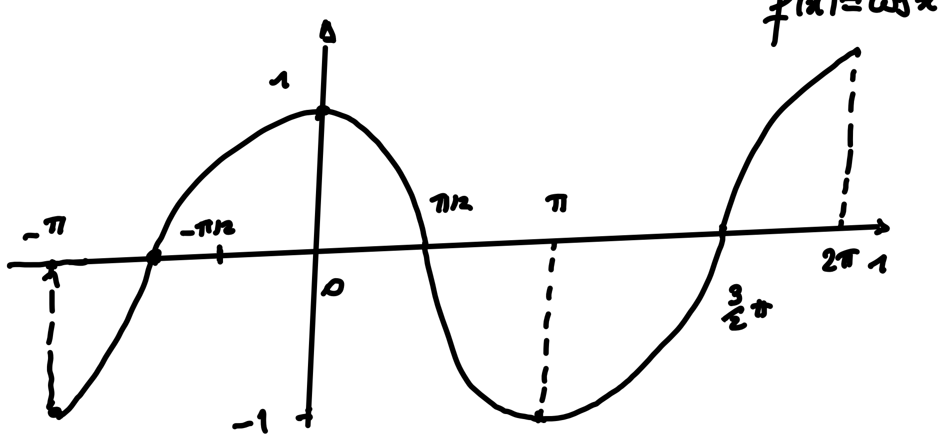


$\cos(x + 2\pi) = \cos x \quad \forall x \in \mathbb{R}$

$\cos 0 = 1$

coseno è periodica di periodo 2π

$f(x) = \cos x$



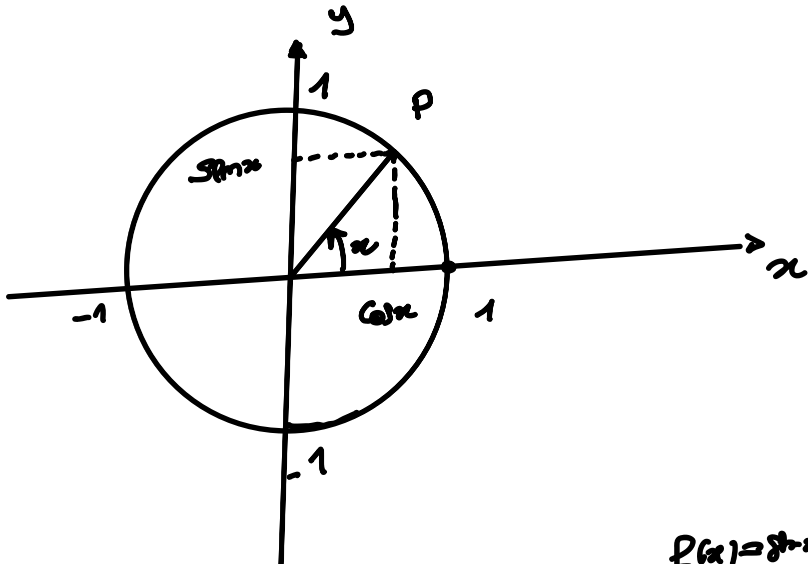
$D_f = \mathbb{R}$ Codominio $f(\mathbb{R}) = [-1, 1]$

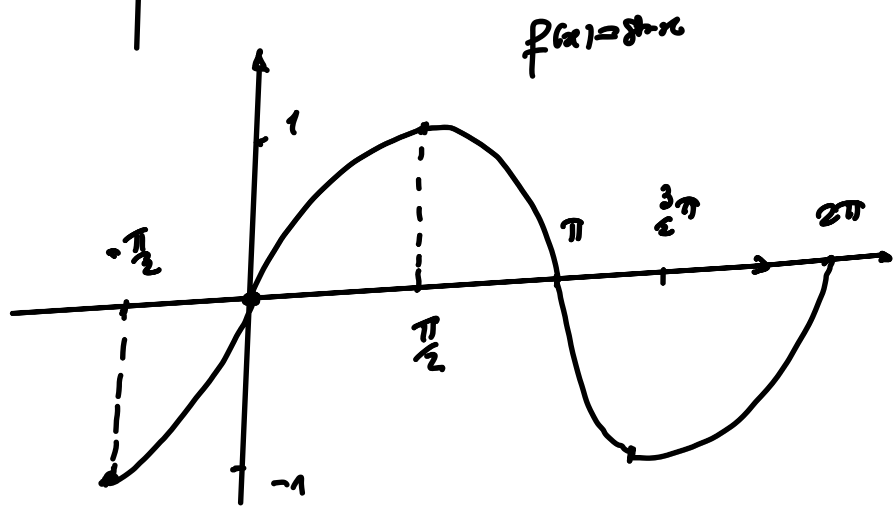
$\min_{\mathbb{R}} \cos x = -1$

$\max_{\mathbb{R}} \cos x = 1$

$\cos(-x) = \cos x$ è pari

$\cos(x + 2k\pi) = \cos x \quad \forall x \in \mathbb{R}, \quad \forall k \in \mathbb{Z}$





$|\sin x| \leq 1$

$\sin(x + 2\pi) = \sin x$

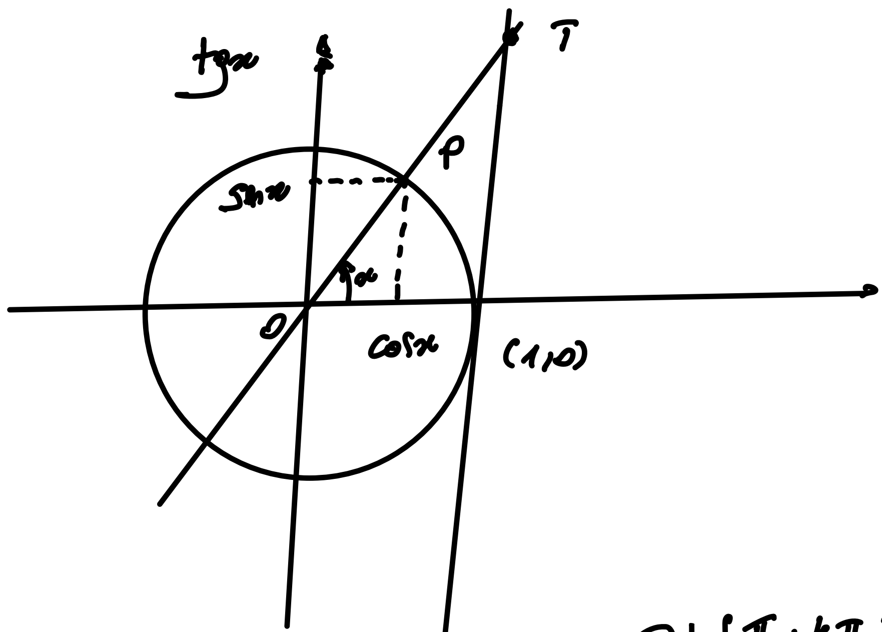
La funzione seno è periodica di periodo 2π

$\min_{\mathbb{R}} \sin x = -1, \quad \max_{\mathbb{R}} \sin x = 1 \quad ; \quad D_f = \mathbb{R}$

$f(\mathbb{R}) = [-1, 1]$

Poiché $\sin(-x) = -\sin x$, la funzione seno è dispari!

$$\sin(x+2k\pi) = \sin x \quad \forall k \in \mathbb{Z}$$



$$f(x) = tg\,x = \tan x = \frac{\sin x}{\cos x} \; ; \; D_f = \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}$$

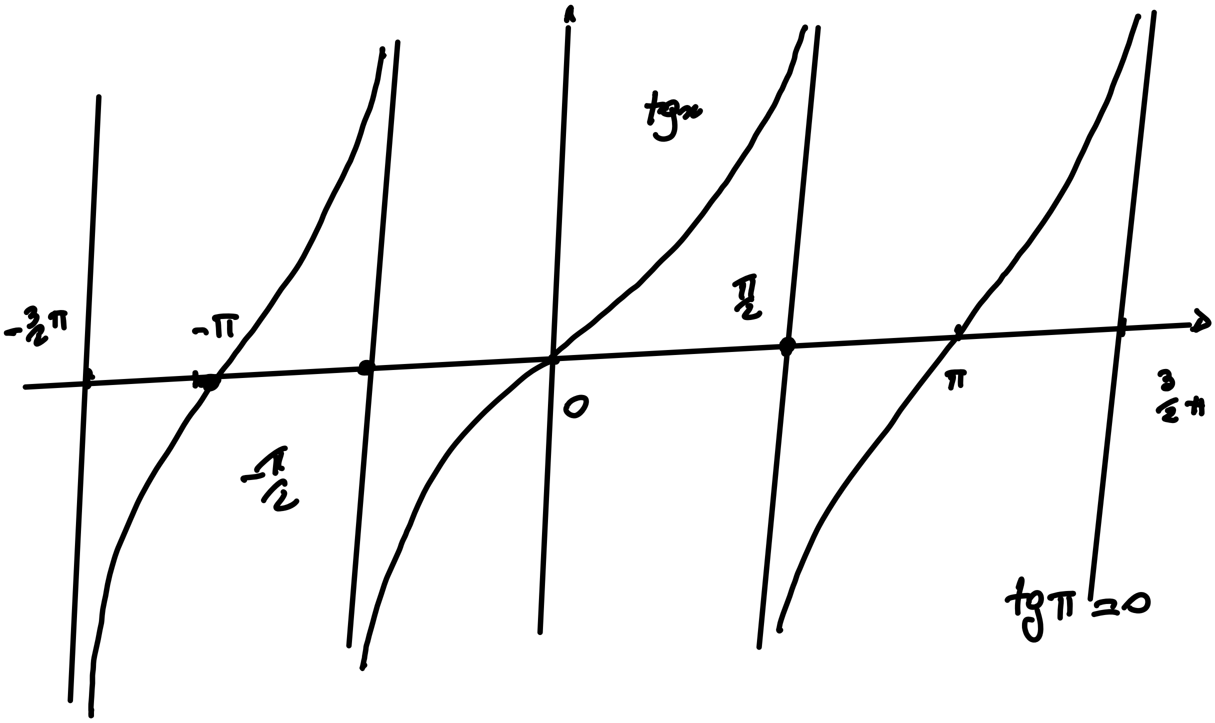
$$\cos x \neq 0 \iff x \neq \frac{\pi}{2} + k\pi \, , \; k \in \mathbb{Z}$$

$f(x)$

$$f(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \frac{\sin x}{\cos x}$$

$$= tg\,x = f(x)$$

la funzione tangente è periodica di periodo π.



Codominio di $tg\,x = \mathbb{R}$

$\sup tg\,x = +\infty$

$\inf tg\,x = -\infty$

tangente è dispari : $f(-x) = \dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin x}{\cos x}$

$= -f(x)$