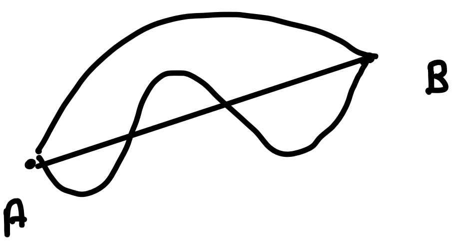
Lezione del 29/11/2022

Massimi e minimi relativi
(Estremi relativi)

__Ottica geometrica__ (Principio di Fermat): il percorso seguito da un raggio luminoso per collegare due punti A e B è quello che richiede il minor tempo di percorrenza



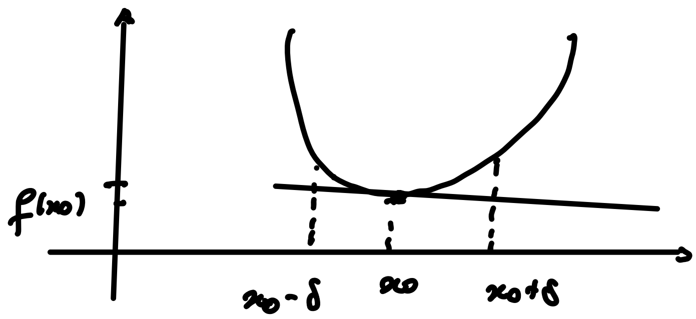
- Economia elementare p $C(p)$

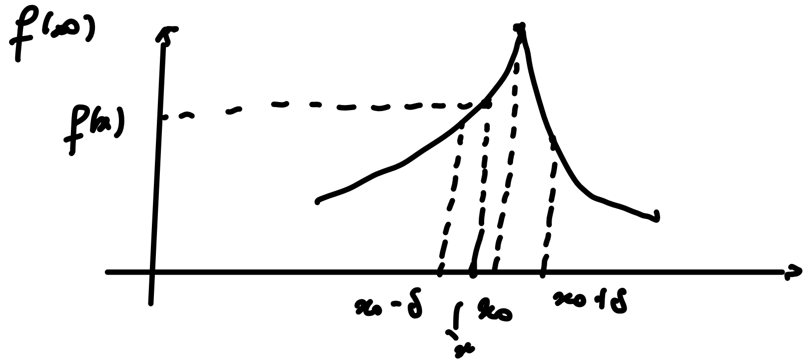
$$\frac{C(p)}{p} = \text{costo di produzione medio per unità di prodotto}$$

__Def.__ $f : I \subseteq \mathbb{R} \to \mathbb{R}$, $x_0 \in I$: si dice che x_0 è di minimo (risp. massimo) relativo per f se

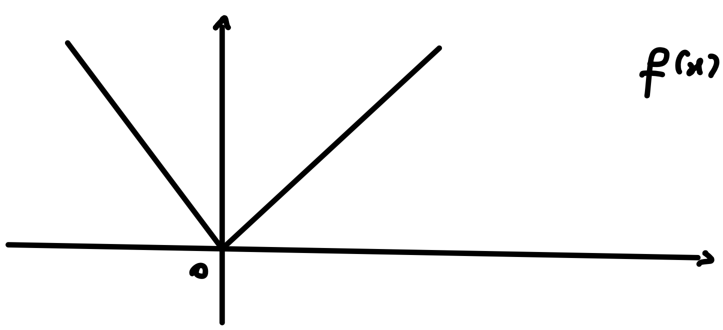
$$\exists\, \delta > 0 \text{ tale che } \underset{(\text{risp. } \leq)}{f(x) \geq f(x_0)}, \quad \forall\, \underbrace{x \in I : |x - x_0| < \delta}$$

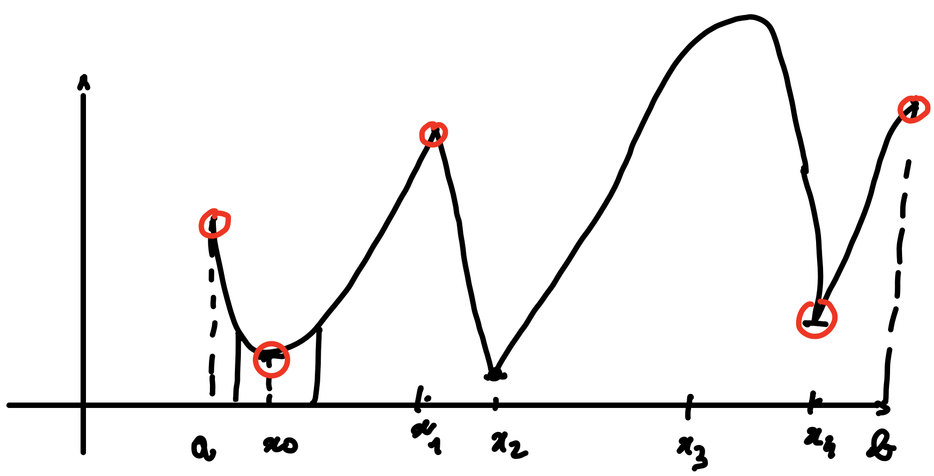
$\forall x \in I \cap \left]x_0-\delta, x_0+\delta\right[$





$f(x) = |x|$





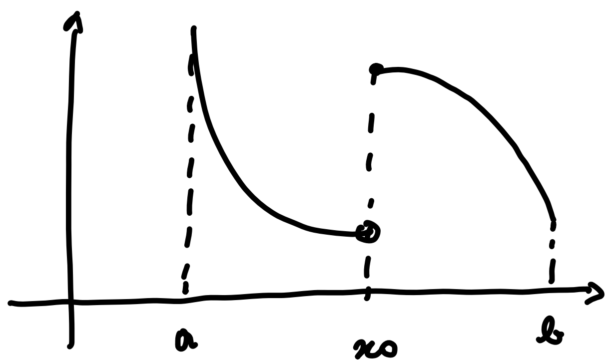
x_0, x_2, x_4 punti di minimo relativo

a, x_1, x_3, b " massimo "

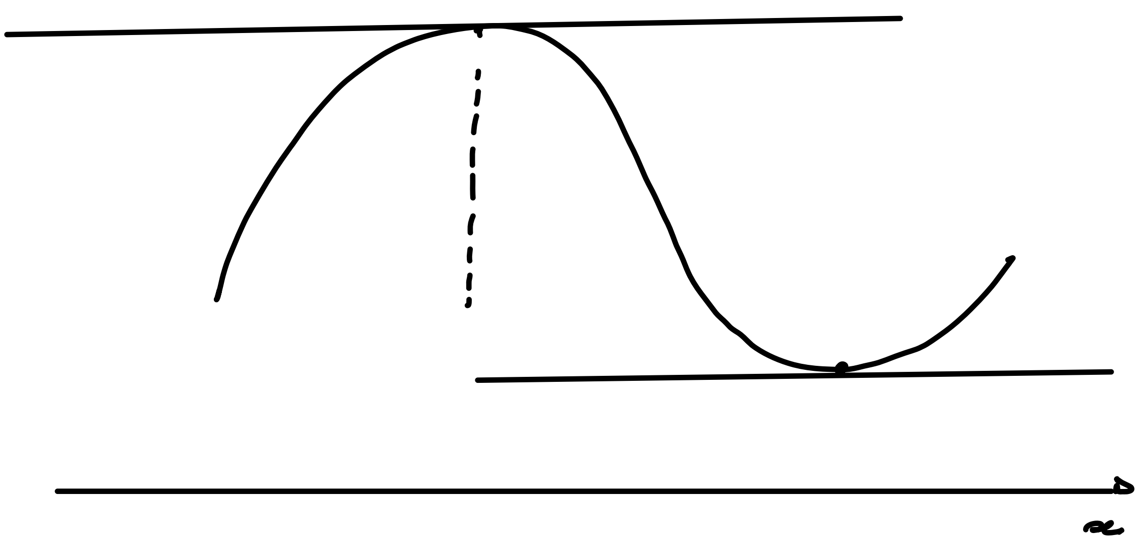
x_2 minimo assoluto

x_3 massimo assoluto

Punti di minimo (risp. massimo) assoluto $\nLeftarrow$ $\Rightarrow$ punti di minimo (risp. massimo) relativo

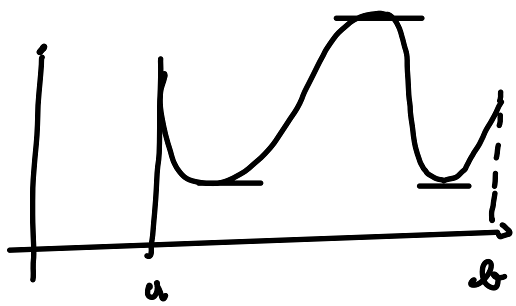


$$y = f(x_0) + f'(x_0)(x - x_0)$$

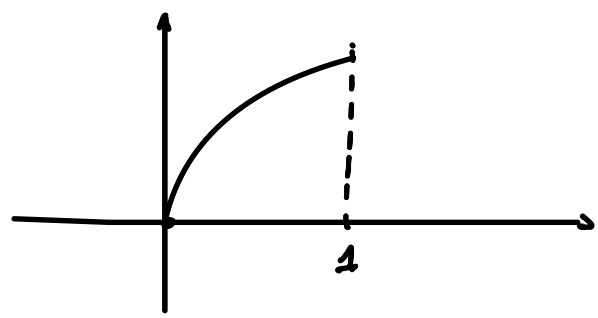


Teorema di Fermat Sia $f: I \to \mathbb{R}$ una funzione derivabile in un punto x_0 interno ad I. Allora, se x_0 è un estremo relativo per f in I, si ha

$$f'(x_0) = 0$$

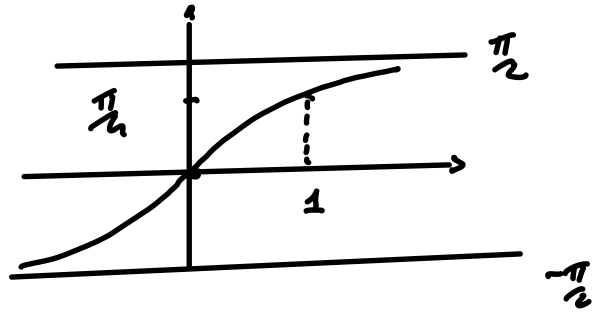


$f'(a) = -\infty$



$f(x) = \sqrt{x}$

$f'(0) = +\infty$



$f(x) = \operatorname{arctg} x \quad , x \in [0,1]$

$\operatorname{arctg} 1 = \frac{\pi}{4}$

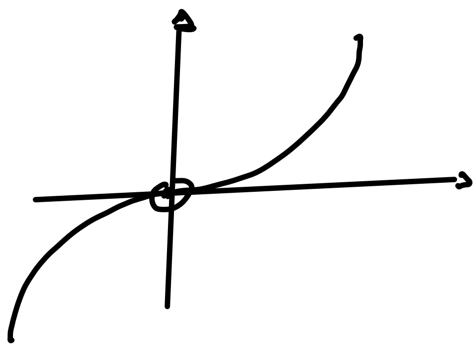
$f'(x) = \dfrac{1}{1+x^2}$ $\qquad f'(0) = 1 \neq 0$

Dal teorema di Fermat si ha:

x_0 interno ad I, estremo locale (massimo o minimo)

$$\Downarrow \quad \Uparrow ? \quad \underline{\text{No}}$$

$$f'(x_0) = 0$$



ES. $f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$

$x_0 = 0$ né di minimo, né di massimo!

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x_0 tale che $f'(x_0) = 0$: x_0 punto <u>critico</u>

o punto <u>stazionario</u>

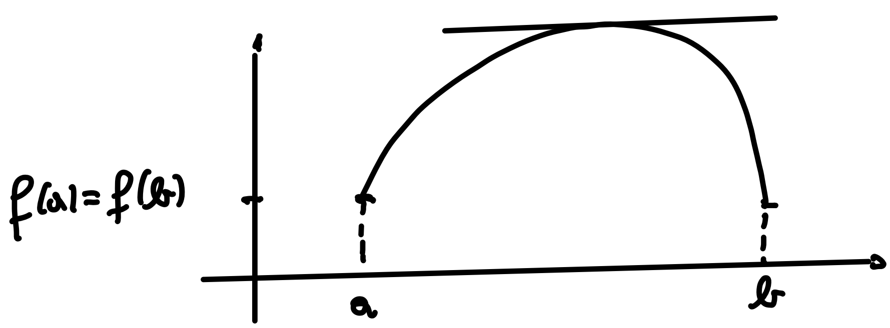
Teorema di Rolle

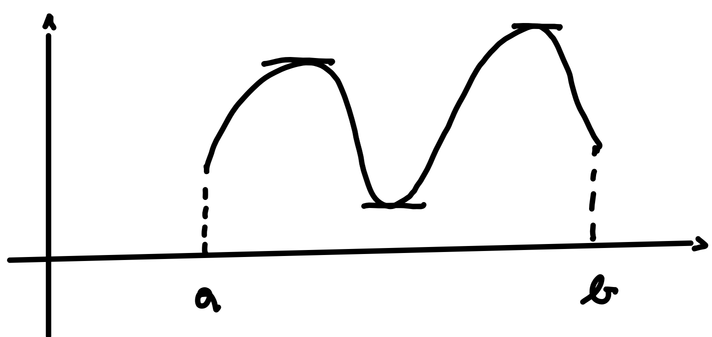
$f : [a,b] \to \mathbb{R}$ continua in $[a,b]$ e derivabile in $]a,b[$. Supponiamo che

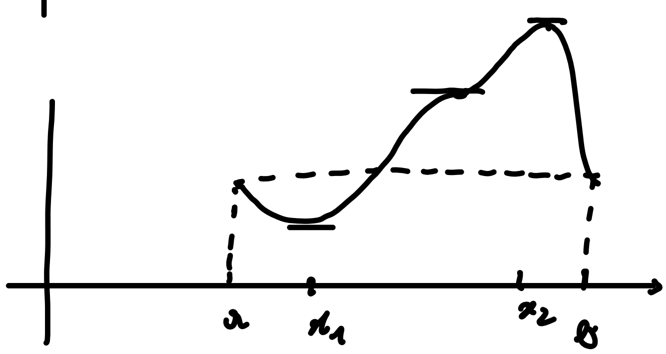
$$f(a) = f(b).$$

Allora esiste un punto $c \in]a,b[$ tale che

$$f'(c) = 0$$







min. ass.

Dim. Per il teorema di Weierstrass: $\exists\, x_1, x_2 \in [a,b]$

max assoluto

tale che $f(x_1) \leq f(x) \leq f(x_2)$
$\forall x \in [a,b]$

Se x_1 o x_2 è interno ad $[a,b]$ si ha dal teorema di Fermat, $f'(x_1)=0$ opp. $f'(x_2)=0$

$c=x_1$, $c=x_2$

Supponiamo che $x_1<x_2$ e che $x_1=a$ e $x_2=b$

$\Downarrow$

$$f(a) = f(x_1) \leq \underline{\underline{f(x)}} \leq f(x_2) = f(b) = f(a)$$

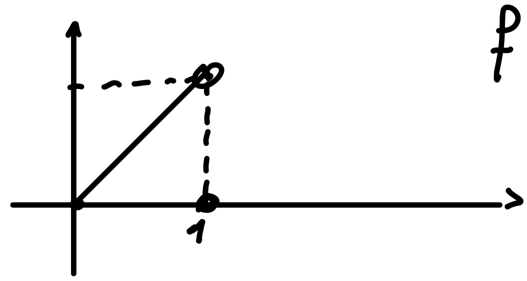
$$f(x) \geq f(a), \ \leq f(a) \Rightarrow \underline{f(x)=f(a) \ \forall x}$$

$\Rightarrow$ $f(x)$ è costante $\Rightarrow$ $f'(x)=0 \ \forall x$

<u>OSS.</u> La continuità agli estremi è indispensabile.

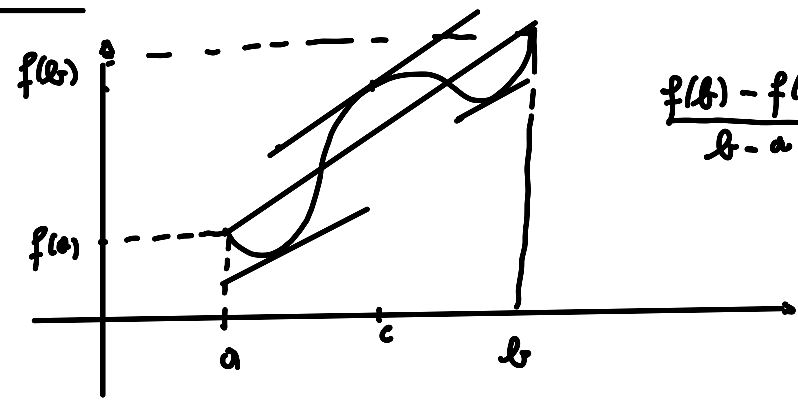
Ad esempio: $f(x)=\begin{cases} x & \text{se } x\in[0,1[\\ 0 & \text{se } x=1 \end{cases}$

$f(0)=f(1)=0$



$f'(x)=1$
$\underline{\underline{\forall x\in\,]0,1[}}$

Teorema di Lagrange



$$\frac{f(b)-f(a)}{b-a}$$

$f:[a,b] \longrightarrow \mathbb{R}$: f continua in $[a,b]$

f derivabile in $]a,b[$.

Allora $\exists\, c \in]a,b[$ tale che

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Dim.

$$g(x) = f(x) - \Big[\underbrace{f(a) + \frac{f(b)-f(a)}{b-a}(x-a)}_{=y}\Big]$$

- g è continua in $[a,b]$;
- g è derivabile in $]a,b[$;

$$g(a) = \overbrace{f(a) - \Big[f(a)}^{=0} + \frac{f(b)-f(a)}{b-a}\underbrace{(a-a)}_{=0}\Big] = 0$$

$$g(b) = f(b) - \left[f(a) + \frac{f(b)-f(a)}{\cancel{b-a}} \cancel{(b-a)} \right]$$

$$= \cancel{f(b)} - \cancel{f(a)} - \cancel{f(b)} + \cancel{f(a)} = 0$$

- $g(a) = g(b) = 0$

$$\Downarrow$$

da Rolle, $\exists\ c \in]a,b[$ tale che

$$g'(c) = 0$$

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a};$$

<u>allora</u> $g'(c) = 0 \implies$

$$f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\implies f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Criterio di monotonia

Sia f continua in $[a,b]$, derivabile in $]a,b[$. Allora:

(a) $f'(x) \geq 0 \quad \forall x \in]a,b[\iff$
$\iff f$ è crescente in $[a,b]$

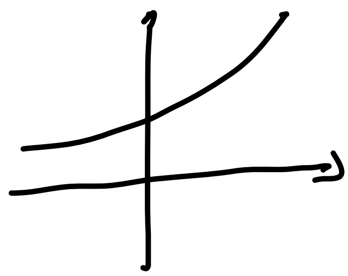
(b) $f'(x) \leq 0 \quad \forall x \in]a,b[\iff$
$\iff f$ è decrescente in $[a,b]$.

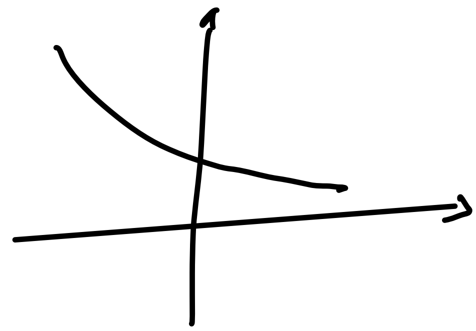
• ES.
$$f(x) = e^x$$
$$f'(x) = e^x > 0$$
$\Rightarrow$ l'esponenziale è crescente (strett.)

$f(x) = a^x$ $\quad$ $f'(x) = a^x \log a$

$a > 1$ $\quad$ $f'(x) > 0$

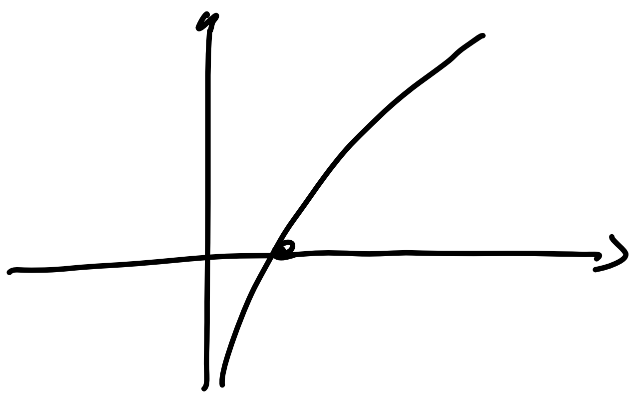
$0 < a < 1$ $\quad$ $f'(x) < 0$



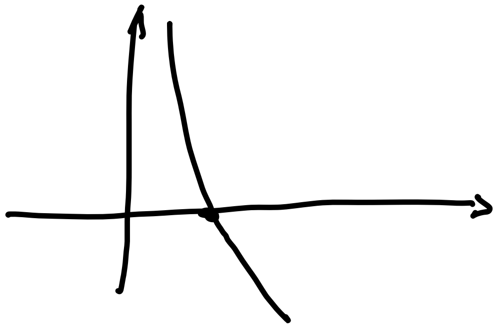


$f(x) = \log x$: $\quad$ $f'(x) = \frac{1}{x} > 0$

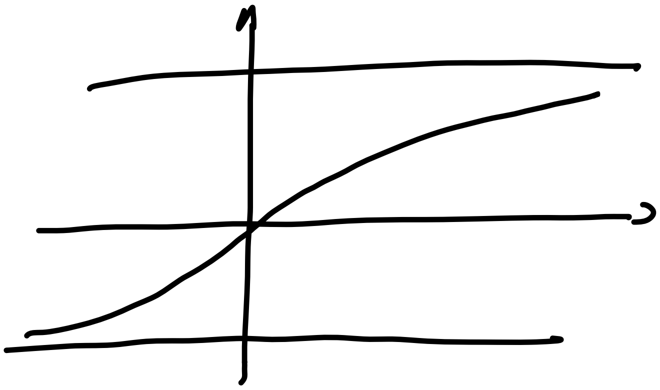
$x > 0$



$f(x) = \log_a x$ $\quad$ $f'(x) = \frac{1}{x} \log_a e$

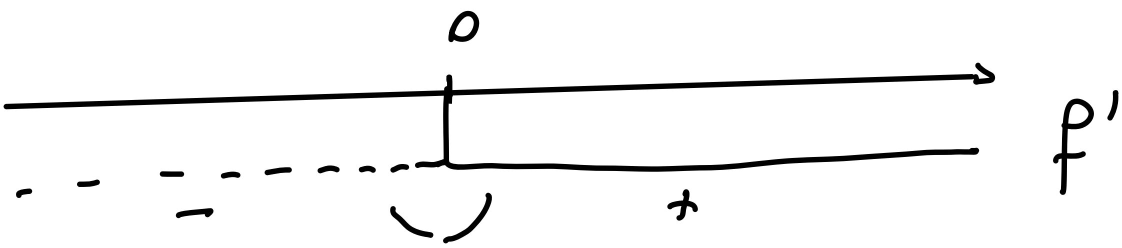


$f(x) = \arctan x \quad : \quad f'(x) = \frac{1}{1+x^2} \geq 0$

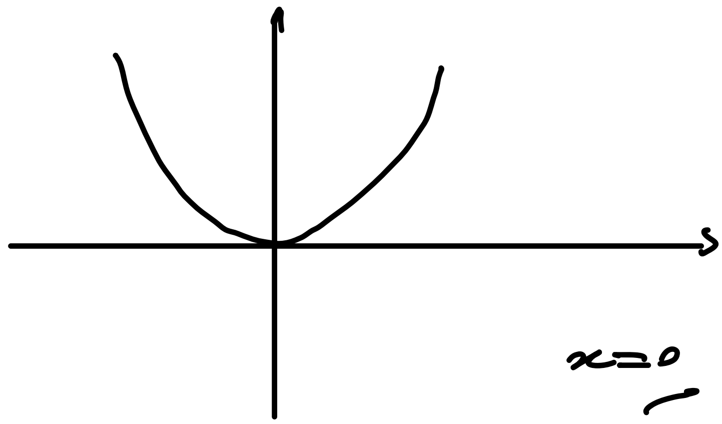


$f(x) = x^2 \quad : \quad f'(x) = 2x \geq 0$

$\iff x \geq 0$



$x_0 = 0$ p.t. di minimo

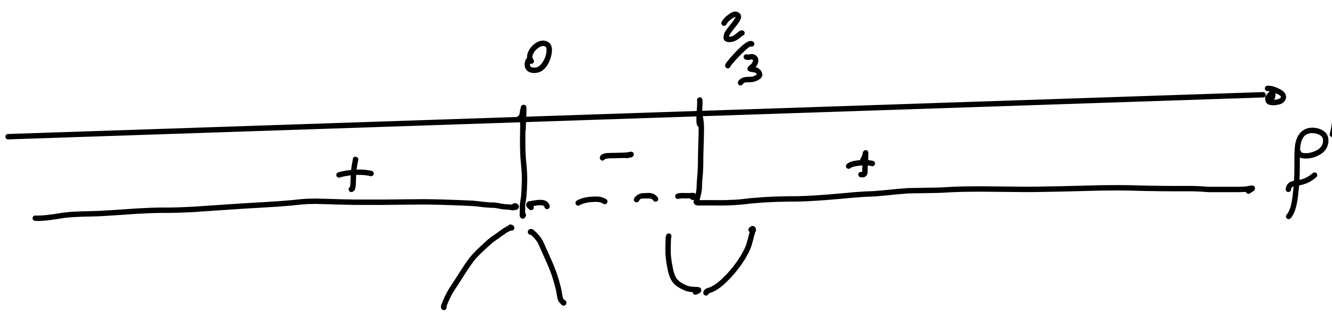


$x=0 \quad x=1$

$$f(x) = x^3 - x^2 = 0 \qquad x^2(x-1)=0$$

$$f'(x) = \underbrace{3x^2 - 2x} \geq 0 \iff x \leq 0,\ x \geq \frac{2}{3}$$

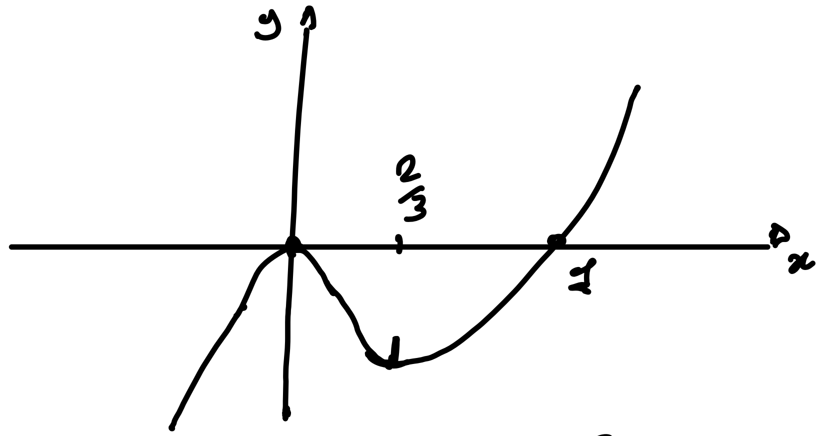
$$x_0 = 0,\ x_1 = \frac{2}{3}$$



$x_0 = 0$ p.to di max relativo $\qquad f(0) = 0$

$x_1 = \frac{2}{3}$ " min. relativo

$$f\left(\frac{2}{3}\right) = \frac{8}{27} - \frac{4}{9} = \frac{8-12}{27} = -\frac{4}{27} < 0$$

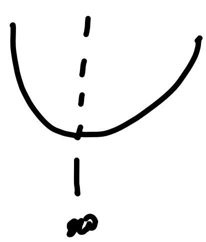


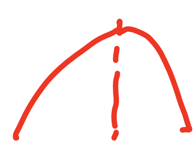
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(x^3-x^2\right) = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(x^3-x^2\right) = +\infty$$

In generale: se esiste $\delta>0$ tale che

$$\left.\begin{array}{ll} f'(x) \le 0 \;\; (\ge 0) & \forall x \in [x_0-\delta, x_0] \\ f'(x) \ge 0 \;\; (\le 0) & \forall x \in [x_0, x_0+\delta] \end{array}\right\} \Rightarrow x_0 \text{ di min. relativo (max. relativo)}$$





Caratterizzazione delle funzioni costanti in un intervallo

Una funzione è <u>costante</u> in un intervallo $[a,b]$ se e solo se la derivata è ovunque nulla:

$$f \text{ costante in } [a,b] \iff f'(x)=0 \quad \forall x \in [a,b]$$

OSS. $f(x) = \operatorname{arctg} x + \operatorname{arctg} \frac{1}{x} \quad \forall x \neq 0$

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) =$$

$$= \frac{1}{1+x^2} - \frac{1}{\frac{x^2+1}{x^2}} \cdot \frac{1}{x^2}$$

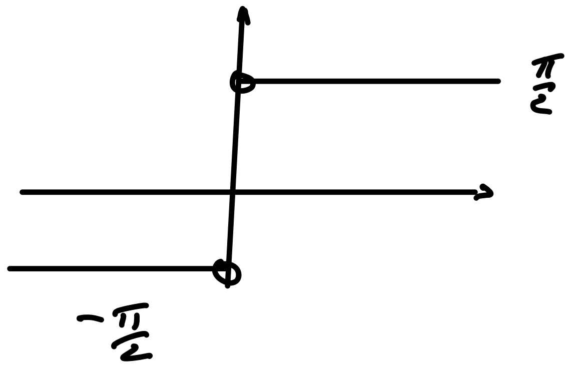
$$= \frac{1}{1+x^2} - \frac{\cancel{x^2}}{1+x^2} \cdot \frac{1}{\cancel{x^2}} = 0$$

$$f(x) = \begin{cases} c_1 & \text{se } x<0 \quad (-\infty,0) \\ c_2 & \text{se } x>0 \quad (0,\infty) \end{cases}$$

$$f(1) = \operatorname{arctg} 1 + \operatorname{arctg} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

$$f(-1) = \operatorname{arctg}(-1) + \operatorname{arctg}(-1) = -2 \operatorname{arctg} 1 = -\frac{\pi}{2}$$

$$f(x) = \begin{cases} \frac{\pi}{2} & \text{se } x \geq 0 \\ -\frac{\pi}{2} & \text{se } x < 0 \end{cases}$$



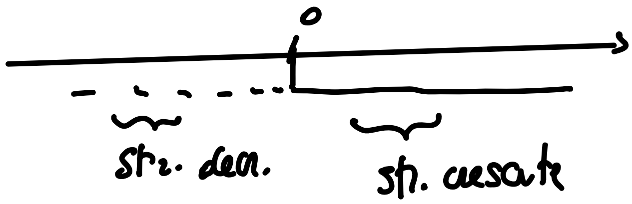
Criterio di stretta monotonia

f continua in $[a,b]$ e derivabile in $]a,b[$. Allora:

(a) $f'(x) \geq 0 \quad \forall x \in]a,b[$
f' non si annulla identicamente in alcun intervallo contenuto in $]a,b[$ $\Rightarrow$ f è strettamente crescente

(b) $f'(x) \leq 0 \quad \forall x \in]a,b[$
f' non si annulla identicamente in alcun intervallo contenuto in $]a,b[$ $\Rightarrow$ f è strettamente decrescente

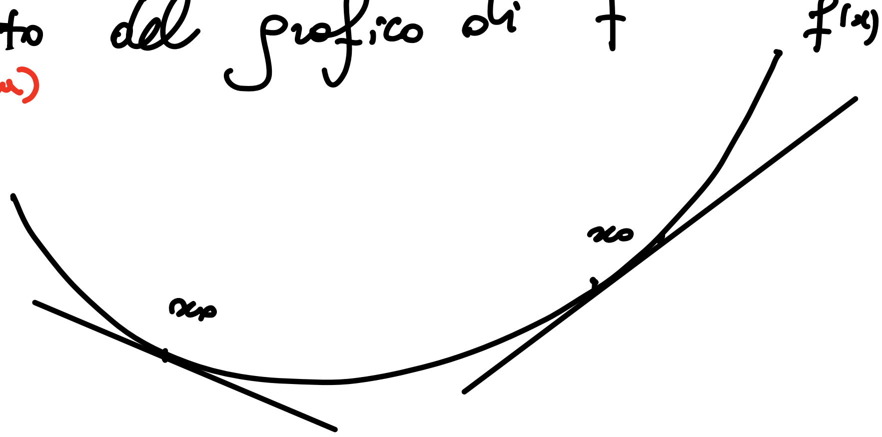
$f(x)=x^2$ $\qquad$ $f'(x)=2x \geq 0 \iff x \geq 0$



Concavità e convessità

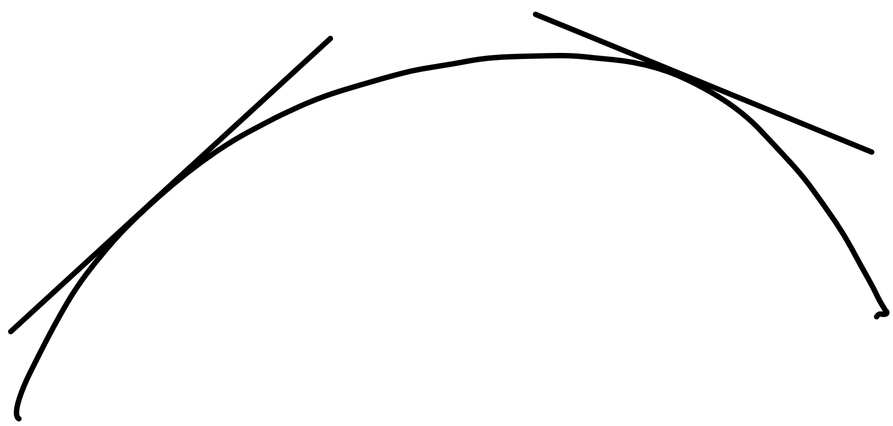
f''

Def. $f:[a,b] \to \mathbb{R}$ (ad esempio) derivabile in $[a,b]$: si dice che f è
(concava)
convessa in $[a,b]$ se, per ogni $x_0 \in [a,b]$ la retta tangente al grafico di f in $(x_0, f(x_0))$ sta al di sotto del grafico di f
(sopra)



f è convessa $\Leftrightarrow$

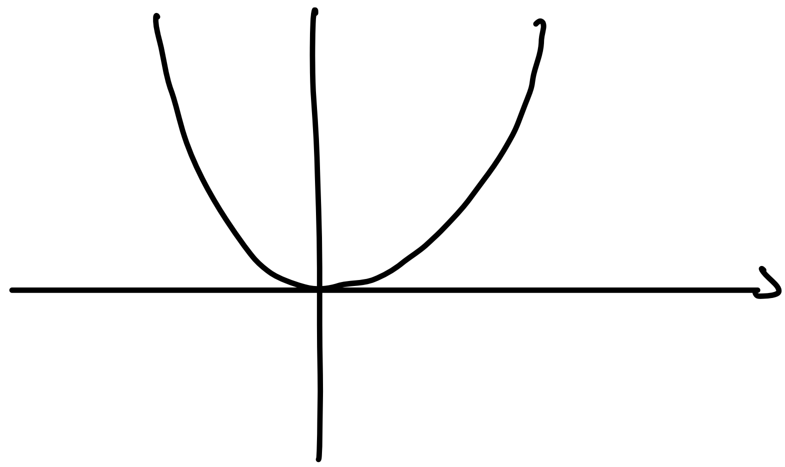
$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$
$$\forall x, x_0 \in [a, b]$$



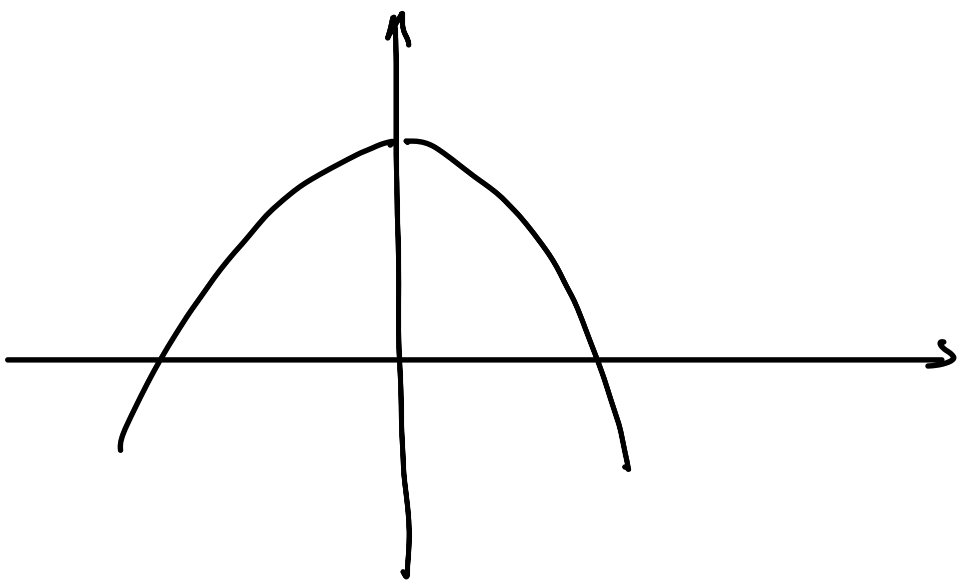
f è concava $\Leftrightarrow$

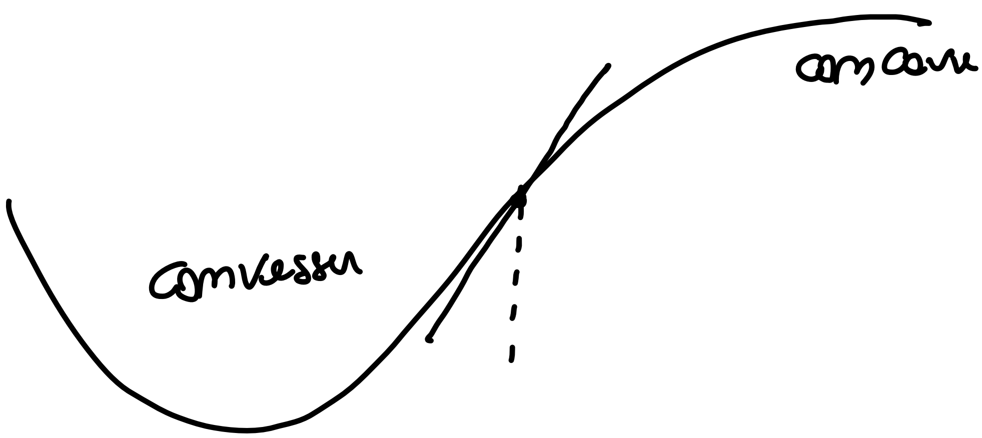
$$f(x) \leq f(x_0) + f'(x_0)(x - x_0)$$
$$\forall x, x_0 \in [a, b]$$

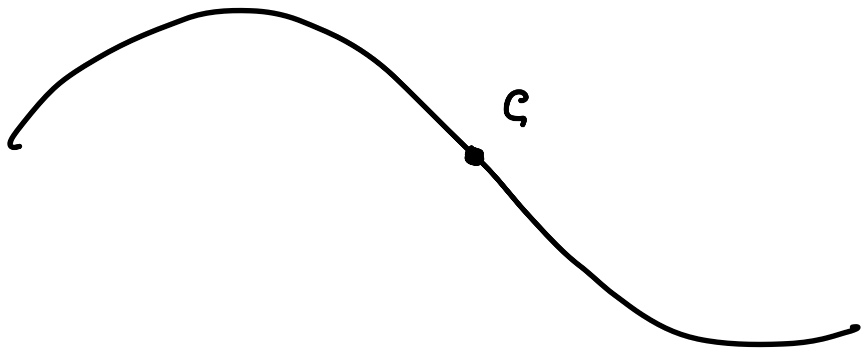
$f(x) = x^2$



$f(x) = 1 - x^2$



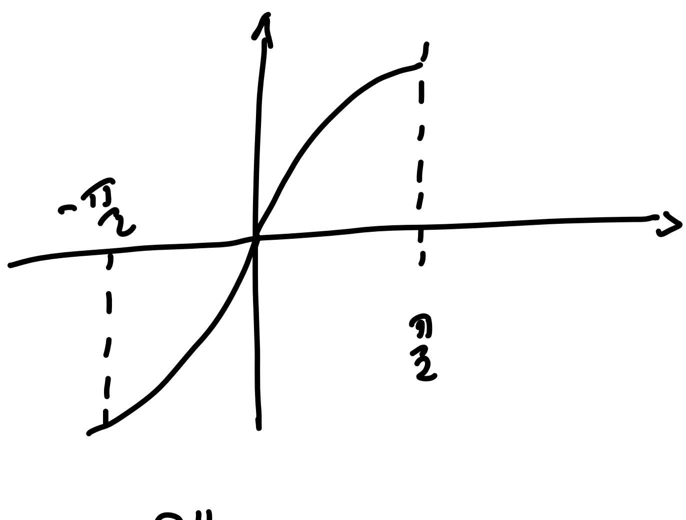




Oss. Se f è derivabile due volte in c con c flesso, allora

$$f''(c)=0$$

$f(x) = \sin x$

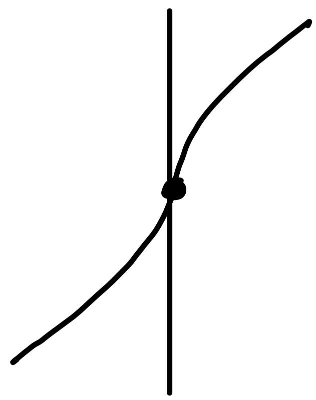


$f'(x) = \cos x$

$f'(0) = 1 \neq 0$

$f''(x) = -\sin x$

$f''(0) = 0$

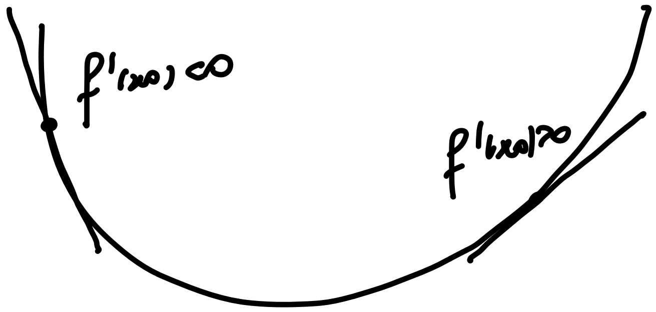


<u>Criterio di convessità</u> f derivabile in $[a,b]$ e ammette derivata seconda in $]a,b[$: le seguenti condizioni sono fra loro equivalenti:

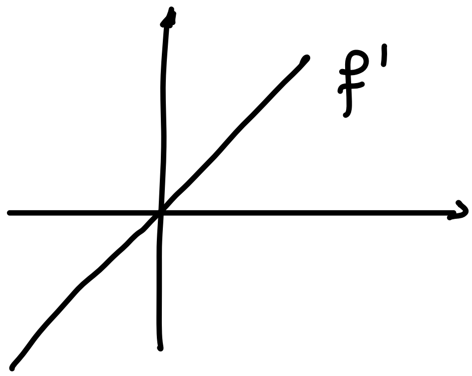
(a) f è convessa in $[a,b]$; (concava)

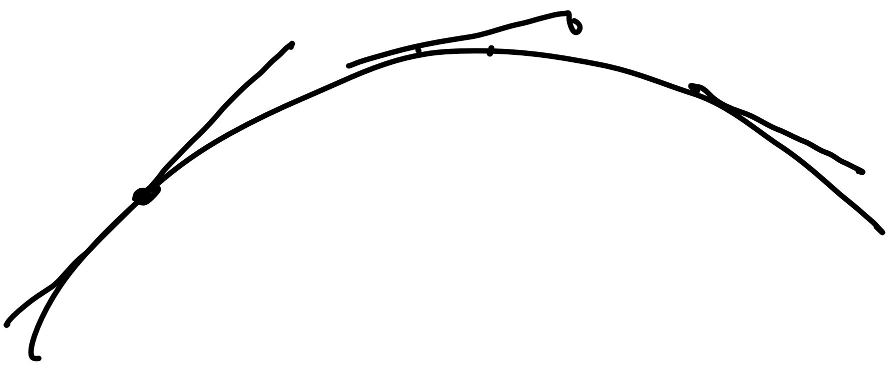
(b) f' è crescente in $[a,b]$; (decrescente)

(c) $f''(x) \geq 0 \quad \forall x \in]a,b[$ $(f''(x) \leq 0)$



$f(x) = x^2$ $\quad$ $f'(x) = 2x$





Es.

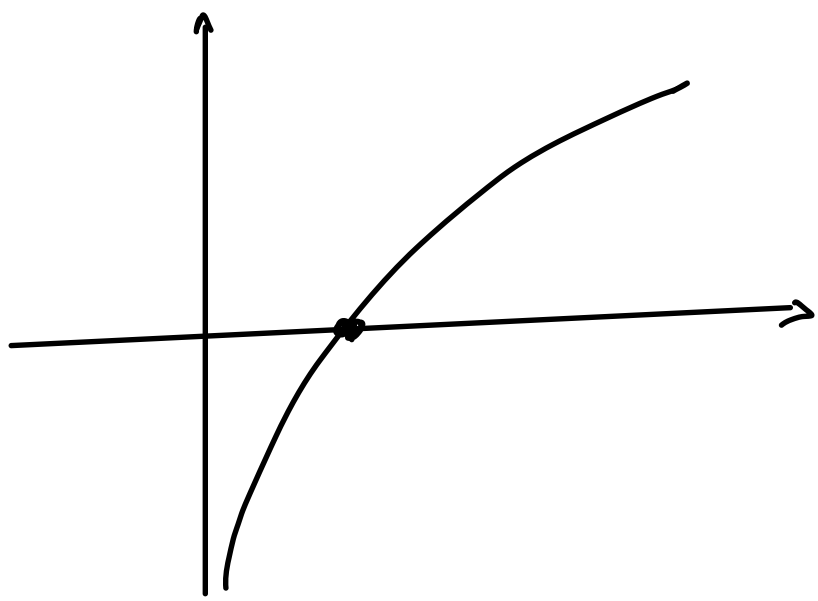
$f(x) = e^x$ $\quad$ $f'(x) = f''(x) = e^x > 0$

$\Rightarrow f(x)$ convessa

$f(x) = \log x \qquad x > 0$

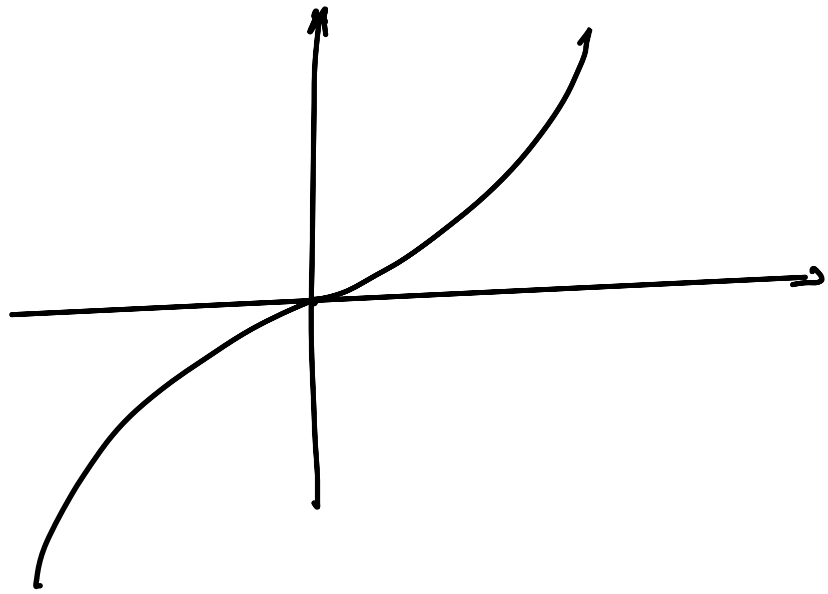
$f'(x) = \frac{1}{x} \qquad f''(x) = -\frac{1}{x^2} < 0$

$\Longrightarrow f$ è concava



$f(x) = x^3 \qquad f'(x) = 3x^2$

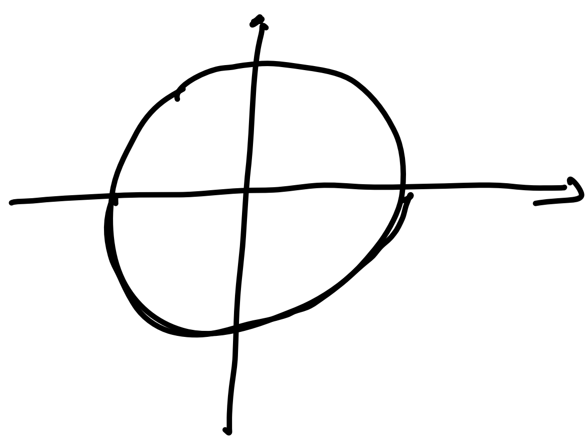
$f''(x) = 6x \geq 0 \iff x \geq 0$



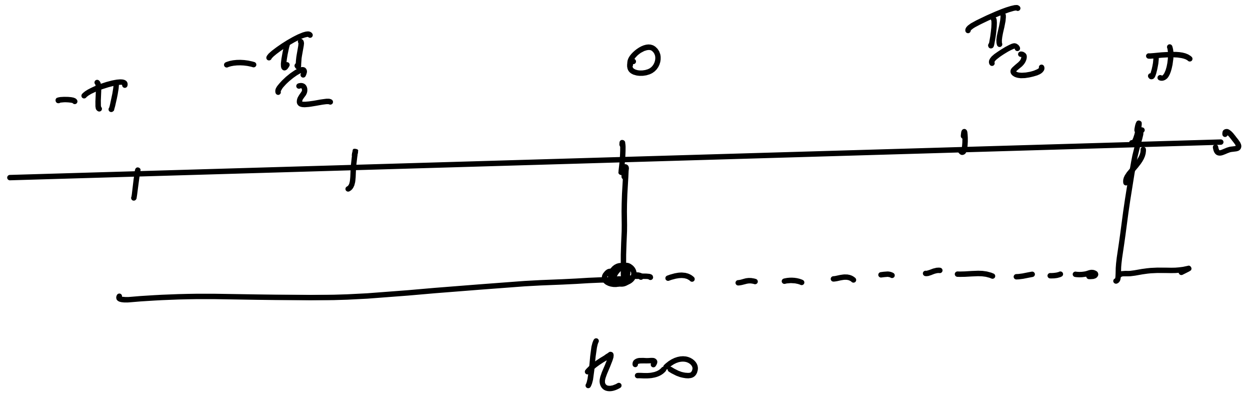
$f(x) = \sin x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$f' = \cos x$, $f'' = -\sin x$

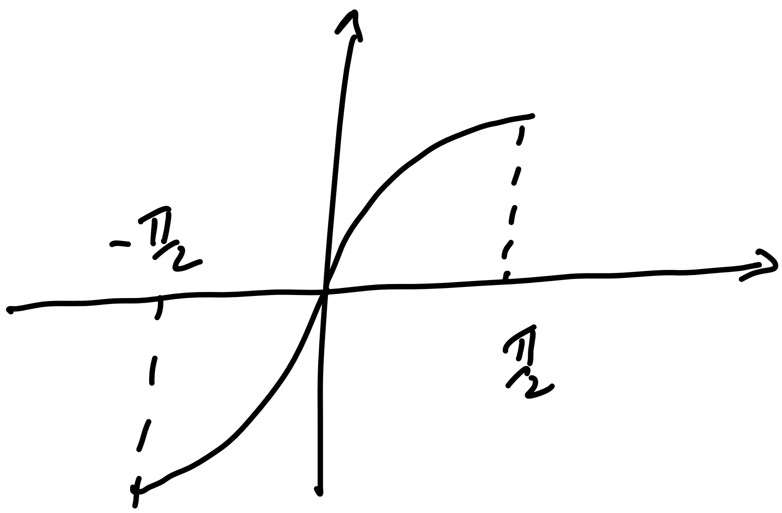
$f''(x) \geq 0 \iff \sin x \leq 0$



$\Leftrightarrow \quad -\pi + 2k\pi \leq x \leq 2k\pi, \quad k \in \mathbb{Z}$



$-\pi \leq x \leq 0$



$f(x) = \underbrace{\text{arctg}}\, x \qquad f'(x) = \frac{1}{1+x^2}$

$f''(x) = -\frac{1}{(1+x^2)^2} \cdot 2x \geq 0$

$\Longleftrightarrow \quad 2x \leq 0 \quad \Longleftrightarrow \quad x \leq 0$