Lezione del 25/10/2022

$\mathbb{N}_0$ = insieme dei numeri naturali $= \{0, 1, 2, 3, 4, \dots, n, \dots\}$

$\mathbb{N}$ = " " " non nulli $= \{1, 2, 3, 4, \dots, n, \dots\}$

$\mathbb{Z}$ = insieme dei numeri relativi $= \{0, \pm 1, \pm 2, \pm 3, \dots, \pm n, \dots\}$

$$\mathbb{N} \subsetneq \mathbb{N}_0 \subsetneq \mathbb{Z} \subsetneq \mathbb{Q}$$

$A \subseteq B$

$\subset$ "inclusione propria"

$A \subset B$ opp. $A \subsetneq B$

$+ \quad \cdot$

$0 \quad 1$

$n \in \mathbb{Z} \quad -n \in \mathbb{Z}$

$n + (-n) = 0$

$n \neq 0 \quad \nexists\, m : nm = 1$

$\frac{1}{n} \notin \mathbb{Z}$

$\mathbb{Q}$ = insieme dei numeri razionali $= \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0 \right\}$

"tale che"

$n \in \mathbb{Z}, \quad n = \frac{n}{1}$

$+ \quad \cdot$

0

1

$m, n \neq 0$

$\frac{m}{n} + \left(\frac{-m}{n}\right) = 0$

$\left(\frac{m}{n}\right)^{-1} = \frac{n}{m}$

$$\mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

campi

$\mathbb{Q}, \mathbb{R}$ $\quad + \;, \;\cdot$

1) Prop. commutativa $\quad a+b=b+a$
$\quad ab=ba$

2) " associativa $\quad (a+b)+c = a+(b+c)$
$\quad (ab)c = a(bc)$

3) Esistenza degli elementi neutri: $\quad a+0=a \quad \forall a$
$\quad a\cdot 1 = a \quad$ "

4) Esistenza dell'opposto: per ogni $a \in \mathbb{Q}$ opp. $\mathbb{R}$
$\exists\, -a$ tale che $\quad a+(-a)=0$

5) Esistenza dell'inverso: per ogni $a \neq 0$
$\exists\, a^{-1} = 1/a$ tale che $\quad a \cdot a^{-1} = 1$

6) Prop. distributiva $\quad (a+b)\cdot c = a\cdot c + b\cdot c$

$\mathbb{Q}$, $\mathbb{R}$ campi

$\leq$ "relazione d'ordine" $\quad a \underset{\geq}{\leq} b$

7) Riflessiva : $a \leq a \quad \forall a$

8) Asimmetrica : $a \leq b$ e $b \leq a \Rightarrow a = b$

9) Transitiva : $a \leq b$ e $b \leq c \Rightarrow a \leq c$

Relazione di ordine totale: $\forall a, b \in \mathbb{Q}, \mathbb{R}$

o $a \leq b$ opp. $b \leq a$

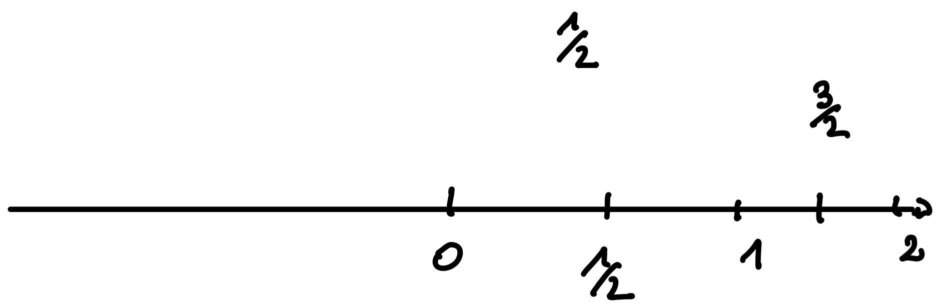
$\mathbb{Q}, \mathbb{R}, +, \cdot, \leq$ campi totalmente ordinati

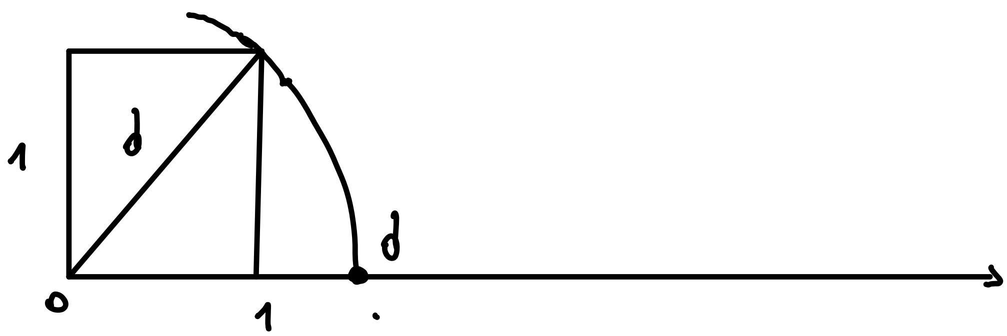
10) $a \leq b \Rightarrow a + c \leq b + c \quad \forall c$

11) $a \leq b \Rightarrow a \cdot c \leq b \cdot c$
$\forall c > 0$

Proprietà di compatibilità di $\leq$ rispetto a $+, \cdot$

$\mathbb{Q}$





$$d^2 = 1 + 1 = 2$$

$$\underline{\underline{d = \sqrt{2}}} \qquad d^2 = 2$$

$$\nexists\, d \in \mathbb{Q} \text{ t.c. } d^2 = 2$$

<u>Estremo superiore, estremo inferiore</u>

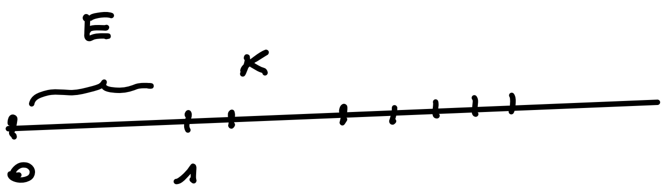
X campo totalmente ordinato $(X = \mathbb{Q},\ X = \mathbb{R})$

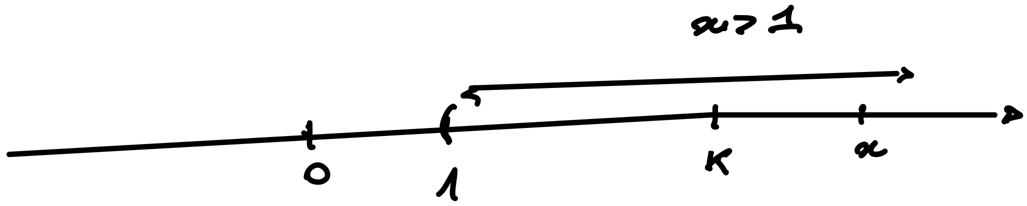
Def. (Insieme limitato superiormente, inferiormente)

$E \subseteq X$: si dice che E è <u>limitato superiormente</u> se

$\exists\, K \in X$ tale che

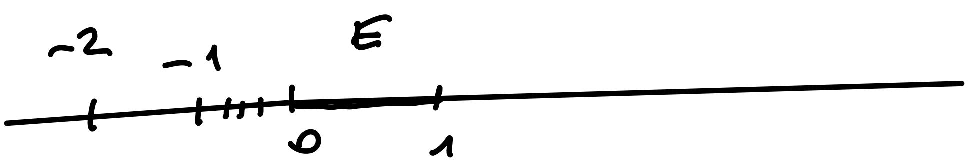
$$x \le K, \quad \forall x \in E$$



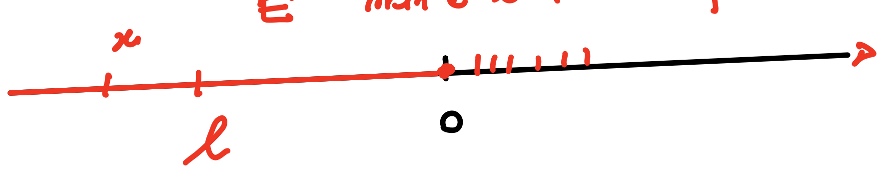


E **limitato inferiormente** se $\exists\, \ell \in X$ tale che

$$x \ge \ell, \quad \forall x \in E$$



non è limitato inf., limit. sup.

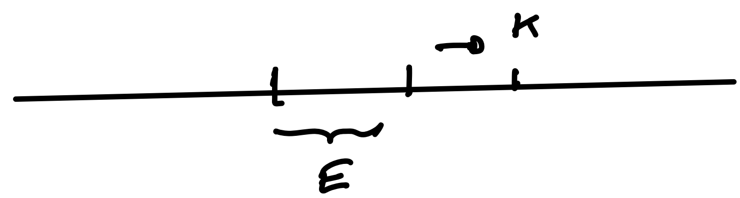


$E \subseteq X$ si dice limitato se è **limitato superiormente ed inferiormente**: $\exists\, K, \ell \in X$ tali che

$$\ell \le x \le K, \quad \forall x \in E$$

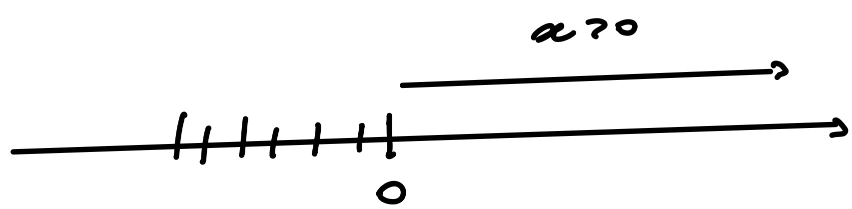
Def. $E \subseteq X$ limitato superiormente, un maggiorante di E è un numero $K \in X$ tale che

$$x \le K, \quad \forall x \in E$$



$E \subseteq X$ limitato inferiormente, un minorante di E è un numero $\ell \in X$ tale che

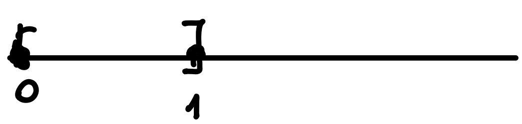
$$x \ge \ell, \quad \forall x \in E$$

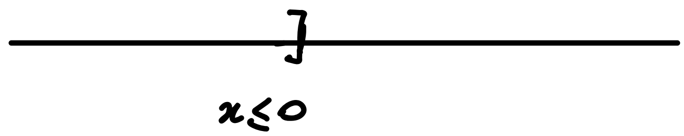


Def. $E \subseteq X$ si dice massimo di E, un numero $M \in E$ tale che

$$x \le M, \quad \forall x \in E$$

massimo di $E = 1$



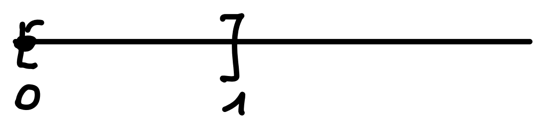


massimo di $E = 0$

$E \subseteq X$, si dice <u>minimo di E</u>, un numero $m \in E$, tale che

$$x \geq m \quad , \forall x \in E$$

minimo di $E = 0$



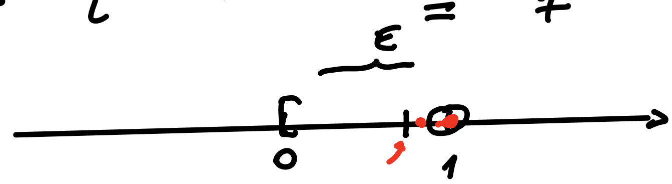
max E = massimo di E

min E = minimo di E

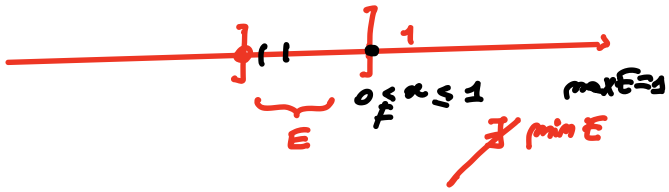
OSS. Se E ha minimo, esso è limitato inferiormente (min E = minorante di E)

E ha massimo, esso è limitato superiormente

ES. $E = \{ x \in \mathbb{R} : 0 \underset{=}{\leq} x \underset{\neq}{<} 1 \}$



max E non esiste!



Es.

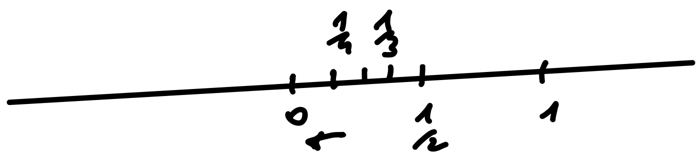
$\mathbb{N}_0 = \{0, 1, 2, 3, \dots, n, \dots\}$

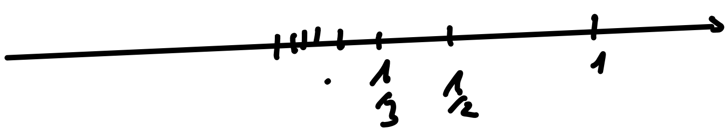
$\nexists \max \mathbb{N}_0$ $\qquad$ $\min \mathbb{N}_0 = 0$

~~lim sup~~ $\qquad$ limitato inferiormente: SI

$E = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$ $\qquad$ $\max E = 1$

$\nexists \min E$





$n=0$ $\qquad$ $n=2$ $\qquad$ $\frac{3}{5}$

$E = \left\{ \frac{n-1}{n+1} : n \in \mathbb{N}_0 \right\} = \left\{-1, 0, \frac{1}{3}, \frac{1}{2}, \dots\right\}$

$n=1$ $\qquad$ $n=3$

limitato inferiormente: $\min E = -1$

|| superiormente

$n \to +\infty$

$\frac{n-1}{n+1} \longrightarrow 1$

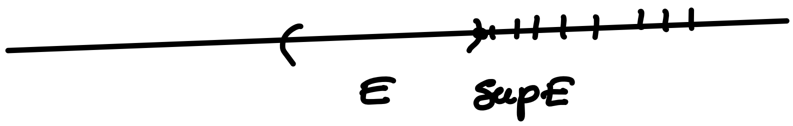
$\frac{n-1}{n+1} < 1$

$\max E = 1$?

$\frac{n-1}{n+1} = 1 \iff \cancel{n} - 1 = \cancel{n} + 1$

Estremo superiore, estremo inferiore

$E \subseteq X$ limitato superiormente

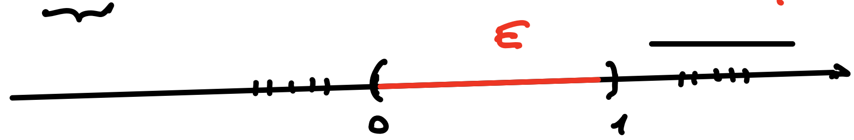


Il minimo dell'insieme dei maggioranti (se esiste) si chiama estremo superiore di E e si denota con

$$\sup E$$

$E = \{x \in \mathbb{R} : 0 < x < 1\}$

$\nexists \max E$



$\inf E = 0$ $\qquad \sup E = 1$

$\nexists \min E$

E limitato inferiormente, il massimo dell'insieme dei minoranti (se esiste) si chiama estremo inferiore di E e si denota con

$$\inf E$$

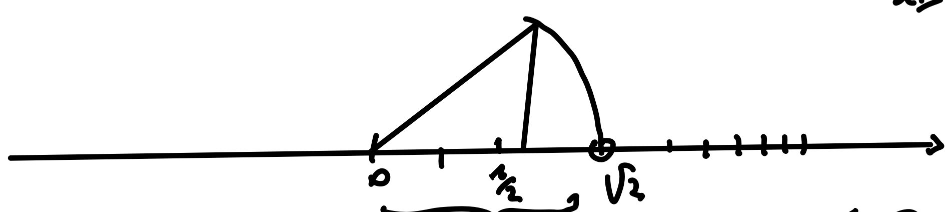
(Assioma di completezza): $\forall E \subseteq X$, $E \neq \emptyset$ e limitato superiormente, possiede estremo superiore

(" inferiormente " " inferiore)

ES $E = \{x \in \mathbb{Q} : \underbrace{x \geq 0}_{\text{"x non negativo"}},\ x^2 < 2\} = \{x \in \mathbb{Q} : 0 \leq x < \sqrt{2}\}$ $\subseteq \mathbb{Q}$

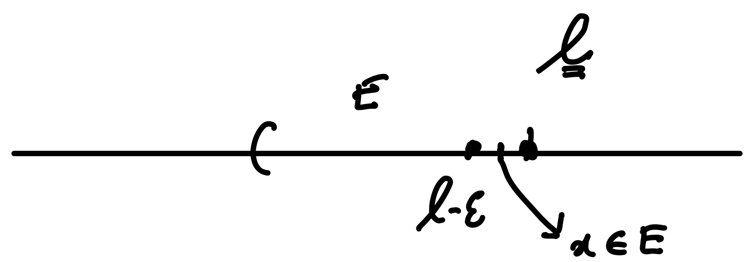
$x^2 < 2 \iff x^2 - 2 < 0 \iff -\sqrt{2} < x < \sqrt{2}$

$x \geq 0$



$\sqrt{2} \notin \mathbb{Q}$

Def (Assiomatica di $\mathbb{R}$) Chiamiamo $\mathbb{R}$ un campo totalmente ordinato che ha la proprietà dell'estremo superiore.

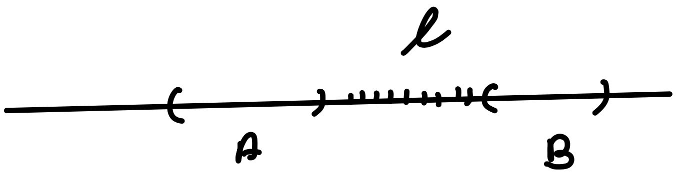


$\forall \varepsilon > 0\ \exists x \in E : x > \ell - \varepsilon$

"Epsilon"

Def. $A, B \subseteq \mathbb{R}$ separati se

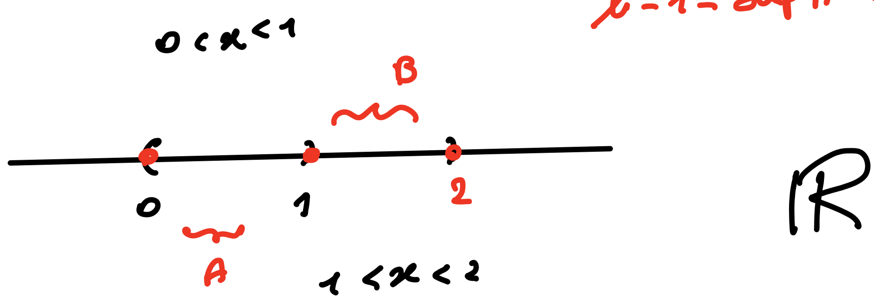
$$a \le b \;, \; \forall\, a \in A,\; b \in B$$



$$\exists\, \ell \in \mathbb{R} : \quad a \le \underbrace{\ell}_{\text{elementi di separazione}} \le b \quad \forall\, a \in A,\; \forall\, b \in B$$

Se $\underbrace{\exists\,!}_{\text{"unico"}}$ elemento di separazione, A e B si dicono contigui

$\ell = 1 = \sup A = \inf B$



Def. (Valore assoluto)

$$\forall\, x \in \mathbb{R}, \qquad |x| = \begin{cases} x & \text{se } x \ge 0 \\ -x & \text{se } x \le 0 \end{cases}$$

$$|-2| = 2 = |2|$$

$|x| \geq 0 \quad \forall x \in \mathbb{R}; \quad |x| = 0 \Leftrightarrow x = 0$

$$|\lambda x| \underbrace{=}_{\text{val. ass. prodotto} \,=\, \text{prodotto valori assoluti}} |\lambda|\,|x| \qquad \forall \lambda \in \mathbb{R},\ \forall x \in \mathbb{R}$$

$$|x+y| \leq |x| + |y| \qquad \forall x, y \in \mathbb{R}$$

$$\big||x| - |y|\big| \leq |x-y|$$

(Radici n-esime aritmetiche)

<u>Teorema</u> $n \in \mathbb{N} = \{1, 2, 3, 4, \dots\}$. Sia $y \geq 0$: allora $\exists!\ x \geq 0$ tale che $x^n = y$

$n = 2$, $\quad x =$ radice quadrata di $y = \sqrt{y}$

$n = 3$ $\quad$ " " cubica " " $= \sqrt[3]{y}$

$$x = \underbrace{\sqrt[n]{y}}_{} = \underbrace{y}_{\text{lo stesso}}^{1/n}$$

<u>Potenze ad esponente reale</u>

$m, n \in \mathbb{N}$; se $a > 0$

$$\underbrace{a^{m/n}}_{\frac{m}{n} \in \mathbb{Q}} := \underbrace{\sqrt[n]{a^m}}_{>0} \qquad \underline{\underline{\text{ben definita}}}$$

$$a^x, \ \forall x \in \mathbb{R} \qquad a^{\sqrt{2}}, \ a^{\pi}$$

$$a^x \in \mathbb{R}$$
$$a^x > 0$$

n $\underline{\underline{\text{dispari}}}$ $\qquad \sqrt[n]{a}$

$a < 0$

$$\sqrt[3]{-8} = -\sqrt[3]{8} = -2$$

$$a^{1/n} = \sqrt[n]{a} = -\sqrt[n]{-a}$$

$$\sqrt[4]{\underbrace{-8}_{<0}} \quad \underline{\text{non esiste}}!$$

$$\underbrace{x^4}_{\geq 0} = -8 \quad \underline{\text{ASSURDO}}!$$

$$\sqrt{-1} \longleftrightarrow \mathbb{C} \quad i$$

Proprietà $a, b > 0$, $c, d \in \mathbb{R}$

1) $a^0 = 1 \quad \forall a \neq 0$, $1^c = 1 \quad \forall c$

2) $a^c > 0$, $\forall c \in \mathbb{R}$

$$\begin{cases} a^c < 1 & \text{se } a < 1 \\ a^c > 1 & \text{se } a > 1 \end{cases} \quad \text{e } c > 0$$

3) $a^{c+d} = a^c \cdot a^d$

4) $(ab)^c = a^c \cdot b^c$

5) $(a^b)^c = a^{bc}$

6) $c < d \Rightarrow \begin{array}{ll} a^c < a^d & a > 1 \\ a^c > a^d & 0 < a < 1 \end{array}$

7) $0 < a \leq b \Rightarrow a^c \leq b^c$, $\forall c > 0$

Logaritmi $a > 0$ $\quad a^x = y$

$a \neq 1$

se $a = 1$ $\quad a^x = 1^x = 1$

$y = 1$

$\underline{a \neq 1}$, $a > 0$

$$\underbrace{a^x} = y$$

$y > 0$

$\exists!\, x \in \mathbb{R}$ tale che $a^x = y$

$$x = \log_a y$$

$$a^{\log_a y} = y$$

$\log_2 \frac{1}{2} = -1 \qquad 2^x = \frac{1}{2}$

$$2^{-1} = \frac{1}{2}$$

$$\log_3 9 = 2$$

$$3^{x} = 9$$

2

Proprietà logaritmi

$x, y > 0$

1) $\log_a (xy) = \log_a x + \log_a y$

2) $\log_a \frac{x}{y} = \log_a x - \log_a y$

3) $\log_a x^{\alpha} = \alpha \log_a x \quad \forall \alpha \in \mathbb{R}$

Sistemi lineari

$m \times n$

$$
(S)\ \begin{cases}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\
\dots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m
\end{cases}
$$

$$\underline{X} = (x_1, \dots, x_n) \in \mathbb{R}^n$$

$$A\underline{X} = B$$

$$A = (a_{ij}) \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\rho(A) = \rho(A') \iff S \text{ compatibile}$$

$$A' = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix}$$

Se $b_1 = b_2 = \dots = b_m = 0$, il sistema si dice omogeneo

$$A\underline{X} = \underline{0}$$

$\underline{X} = (0,0,0,..,0)$ sol. banale

$$\rho(A) = \rho(A')$$

Se $m=n$ $(S_1)\begin{cases} a_{11}x_1 + \dots + a_{1n} = 0 \\ - \ - \ - \ - \ - \\ a_{m1}x_1 + \dots + a_{mn} = 0 \end{cases}$

(S_1) è sempre compatibile

$\exists !\ (0,0,..,0)$ se e solo se

$$|A| \neq 0$$

<u>**OSS.**</u> (S_1) ha infinite soluzioni $\Leftrightarrow$ $|A| = 0$

$$\begin{cases} x - 2y = 0 \\ 3x - 6y = 0 \end{cases} \Leftrightarrow x - 2y = 0$$

$$\Leftrightarrow x = 2\underline{y}$$

$$(x,y) = (2y, y) = y(2,1)$$

$\forall y \in \mathbb{R}$

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix} \begin{array}{l} 3 \\ \leftarrow \end{array} \qquad \underline{\underline{\rho(A) = 1}}$$

$$|A| = 0 \qquad \infty^{n-\rho(A)} = \infty^{2-1} = \infty^{1}$$

---

2 equazioni
3 incognite

$$\begin{cases} x - 3y + z = 0 \\ 5x + 2z = 0 \end{cases} \iff$$

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 5 & 0 & 2 \end{pmatrix} \qquad \rho(A) = 2$$

$$\begin{vmatrix} 1 & -3 \\ 5 & 0 \end{vmatrix} = 15 \neq 0$$

$$\iff \begin{cases} x - 3y = -z \\ 5x = -2z \end{cases} \qquad \underset{\smile}{z = t}$$

$$\begin{cases} z = t \\ x - 3y = -t \\ 5x = -2t \end{cases}$$

$$\begin{cases} z = t \\ x = -\frac{2}{5}t \\ x - 3y = -t \end{cases}$$

$$\begin{cases} z = t \\ x = -\frac{2}{5}t \\ -\frac{2}{5}t - 3y = -t \end{cases} \Leftrightarrow \frac{2}{5}t + 3y = t$$

$$\Leftrightarrow \quad 3y = t - \frac{2}{5}t$$

$$= \frac{3}{5}t$$

$$y = \frac{1}{5}t$$

$$\begin{cases} x = -\frac{2}{5}t \\ y = \frac{t}{5} \\ z = t \end{cases}$$

$$(x, y, z) = \left(-\frac{2}{5}t, \frac{t}{5}, t\right) =$$

$$= t\underbrace{\left(-\frac{2}{5}, \frac{1}{5}, 1\right)} \quad \forall t \in \mathbb{R}$$

$$\infty^1$$

$$(S_2) \begin{cases} x - y + t - z = 0 \\ 2x - 2y + 2t - 2z = 0 \end{cases} \Bigg\|$$