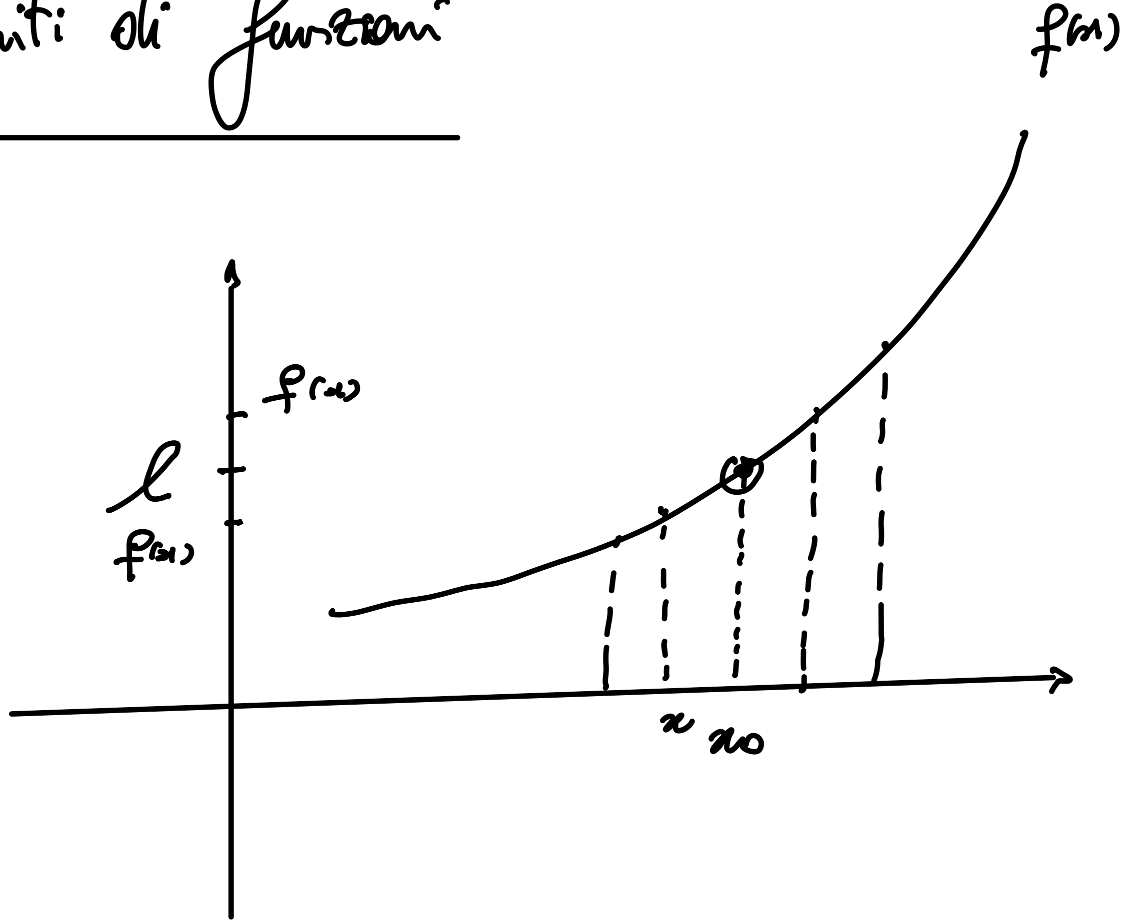
Lezione dell'11/11/2022

Limiti di funzioni

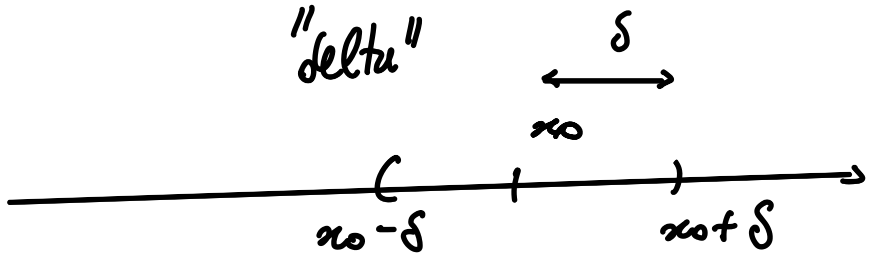


Intorno di un punto

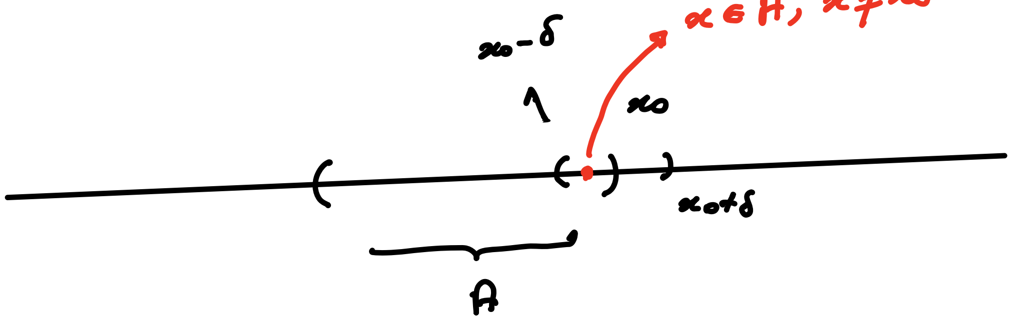
$x_0 \in \mathbb{R}$: un intorno di x_0 è un qualsiasi intervallo del tipo

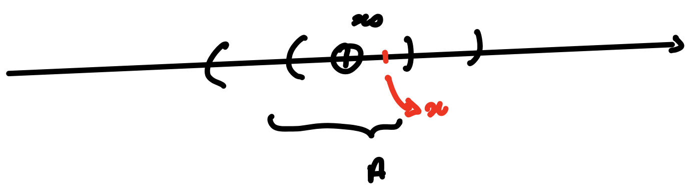
$$]x_0 - \delta \, , x_0 + \delta[$$

"delta"



Def. $A \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$: si dice che x_0 è di accumulazione per A se "in ogni intorno di x_0 cade almeno un punto di A, <u>diverso</u> da x_0"



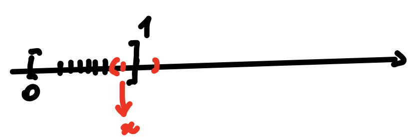


$x_0 \in \mathbb{R}$ è di accumulazione per A

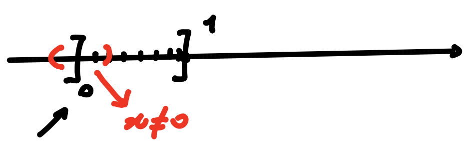
$$\Leftrightarrow \quad \forall \delta > 0 \quad \exists x \in A : 0 < |x - x_0| < \delta$$

$$x_0 - \delta < x < x_0 + \delta$$
$$\Leftrightarrow \quad |x - x_0| < \delta$$

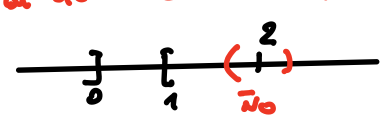
$A = [0,1]$



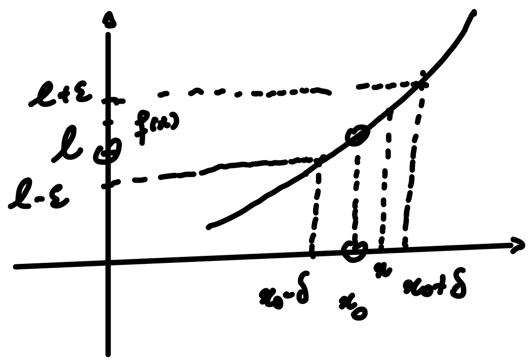
$A =]0,1]$



Punti di accumulazione $= [0,1]$



$[0,1]$



$f: A \subseteq \mathbb{R} \to \mathbb{R}$ x_0 di accumulazione per A

ε "epsilon"

$\varepsilon > 0$ piccolo a piacere

Def Si dice che $f(x)$ tende ad $l \in \mathbb{R}$ per x che tende ad x_0 ($x \to x_0$) e scriveremo

$$\lim_{x \to x_0} f(x) = l \overset{def.}{\iff}$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \text{ t.c. } 0 < |x - x_0| < \delta \quad ?$$

$$\Downarrow$$

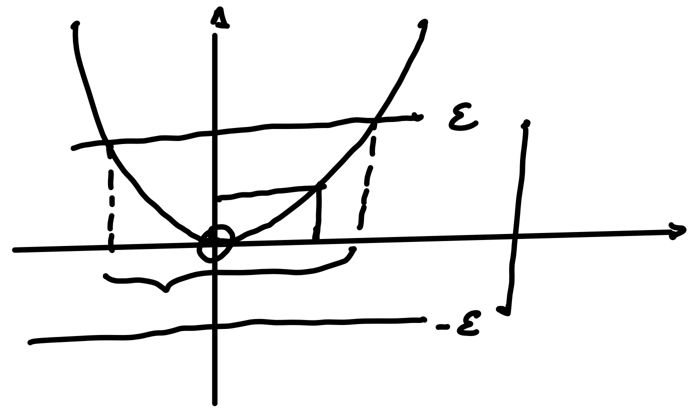
$$|f(x) - l| < \varepsilon$$

$$\iff l - \varepsilon < f(x) < l + \varepsilon$$

ES $f(x) = x^2$ se $x \neq 0$ $\quad \mathbb{R} \setminus \{0\} = D_f$

$y = x^2$?

parabola



$\lim_{x \to 0} f(x) = 0$

Capiamolo dalla definizione (ε, δ)

Vogliamo $\varepsilon > 0$ fissato : allora

$$|f(x) - \ell| = |x^2| = x^2 < \varepsilon \;?$$

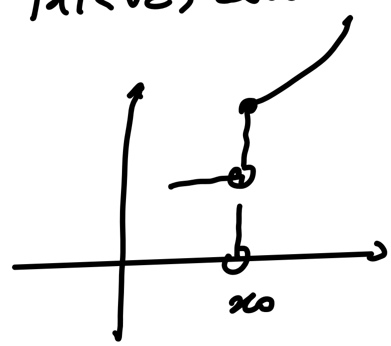
$$\Longleftrightarrow \quad x^2 < \varepsilon \Longleftrightarrow x^2 - \varepsilon < 0 \Longleftrightarrow$$

$$\Longleftrightarrow \quad -\sqrt{\varepsilon} < x < \sqrt{\varepsilon}$$

$\delta = \delta(\varepsilon) = \sqrt{\varepsilon}$: perché quando $|x| < \sqrt{\varepsilon}$, abbiamo

$$|f(x) - \ell| = |x^2| < \varepsilon$$

$\ell = 0$



Teorema di unicità del limite

Se $\exists \lim_{x \to x_0} f(x) = \ell$, ℓ è unico

Dim. Supponiamo che $\exists\, l_1, l_2$ due limiti di

$f(x)$: $\varepsilon > 0 \quad \exists \delta > 0 : \forall x : 0 < |x - x_0| < \delta$

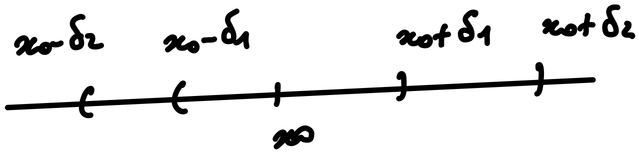
$$\Rightarrow \quad |f(x) - l| < \varepsilon$$

l_1 " $\exists \delta_1 > 0 : \forall x \in A : 0 < |x - x_0| < \delta_1$

$$\Rightarrow \quad |f(x) - l_1| < \varepsilon$$

l_2 " $\exists \delta_2 > 0 : \forall x \in A : 0 < |x - x_0| < \delta_2$

$$\Rightarrow \quad |f(x) - l_2| < \varepsilon$$



$\delta_1 < \delta_2$

$\delta = \min\{\delta_1, \delta_2\}$: $\forall x \in A$ t.c. $0 < |x - x_0| < \delta$ si

ha
$$|f(x) - l_1| < \varepsilon$$
$$|f(x) - l_2| < \varepsilon$$

Allora $|l_1 - l_2| = |(l_1 - f(x)) + (f(x) - l_2)|$

$\leq$ (dis. triangolare) $\leq |\underbrace{f(x) - l_1}_{\hat{\varepsilon}}| + |\underbrace{f(x) - l_2}_{\hat{\varepsilon}}|$

$< 2\varepsilon \qquad \forall \varepsilon > 0$

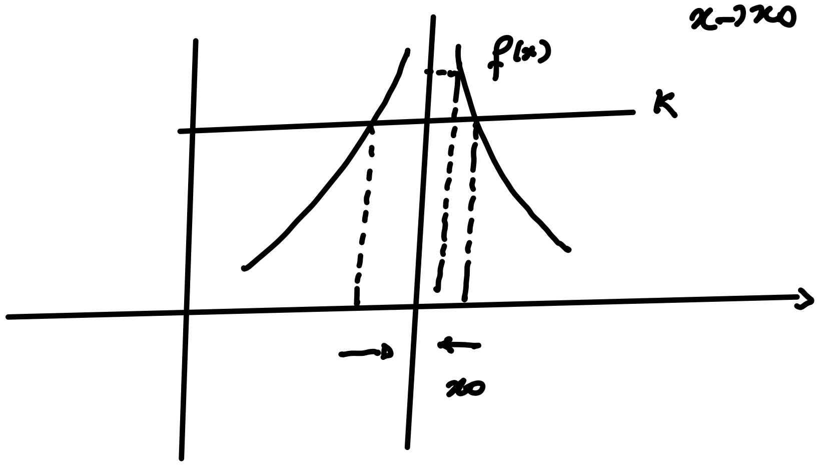
$$\Rightarrow \quad |l_1 - l_2| = 0 \quad \Rightarrow \quad \underline{\underline{l_1 = l_2}}$$

---

1) $l = +\infty$

$$\lim_{x \to x_0} f(x) = +\infty$$

"$f(x)$ diverge positivamente per $x \to x_0$"

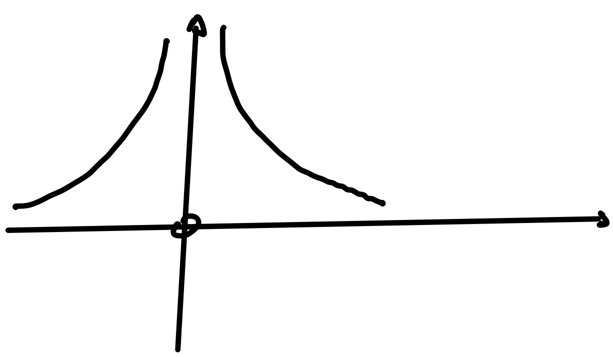


$$\lim_{x \to x_0} f(x) = +\infty \iff \forall K > 0 \ \exists \delta > 0 : \forall x \in A, \ 0 < |x - x_0| < \delta \Rightarrow f(x) > K$$

<u>ES.</u> $f(x) = 1/|x|$ $\underline{\underline{\text{pari}}}$

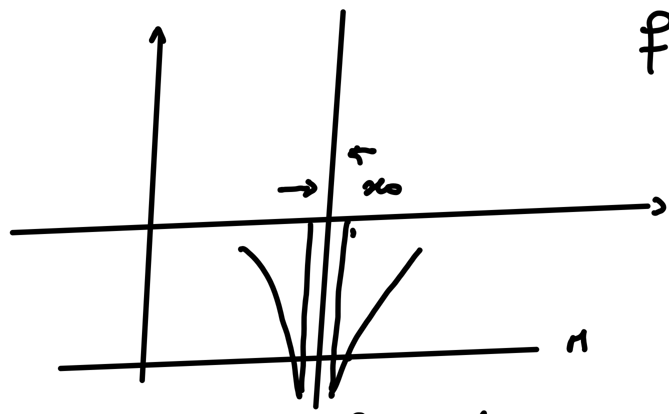
$x = 0$ asint. verticale

$$\lim_{x \to 0} f(x) = +\infty$$



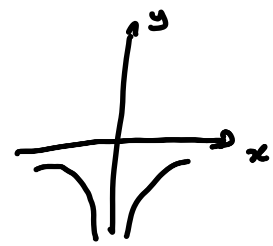
f(x) diverge negativamente in x_0

$$\lim_{x \to x_0} f(x) = -\infty \iff \forall K>0 \ \exists \delta>0 : \forall x \in A \mid 0<|x-x_0|<\delta \Downarrow f(x) < -K$$



$f(x) < M$

$$f(x) = -\frac{1}{|x|} \qquad \lim_{x \to 0} f(x) = -\infty$$

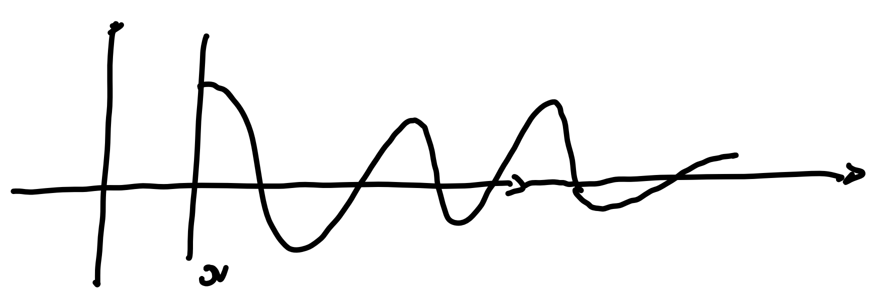


$+\infty \quad -\infty$

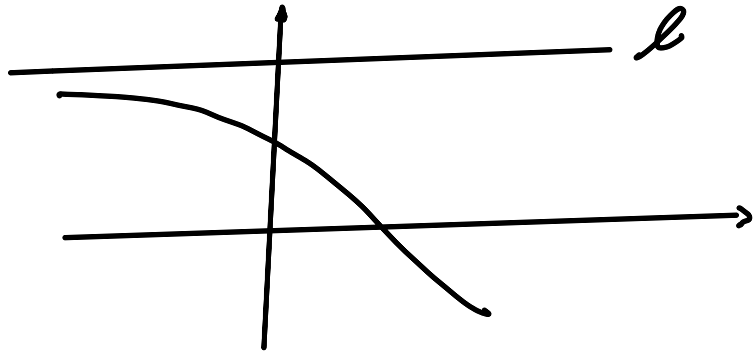
$f: A \subseteq \mathbb{R} \to \mathbb{R}$

A non limitato superiormente o inferiormente

$A =]a, +\infty[$



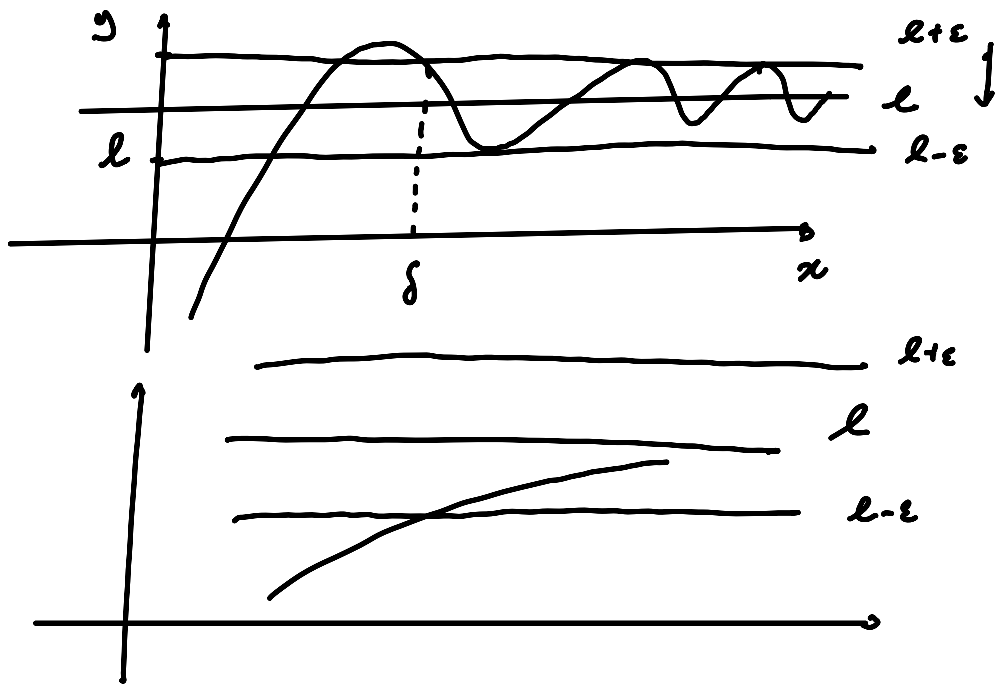
$A =]-\infty, a[$



•) A non sia limitato superiormente $]a, +\infty[$

Diremo che $\lim_{x \to +\infty} f(x) = l$, $l \in \mathbb{R}$

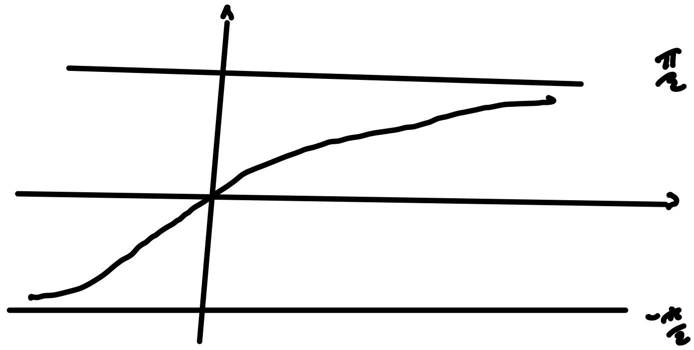
"f(x) converge ad l per $x \to +\infty$" $\overset{def.}{\iff}$



$\iff \forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \mid x > \delta \Rightarrow |f(x) - l| < \varepsilon$

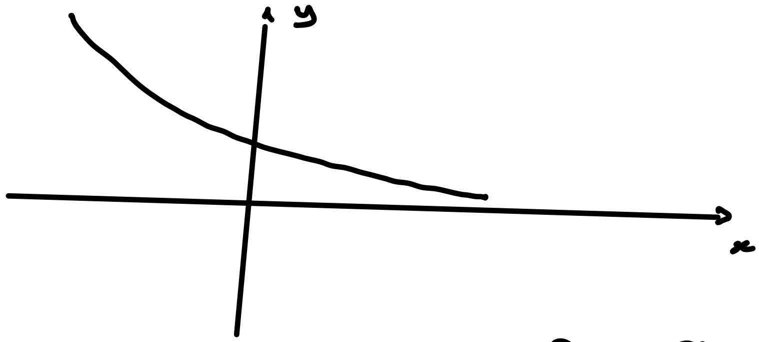
$f(x) = \mathrm{arctg}\, x$

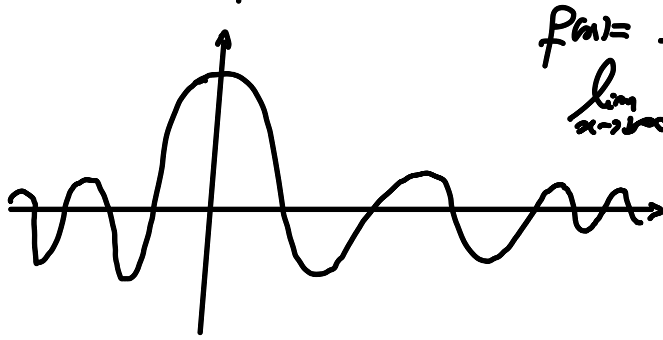
$\lim_{x \to +\infty} f(x) = \frac{\pi}{2}$



$f(x) = \left(\frac{1}{2}\right)^x$

$\lim_{x \to +\infty} f(x) = 0$

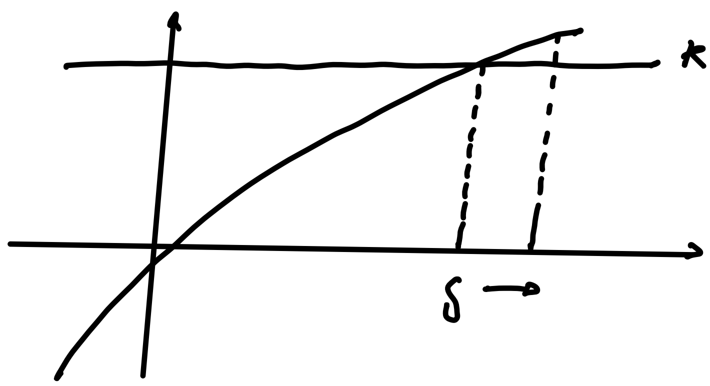




$f(x) = \frac{\sin x}{x}$

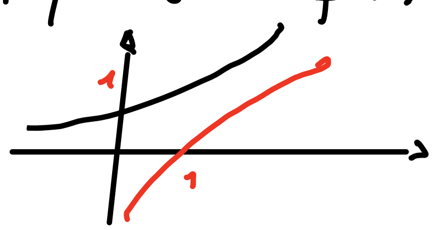
$\lim_{x \to +\infty} f(x) = 0$

$\lim_{x \to +\infty} \underbrace{f(x) = +\infty}_{\text{div. pos. a } +\infty}$

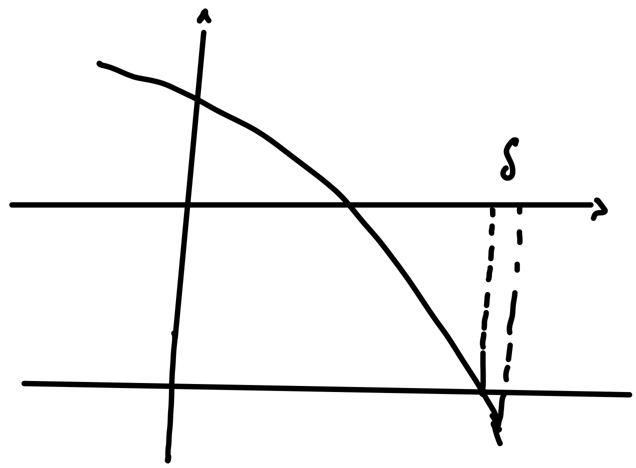


$\Longleftrightarrow \quad \forall K > 0 \ \exists \delta > 0 : \ \forall x \in A \mid x > \delta \Rightarrow f(x) > K$

ES. $f(x) = e^x$, $\log x$



$$\lim_{x \to +\infty} f(x) = -\infty$$

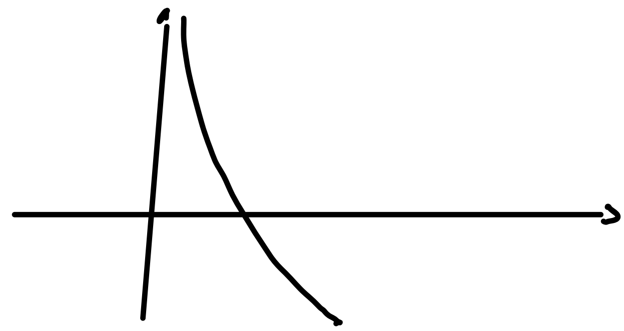


$\iff$

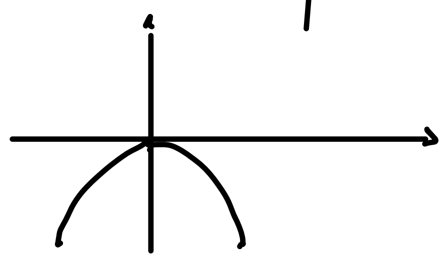
$\forall K > 0 \ \exists \delta > 0 : \forall x \in A \mid x > \delta$

$\Rightarrow f(x) < -K$

$f(x) = \log_{\frac{1}{2}} x$



$f(x) = -x^2$

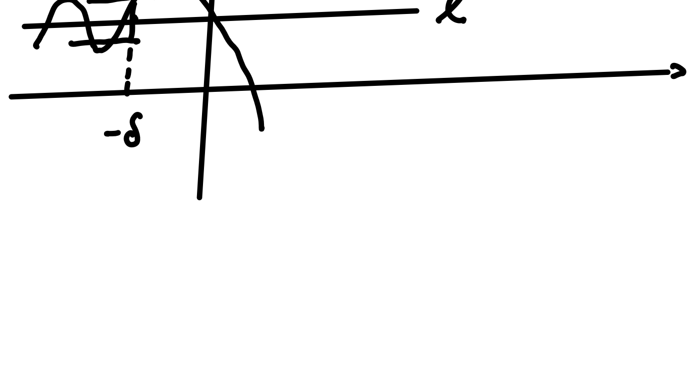


---

se A non limitato inferiormente (e.g. $A =]-\infty, a[$)

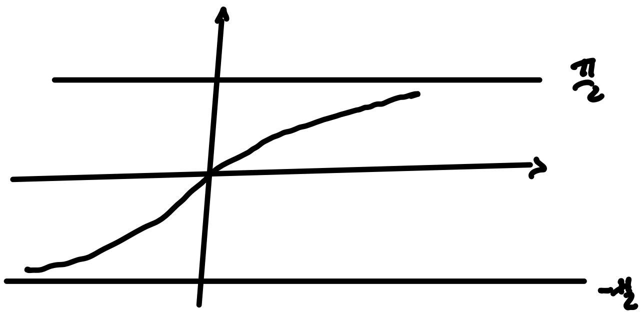
$\lim_{x \to -\infty} f(x) = \ell \in \mathbb{R} \iff \forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \mid x < -\delta$

$\Rightarrow |f(x) - \ell| < \varepsilon$

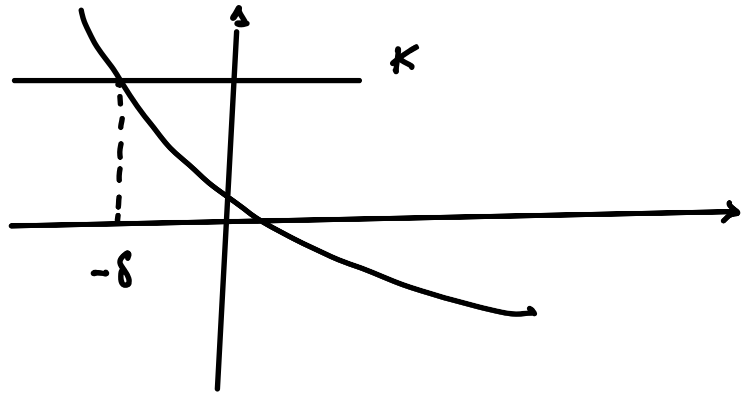


$f(x) = \operatorname{arctg} x$

$\lim\limits_{x\to-\infty} \operatorname{arctg} x = -\frac{\pi}{2}$

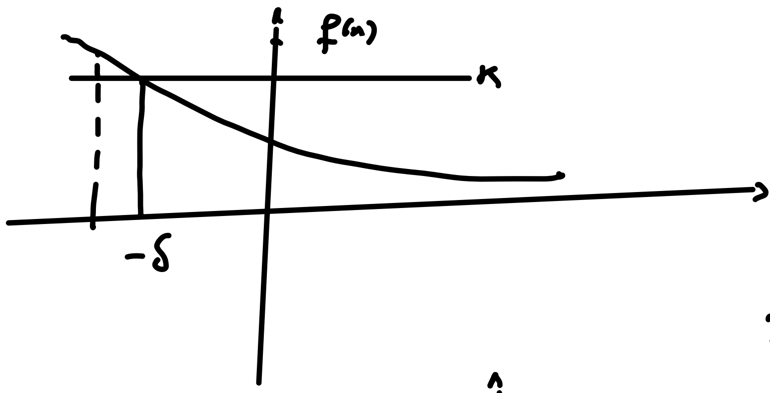


$\lim\limits_{x\to-\infty} f(x) = +\infty$



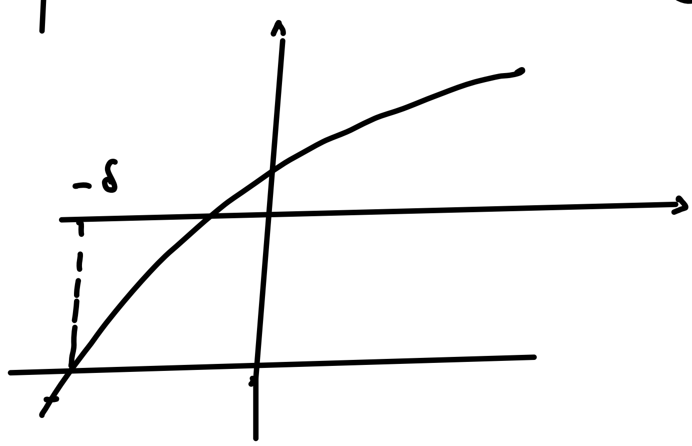
$\Leftrightarrow$

$\forall K>0\ \exists\, \delta>0 : \forall x \in A \mid x<-\delta \Rightarrow f(x)>K$



$f(x) = \left(\frac{1}{2}\right)^x$

$\lim\limits_{x\to-\infty} f(x) = -\infty$



$\Leftrightarrow \forall K>0\ \exists\, \delta>0 : \forall x \in A \mid x<-\delta \Rightarrow f(x)<-K$

Def. (Generale di limite)

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

$x_0 \in \overline{\mathbb{R}}$

$f: A \subseteq \mathbb{R} \to \mathbb{R}$, x_0 di accumulazione per A:

diciamo che $f(x)$ tende ad $\ell \in \mathbb{R} \cup \{\pm\infty\} = \overline{\mathbb{R}}$

$\iff \forall V$ intorno di ℓ $\exists U$ int. di x_0 tale che

$$\forall x \in U \cap A,\ x \neq x_0 \Rightarrow f(x) \in V$$

Se il limite esiste, questo è unico!

Teorema del confronto

$f, g, h: A \subseteq \mathbb{R} \to \mathbb{R}$; x_0 di accumulazione per A

e si suppenga che $f(x) \leq h(x) \leq g(x)$ $\forall x \in A$.

Allora, se $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = \ell \in \mathbb{R}$

<u>si ha</u>

$$\lim_{x \to x_0} h(x) = l$$

<u>Inoltre</u>, se $\quad f(x) \leq g(x)$

e $\quad \lim_{x \to x_0} f(x) = +\infty$, allora

$$\lim_{x \to x_0} g(x) = +\infty$$

e $\quad \lim_{x \to x_0} g(x) = -\infty$, allora

$$\lim_{x \to x_0} f(x) = -\infty$$

<u>Dim.</u> $\quad f(x) \leq h(x) \leq g(x)$

Per la def di limite, $\varepsilon > 0$ fissato, $\exists \delta_1 > 0$ tale che $\quad l - \varepsilon < f(x) < l + \varepsilon \quad \forall x \in A : 0 < |x - x_0| < \delta_1$

$\exists \delta_2 > 0$ tale che $\quad l - \varepsilon < g(x) < l + \varepsilon, \quad \forall x \in A : 0 < |x - x_0| < \delta_2$

Allora se $\delta = \min\{\delta_1, \delta_2\}$ e $0 < |x - x_0| < \delta$

si ha $\quad \ell - \varepsilon < f(x) \leq h(x) \leq g(x) < \ell + \varepsilon$

$$\Downarrow$$

$$\ell - \varepsilon < h(x) < \ell + \varepsilon$$

$$\Rightarrow \quad \lim_{x \to x_0} h(x) = \ell$$

c.v.d.

Limite sinistro, destro

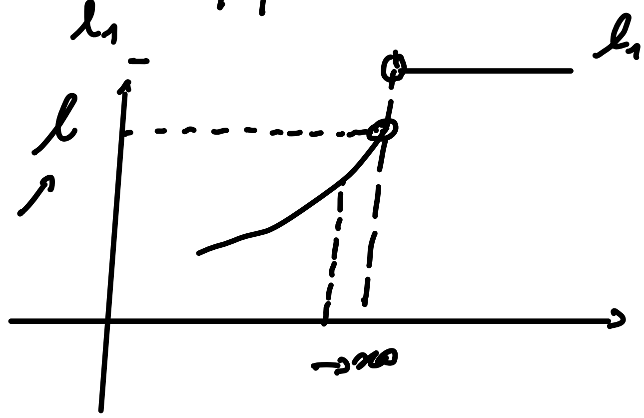
$f = f(x)$, $\quad f : A \subseteq \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ di accumulazione per A.

$$\lim_{x \to x_0^-} f(x) = \ell \in \mathbb{R} \overset{\text{def.}}{\iff}$$

$$\iff \forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \ | \ x_0 - \delta < x < x_0$$

$$\Downarrow$$

$$|f(x) - l| < \varepsilon$$

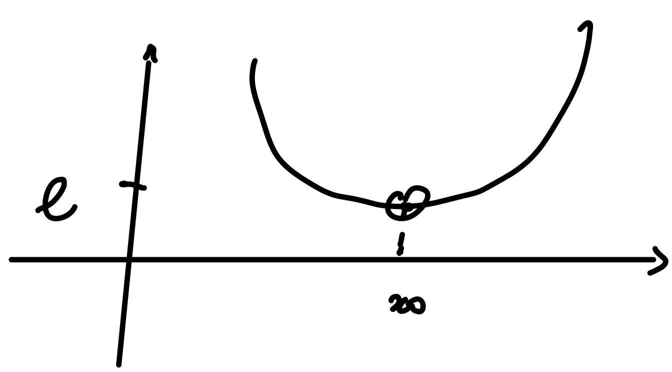


$$\lim_{x \to x_0^+} f(x) = l \iff$$

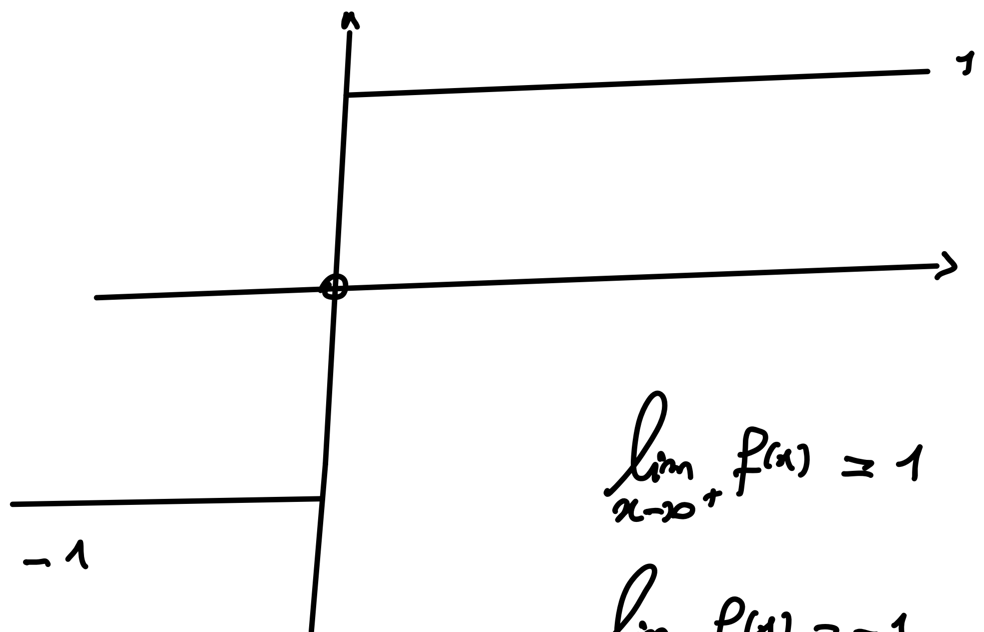
$$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \mid x_0 < x < x_0 + \delta \Rightarrow |f(x) - l| < \varepsilon$$

<u>OSS.</u> $\lim_{x \to x_0} f(x) = l$

$$\iff \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = l$$



$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$$



$$\lim_{x\to 0^+} f(x) = 1$$

$$\lim_{x\to 0^-} f(x) = -1$$

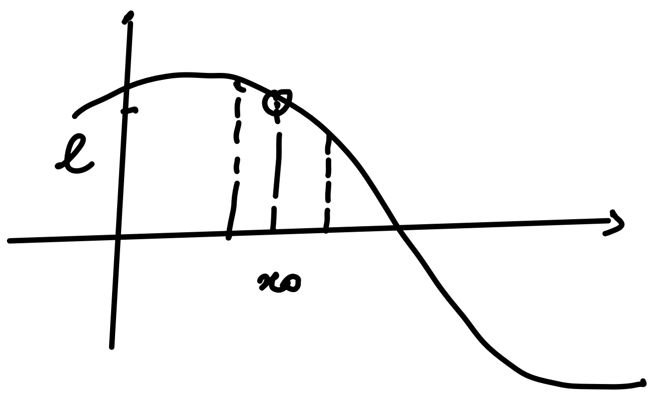
$$\lim_{x\to 0} \frac{|x|}{x} \quad \underline{\text{non esiste}}$$

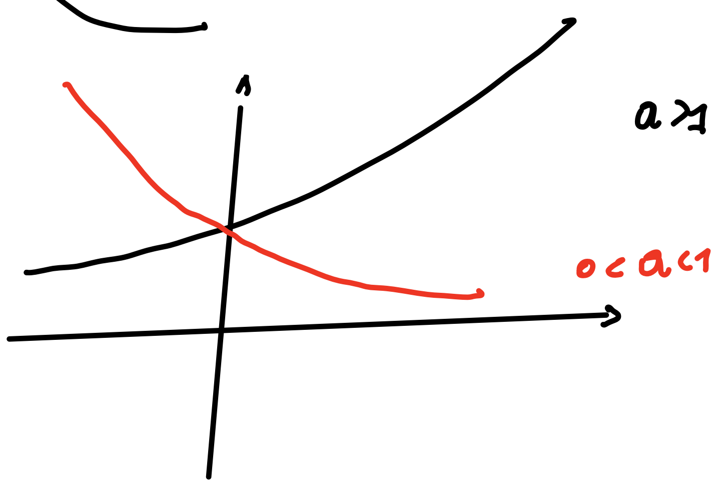
Teorema della permanenza del segno

$f: A \subseteq \mathbb{R} \to \mathbb{R}$, $x_0 \in \overline{\mathbb{R}}$ di accumulazione per A

Se $\lim_{x \to x_0} f(x) = \ell > 0$ (<0), $\exists\, U$ int. di x_0 tale che

$f(x) > 0$ (<0), $\forall x \in U \cap A$, $x \neq x_0$





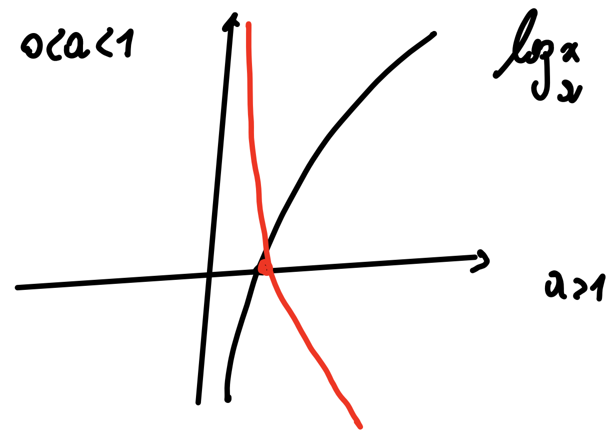
Limiti notevoli

$$\lim_{x \to +\infty} a^x = \begin{cases} +\infty & \text{se } a > 1 \\ 0 & \text{se } 0 < a < 1 \end{cases}$$

$$\lim_{x\to -\infty} a^x = \begin{cases} 0 & a>1 \\ +\infty & 0<a<1 \end{cases}$$

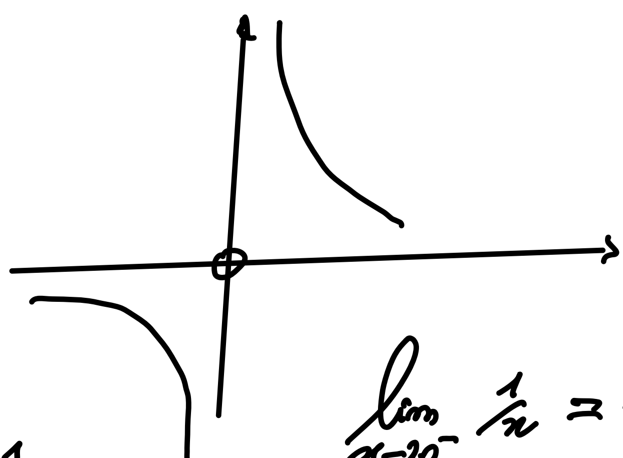
$$\lim_{x\to +\infty} \log_a x = \begin{cases} +\infty & \text{se } a>1 \\ -\infty & \text{se } 0<a<1 \end{cases}$$

$\underline{x>0}$



$$\lim_{x\to 0^+} \log_a x = \begin{cases} -\infty & \text{se } a>1 \\ +\infty & \text{se } 0<a<1 \end{cases}$$

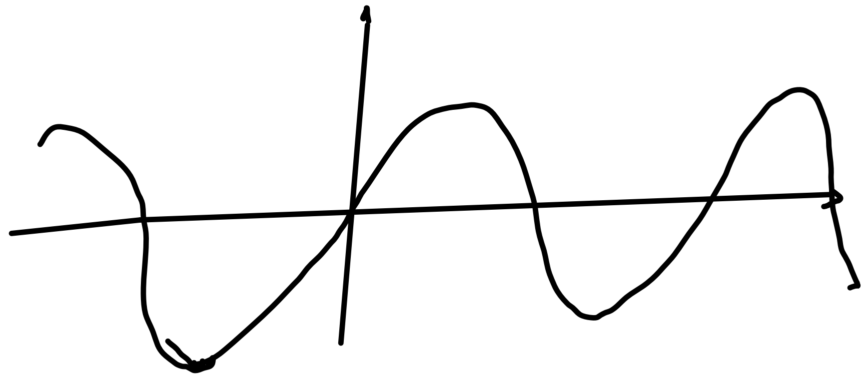
$f(x) = 1/x \quad , \forall x \neq 0$



$$\lim_{x\to 0^+} 1/x = +\infty$$

$\lim_{x\to 0} 1/x$ $\underline{\underline{\text{non}}}$ esiste

$$\lim_{x\to 0^-} 1/x = -\infty$$



$\lim_{x\to\pm\infty} \sin x$, $\cos x$ non esiste

Fondamentale $\qquad \lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^{x} = e$

$1^{\pm\infty}$

e Numero di Nepero

$2 < e < 3$

Operazioni con i limiti di funzioni

1) $\lim_{x\to x_0}\left(f(x)\pm g(x)\right) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)$

$+\infty \quad -\infty$ forma indeterminata

$$\lim_{x\to x_0} (f(x)\cdot g(x)) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$$

$0 \cdot \pm\infty$ forma indeterminata

$f(x) \to 0$ e $g(x) \to \pm\infty$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} \qquad \frac{0}{0} \quad \frac{\pm\infty}{\pm\infty}$$

$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$ $\left(\frac{\infty}{\infty}\right)$ forme indeterminate

<u>Limiti di funzioni composte</u>

$f: A \to f(A) \subseteq B$

$g: B \to \mathbb{R}$

$g \circ f: A \to \mathbb{R}$

$f = f(x)$, $g = g(y)$

$f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$

$f(A) \subseteq B$; $x_0 \in \overline{\mathbb{R}}$ di accumulazione per A

e $\lim_{x\to x_0} f(x) = y_0 \in \overline{\mathbb{R}}$.

Allora, se y_0 è di accumulazione per B e

$f(x) \neq y_0$ in $]x_0-\delta, x_0+\delta[$

e $\lim\limits_{y\to y_0} g(y) = \ell \in \overline{\mathbb{R}}$

Si ha che

$$\lim_{x\to x_0} g(\underbrace{f(x)}) \overset{\text{limite per sostituzione}}{=} \lim_{y\to y_0} g(y) = \ell$$

$y = f(x)$

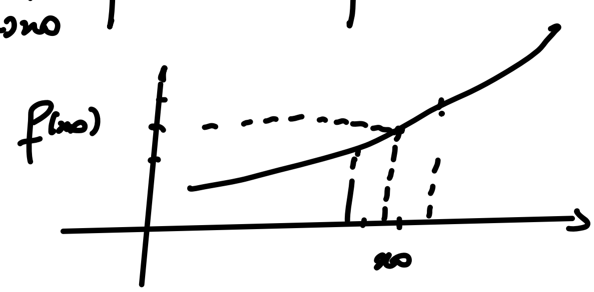
$x\to x_0 \Rightarrow y \to y_0$

Funzioni continue

$f: A\subseteq \mathbb{R}\to\mathbb{R}$, $x_0\in A$ di accumulazione per A:

diremo che f è continua in x_0 se

$$\lim_{x\to x_0} f(x) = f(x_0)$$



Quindi f è continua in x_0

$$\Longleftrightarrow \left[\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \mid |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \right]$$

Se $x_0 \in A$ è isolato, supponiamo che f è continua in x_0 per convenzione.

<u>Def.</u> f continua in A se è continua in ogni punto di A.

<u>OSS.</u> Somme, differenze, prodotti e rapporti di funzioni continue sono funzioni continue

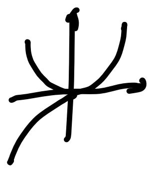
$$\frac{f(x)}{g(x)} \qquad g(x) \neq 0$$

$$f : A \to \mathbb{R} \ \underset{x_0}{\text{continua}} \quad e \quad g : B \to \mathbb{R} \ \underset{y}{\text{continua}}$$

$$\Downarrow$$

$$g \circ f : A \to \mathbb{R} \ \text{è continua}$$

Funzioni continue (esempi)



$f(x) = x^\alpha$, $\alpha \in \mathbb{R}$

$f(x) = x^n$, $n \in \mathbb{N}$, $\forall x \in \mathbb{R}$

$f(x) = a^x$, $a \neq 1$, $a > 0$
$\forall x \in \mathbb{R}$

$f(x) = \log_a x$ $\forall x > 0$

$f(x) = \arcsin x$, $\forall x \in [-1, 1]$
$= \arccos x$ " "

$f(x) = \sin x$, $f(x) = \cos x$ $\forall x \in \mathbb{R}$

$f(x) = \operatorname{tg} x$

$f(x) = \operatorname{arctg} x$, $\forall x \in \mathbb{R}$

$f(x) = |x|$