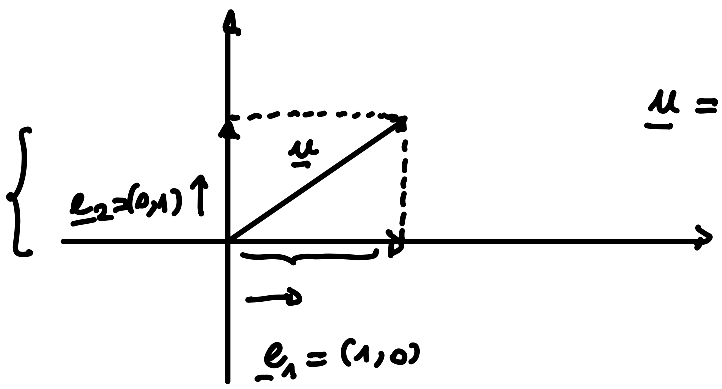
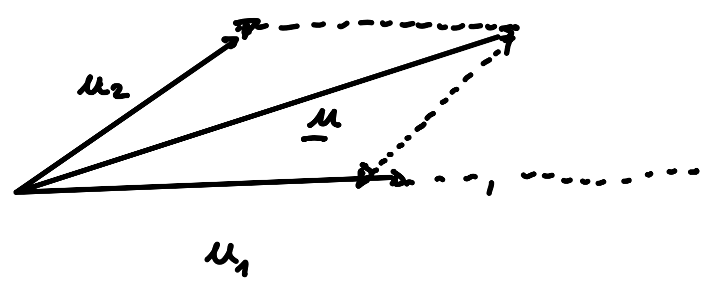
Lezione dell' 11/10/2022

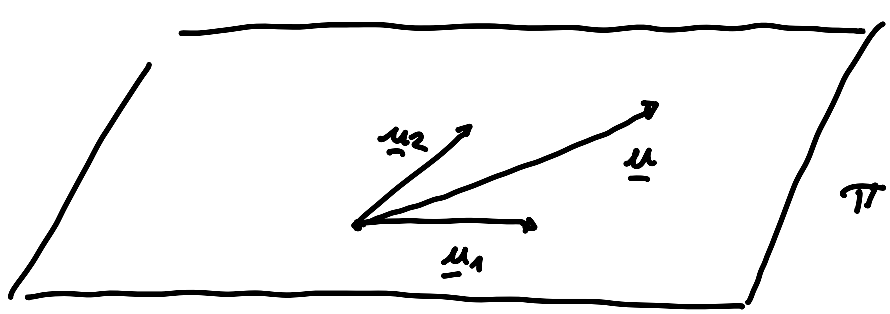
$\vec{u}$ $\underline{u}$

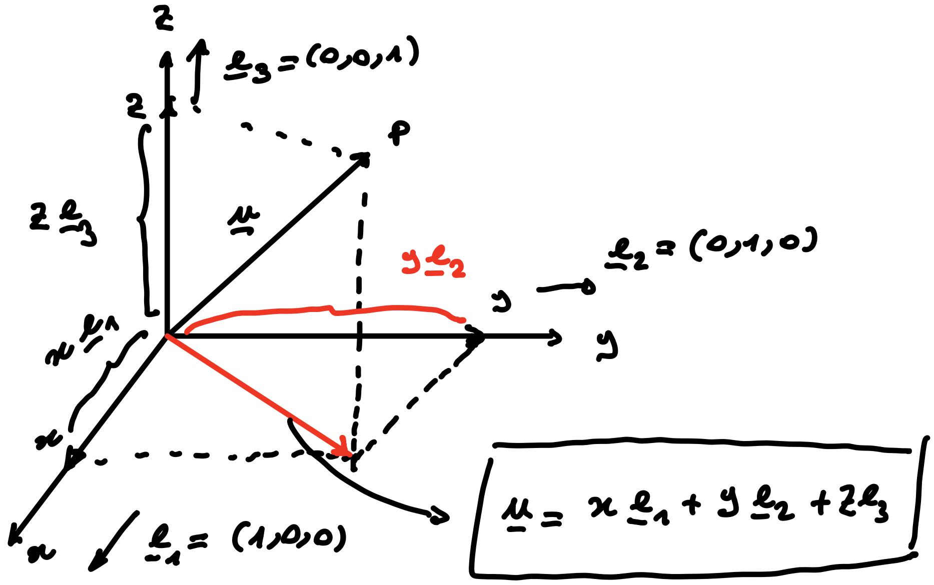


$\underline{u} = x\,\underline{e}_1 + y\,\underline{e}_2$



$\underline{u} = \alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 \,, \quad \underline{\alpha_1, \alpha_2} \in \mathbb{R}$





$$\underline{V}^2 = \text{insieme dei vettori del piano} \equiv \mathbb{R}^2$$

$$\underline{V}^3 = \quad " \quad " \quad \text{dello spazio} \equiv \mathbb{R}^3$$

$$\mathbb{R}^2 = \{ (x,y) : x, y \in \mathbb{R} \}$$

$$\mathbb{R}^3 = \{ (x,y,z) : x, y, z \in \mathbb{R} \}$$

$$\mathbb{R}^n = \{ \underbrace{(x_1, x_2, \dots, x_n)}_{n} : x_i \in \mathbb{R}, \ \forall i = 1, \dots, n \}$$

$$\underline{u} = (x_1, x_2, \dots, x_n) \qquad \underline{v} = (y_1, y_2, \dots, y_n)$$

$$\underline{u} + \underline{v} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in \mathbb{R}^n$$

$\lambda \in \mathbb{R} \qquad \lambda \underline{u} = (\lambda x_1, \lambda x_2, \dots, \lambda x_n) \in \mathbb{R}^n$

$\underline{0} = (0, 0, \dots, 0)$

$\underline{u} + \underline{0} = \underline{u} \qquad \forall \underline{u} \in \mathbb{R}^n$

"per ogni : significa per qualsiasi scelta"

$\underline{u} = (x_1, x_2, \dots, x_n) \qquad \underline{v} = (y_1, y_2, \dots, y_n)$

$-\underline{u} = (-x_1, -x_2, \dots, -x_n)$

$\underline{u} - \underline{v} = (x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$

Prodotto scalare (nel piano)

$\underline{u} = (x_1, x_2) \quad , \quad \underline{v} = (y_1, y_2)$

$\underbrace{\underline{u} \cdot \underline{v}}_{\text{prodotto scalare di } \underline{u} \text{ per } \underline{v}} =$ "$\underline{u}$ scalare $\underline{v}$" $= x_1 y_1 + x_2 y_2 \in \mathbb{R}$

$\underline{u} = (1, 2), \quad \underline{v} = (3, 4) \ : \ \underline{u} \cdot \underline{v} = 1 \cdot 3 + 2 \cdot 4$
$= 3 + 8 = 11$

$\underline{v} = (-2, -4)$ $\qquad$ $\underline{u} \cdot \underline{v} = 1 \cdot (-2) + 2 \cdot (-4)$

$$= -2 - 8 = -10 < 0$$

$\underline{u} = (x_1, x_2, x_3)$, $\quad \underline{v} = (y_1, y_2, y_3)$

$$\underline{u} \cdot \underline{v} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$\underline{u} = (1, 0, -1)$, $\quad \underline{v} = (-2, 4, 5)$

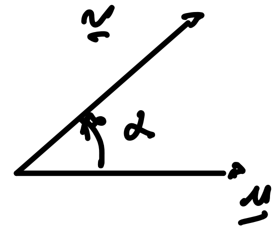
$$\underline{u} \cdot \underline{v} = 1 \cdot (-2) + 0 \cdot 4 + (-1) \cdot 5 =$$
$$= -2 - 5 = -7$$

$\underline{u} \cdot \underline{v} = 0$ $\quad$ i vettori $\underline{u}$ e $\underline{v}$ si dicono ortogonali

Proprietà

$$\underline{u} \cdot \underline{v} = \overbrace{\|\underline{u}\|}^{\text{modulo}} \, \|\underline{v}\| \cos(\underbrace{\widehat{u, v}}_{\text{angolo tra } u \text{ e } v})$$

$$\|\underline{u}\| = \sqrt{x_1^2 + x_2^2}$$

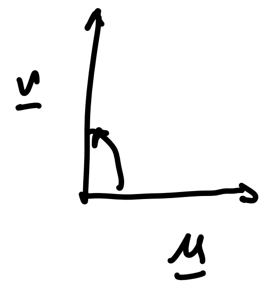


$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\alpha$$

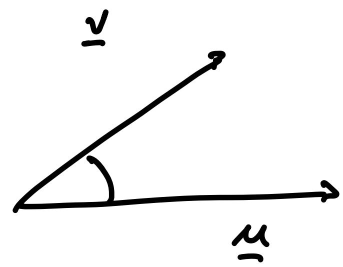
$$\alpha \in [0, \pi]$$

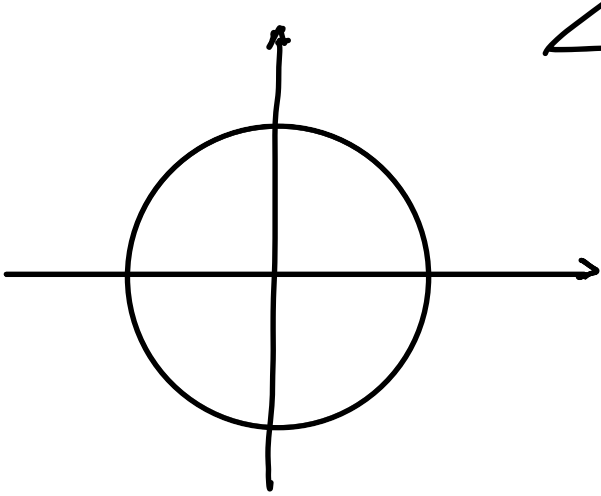
Se $\underline{u} \cdot \underline{v} = 0 \Rightarrow \|\underline{u}\| \, \|\underline{v}\| \cos\alpha = 0$

$\Rightarrow \cos\alpha = 0 \Rightarrow \alpha = \frac{\pi}{2}$

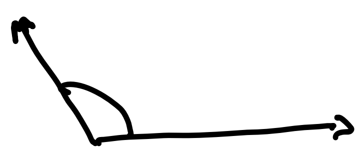


$\underline{u} \cdot \underline{v} > 0$





$\underline{u} \cdot \underline{v} < 0$



Proprietà

$\underline{u} \cdot \underline{u} = x_1 \cdot x_1 + x_2 \cdot x_2 =$

$= x_1^2 + x_2^2 \geq 0$

$\underline{u} = (x_1, x_2)$

$$\underline{u} \cdot \underline{u} = 0 \Rightarrow x_1 = x_2 = 0$$
$$\Rightarrow \underline{u} = \underline{0}$$

$$\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u} \quad \text{(prop. commutativa)}.$$

$$\alpha \in \mathbb{R} : \quad \alpha \underline{u} \cdot \underline{v} = \underline{u} \cdot \alpha \underline{v} = \alpha (\underline{u} \cdot \underline{v}).$$

Distributiva: $\underline{u}, \underline{v}, \underline{w} \in V^2 \ (V^3)$.

$$(\underline{u} + \underline{v}) \cdot \underline{w} = (\underline{u} \cdot \underline{w}) + (\underline{v} \cdot \underline{w})$$

(si dimostra)

con la nozione di prodotto scalare

Spazio vettoriale $\quad V^2, V^3, V^n(\mathbb{R})$

Def- Uno spazio vettoriale è un insieme V non vuoto ($V \neq \emptyset$) nel quale sono introdotte un'operazione di somma $+$ e un'operazione di prodotto esterno per scalari ($\mathbb{R}$) $\cdot$ tale che:

1) $\underline{u}, \underline{v} \in V$, $\quad \underline{u} + \underline{v} = \underline{v} + \underline{u}$ (prop. commutativa)

2) $\underline{u}, \underline{v}, \underline{w} \in V$, $\quad (\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$

(prop. associativa)

3) $\exists\ \underline{0} \in V$ tale che $\quad \underline{u} + \underline{0} = \underline{u}, \ \forall \underline{u} \in V$

$\underline{0}$ = elemento neutro rispetto a $+$

4) $\forall \underline{u} \in V \quad \exists\ -\underline{u} \in V$ tale che

$$\underline{u} + (-\underline{u}) = \underline{0}$$

$-\underline{u}$ = opposto di $\underline{u}$

5) $\forall \lambda \in \mathbb{R}$, $\forall \underline{u}, \underline{v} \in V$ si ha

$$\lambda(\underline{u} + \underline{v}) = \lambda \underline{u} + \lambda \underline{v}$$

6) $\quad (\lambda \beta)\underline{u} = \lambda(\beta \underline{u})$

$\forall \lambda, \beta \in \mathbb{R}$

7) $\quad 1 \cdot \underline{u} = \underline{u} \quad \forall \underline{u} \in V$

Esempi 1) V^2 (insieme dei vettori nel piano)

V^3 (" " nello spazio)

$$V^n,\ \mathbb{R}^n = \underbrace{\{(x_1, x_2, .., x_n) : x_i \in \mathbb{R}\ \forall i = 1, .., n\}}$$

$$\underline{u} = (x_1, .., x_n)$$
$$-\underline{u} = (-x_1, .., -x_n)$$
$$\underline{0} = (0, 0, ., 0)$$

(Sottospazio di uno spazio vettoriale)

Def. V spazio vettoriale

sottoinsieme W di V : $W \subseteq V$

si dice "W contenuto in V"

sottospazio vettoriale se per ogni

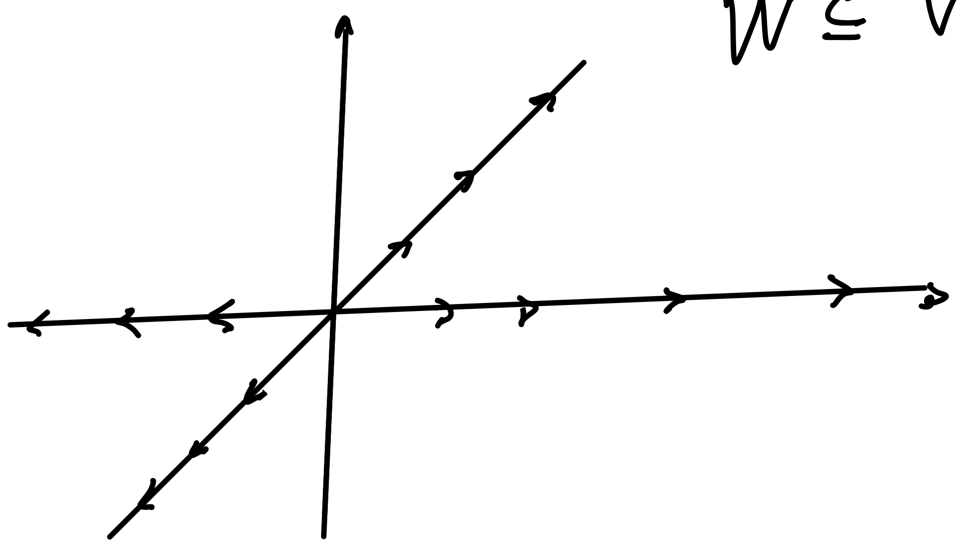
$u_1, u_2 \in W$ e per ogni $\alpha_1, \alpha_2 \in \mathbb{R}$

si ha $\alpha_1 u_1 + \alpha_2 u_2 \in W$

ESEMPIO in V^2

$$W = \{(x, 0) : x \in \mathbb{R}\}$$

$W \subseteq V^2$



$\underline{u}_1, \underline{u}_2 \in W$

$\alpha_1, \alpha_2 \in \mathbb{R}$

$\underline{u}_1 = (x_1, 0)$

$u_2 = (x_2, 0)$

$$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \alpha_1 (x_1, 0) + \alpha_2 (x_2, 0)$$

$$= (\alpha_1 x_1, 0) + (\alpha_2 x_2, 0)$$

$$= (\alpha_1 x_1 + \alpha_2 x_2, 0) \in W$$

Def (Combinazione lineare di vettori)

V spazio vettoriale

$\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n \in V$ vettori

$\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$

$$\underline{u} = \alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \cdots + \alpha_n \underline{u}_n$$

combinazione lineare di $\underline{u}_1, \ldots, \underline{u}_n$.

Def. (Sottospazio generato da vettori)

$$\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n \in V$$

$$W = \{ \alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \dots + \alpha_n \underline{u}_n : \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R} \}$$

sottospazio di V, sottospazio generato da $\underline{u}_1, \dots, \underline{u}_n$

ES. V^2 $\quad \underline{e}_1 = (1, 0)$

sottospazio generato da $\underline{e}_1$ è

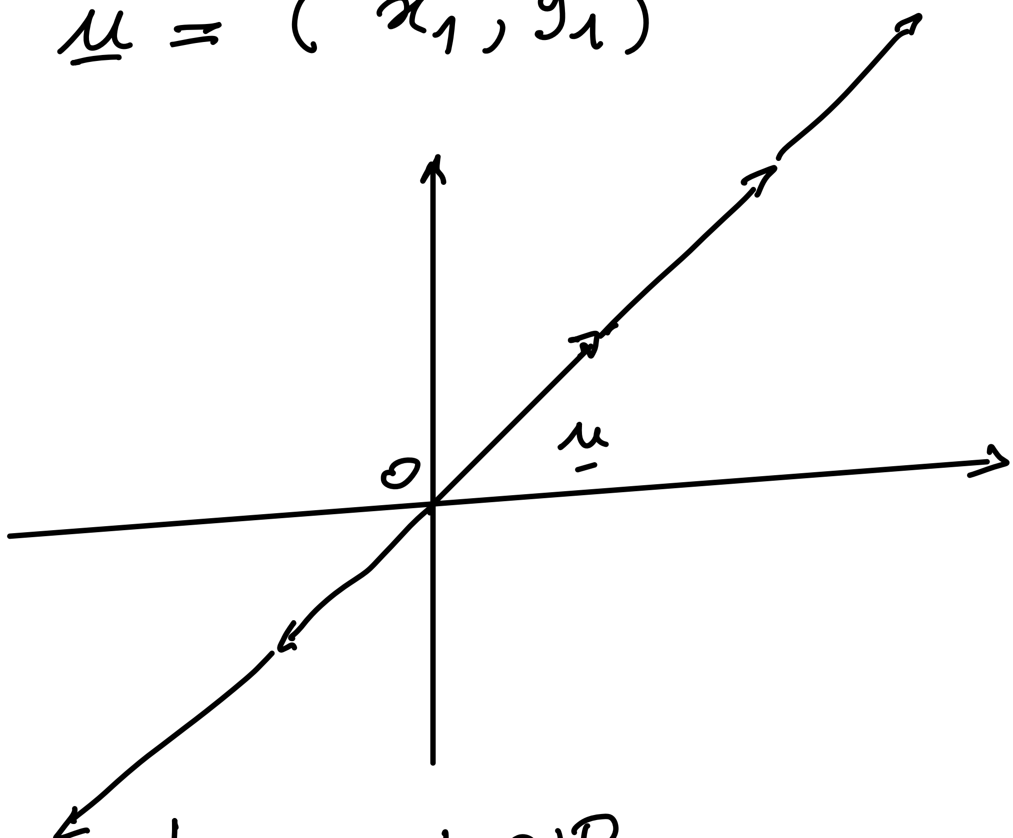
$$W = \{ \alpha \underline{e}_1 : \alpha \in \mathbb{R} \} =$$

$$= \{ (\alpha, 0) : \alpha \in \mathbb{R} \} \quad ??$$

asse x

$\underline{e}_2 = (0,1)$

$\underline{u} = (x_1, y_1)$



$\alpha \underline{u} : \alpha \in \mathbb{R}$

ES.

$\underline{e}_1 = (1,0), \quad \underline{e}_2 = (0,1)$

$W = \langle \underline{e}_1, \underline{e}_2 \rangle$ spazio generato da $\underline{e}_1, \underline{e}_2$

$$= \{ \alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 \ : \alpha_1, \alpha_2 \in \mathbb{R} \}$$

$$= V^2$$

Def (Lineare indipendenza

" dipendenza)

V spazio vettoriale

$\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n \in V$: si dice

che $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ sono linearmente

indipendenti se l'unica combinazione lineare

nulla deve avere coefficienti tutti nulli:

"

$$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \dots + \alpha_n \underline{u}_n = \underline{0}$$

$$\Downarrow$$

$$\alpha_1 = 0,\ \alpha_2 = 0,\ \dots,\ \alpha_n = 0$$

Si dice che $\underline{u}_1, \dots, \underline{u}_n$ sono linearmente dipendenti se $\exists$ $\alpha_1, \alpha_2, \dots, \alpha_n$ non tutti nulli tali che $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \dots + \alpha_n \underline{u}_n = \underline{0}$

"esiste"

$\forall$

ES. $\underline{e}_1 = (1,0)$, $\underline{e}_2 = (0,1)$

$$\alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 = \underline{0}$$

$$\alpha_1 (1,0) + \alpha_2 (0,1) = \underline{0} = (0,0)$$

$$(\alpha_1, \alpha_2) = (0,0)$$

$$\Downarrow$$

$$\alpha_1 = \alpha_2 = 0$$

$\underline{e}_1, \underline{e}_2, \underline{e}_3$ sono lin. indipendenti

$$\begin{array}{ccc} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \parallel & \parallel & \parallel \\ (1,0,0) & (0,1,0) & (0,0,1) \end{array}$$

<u>Proprietà</u> $\underline{u}_1, \underline{u}_2 \in V^3$ sono linearmente dipendenti se e solo se sono paralleli

Se $\underline{u}_1, \underline{u}_2$ linearmente dipendenti : $\exists\, \alpha_1, \alpha_2 \in \mathbb{R}$ non entrambi nulli tali che

$$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0}$$

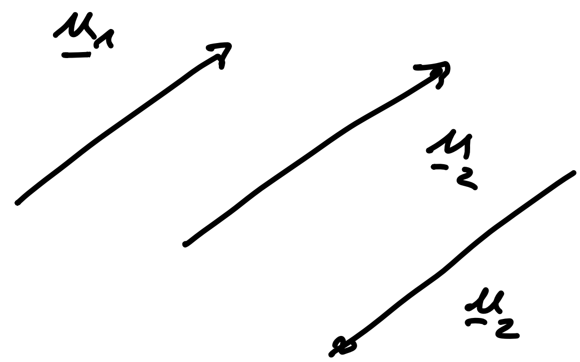
Supponiamo che $\alpha_1 \neq 0$:

$$\alpha_1 \underline{u}_1 = - \alpha_2 \underline{u}_2$$

$\alpha_1 \neq 0$

$$\underline{u}_1 = - \frac{\alpha_2}{\alpha_1} \underline{u}_2 = t \, \underline{u}_2$$

$$t = - \frac{\alpha_2}{\alpha_1}$$



Viceversa, se $\underline{u}_1$ e $\underline{u}_2$ sono paralleli,

$$\underline{u}_1 = t\,\underline{u}_2 \quad \text{con} \quad t \in \mathbb{R}$$

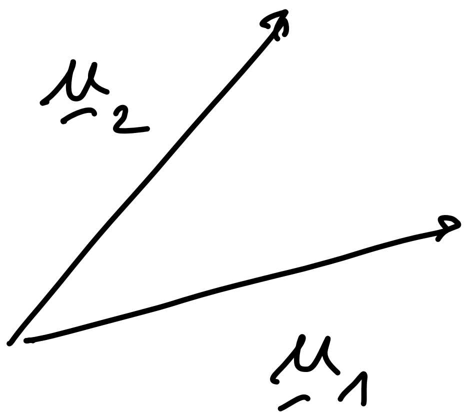
$$1 \cdot \underline{u}_1 + (-t)\,\underline{u}_2 = \underline{0} \qquad \alpha_1 = 1 \neq 0$$

$$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0} \qquad \alpha_2 = -t$$

$\Rightarrow \underline{u}_1, \underline{u}_2$ lin. dipendenti.

<u>OSS.</u> $\underline{u}_1$ e $\underline{u}_2$ sono lin. indipendenti $\Leftrightarrow$ ("se e solo se") non sono paralleli

$\Longleftrightarrow$



Oss. Se $\underline{u}_1 \perp \underline{u}_2$,

$\underline{u}_1$ e $\underline{u}_2$ sono linearmente indipendenti

Def. V spazio vettoriale.

$B = \{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n\} \subseteq V$

Si dice che B è una **base** di V se:

1) $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ sono lin. indipendenti

2) lo spazio vettoriale generato da $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ **coincide** con V:

ossia, $\forall\, \underline{u} \in V \ \exists\, \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$

tali che $\underline{u} = \alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \dots + \alpha_n \underline{u}_n$

ES. V^2 "$\mathbb{R}^2$" $B = \{\underbrace{\underline{e}_1, \underline{e}_2}_{\text{lin. indipendenti}}\}$: **base**

$\underline{u} \in V^2$: $\underline{u} = \alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2$

V^3 $B = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ **base**

$\mathbb{R}^n$

$$\underline{e}_1 = (1, 0, 0, \dots, 0)$$
$$\underline{e}_2 = (0, 1, 0, \dots, 0)$$
$$\vdots$$
$$\underline{e}_n = (\underbrace{0, 0, 0, \dots, 1}_{n})$$

lin. indipendenti

$\forall \underline{u} \in \mathbb{R}^n, \qquad \underline{u} = \alpha_1 \underline{e}_1 + \dots + \alpha_n \underline{e}_n$

ES:

a) $\underline{v} = (1, 5, 7)$ e $\underline{w} = (1, 3, 4)$

lin. indipendenti o lin. dipendenti

$\underline{v} \neq t\,\underline{w}$?

quindi sono indipendenti

b) $\underline{v} = (1, -5, 22)$ $\quad \underline{w} = (0, 0, 0)$

$$0 \cdot \underline{v} + \alpha \cdot \underline{0} = \underline{0}$$

$$\alpha \neq 0$$

c) $\underline{u} = (1, -3, 7)$ $\quad \underline{v} = (2, -1, -1)$

$$\underline{w} = (-4, 2, 2)$$

$$\alpha_1 \underline{u} + \alpha_2 \underline{v} + \alpha_3 \underline{w} = \underline{0}$$

$$(\alpha_1, -3\alpha_1, 7\alpha_1) + (2\alpha_2 - \alpha_2, -\alpha_2) + (-4\alpha_3, 2\alpha_3, 2\alpha_3) = (0,0,0)$$

$$(\alpha_1 + 2\alpha_2 - 4\alpha_3, -3\alpha_1 - \alpha_2 + 2\alpha_3, 7\alpha_1 - \alpha_2 + 2\alpha_3) = (0,0,0)$$

$$\Leftrightarrow \begin{cases} \alpha_1 + 2\alpha_2 - 4\alpha_3 = 0 \\ -3\alpha_1 - \alpha_2 + 2\alpha_3 = 0 \\ 7\alpha_1 - \alpha_2 + 2\alpha_3 = 0 \end{cases}$$

$$\begin{cases} \alpha_1 + 2\alpha_2 - 4\alpha_3 = 0 \\ 3\alpha_1 + \alpha_2 - 2\alpha_3 = 0 \\ 7\alpha_1 - \alpha_2 + 2\alpha_3 = 0 \end{cases}$$

somme

$$\begin{cases} \alpha_1 + 2\alpha_2 - 4\alpha_3 = 0 \\ 3\alpha_1 + \alpha_2 - 2\alpha_3 = 0 \\ 10\alpha_1 = 0 \Leftrightarrow \alpha_1 = 0 \end{cases}$$

$$\begin{cases} \alpha_1 = 0 \\ 2\alpha_2 - 4\alpha_3 = 0 \\ \alpha_2 - 2\alpha_3 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 - 2\alpha_3 = 0 \\ \Leftrightarrow \alpha_2 = 2\alpha_3 \end{cases}$$

α_3 libre

$$\begin{cases} \alpha_1 = 0 \\ \alpha_3 = 1 \\ \alpha_2 = 2 \end{cases}$$

$$\begin{array}{ccc} (0, & 2, & 1) \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array}$$

$$\alpha_1 \underline{u} + \alpha_2 \underline{v} + \alpha_3 \underline{w} = 0$$

<u>lin. dipendenti</u> !