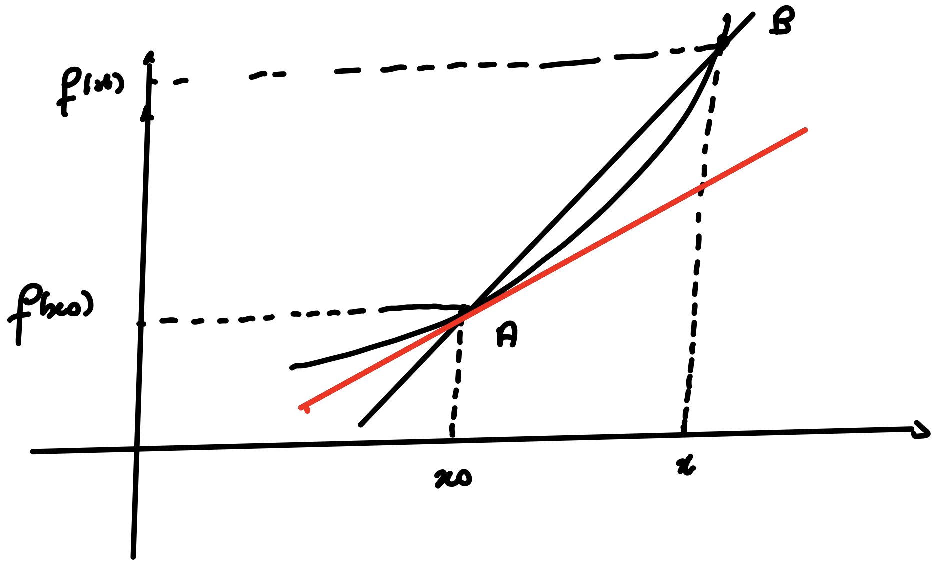
Lezione del 22/11/22

Calcolo differenziale

Derivata di una funzione $\quad f = f(x)$

Def. $\quad f : I \subseteq \mathbb{R} \to \mathbb{R}$ e sia $x_0 \in I$

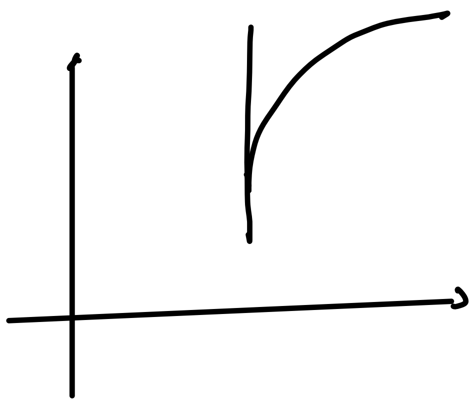


$$\phi(x) = \frac{f(x) - f(x_0)}{x - x_0} \qquad \forall x \in I \setminus \{x_0\}$$

rapporto incrementale di f in x_0

Quando $x \to x_0$, la retta secante $\overline{AB}$ assume una posizione limite, che è quella della retta tangente

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$ = coeff. angolare della retta tangente ad f in $(x_0, f(x_0))$



<u>Def.</u> Diciamo che f è derivabile in x_0 se esiste finito il limite

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

In tal caso, tale limite si chiama derivata di f in x_0, e si denota

$$f'(x_0) = \left.\frac{df}{dx}\right|_{x=x_0} = Df(x)$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

Se f è derivabile in ogni punto di I, si dice che f è derivabile in I.

$$f' : x \in I \longrightarrow f'(x) \in \mathbb{R}$$

<u>OSS.</u> Se f è derivabile in x_0,

$f'(x_0)$ = coefficiente angolare della retta tangente al grafico di f in $(x_0, f(x_0))$.

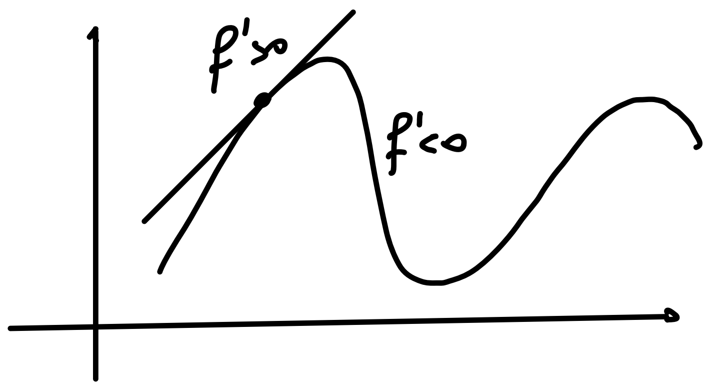
Eq. retta tangente

$$\frac{y-f(x_0)}{x-x_0} = f'(x_0)$$

$$\boxed{y = f(x_0) + f'(x_0)(x-x_0)}$$

retta tangente al grafico di f in $(x_0, f(x_0))$

OSS.



$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \qquad h = x-x_0$$

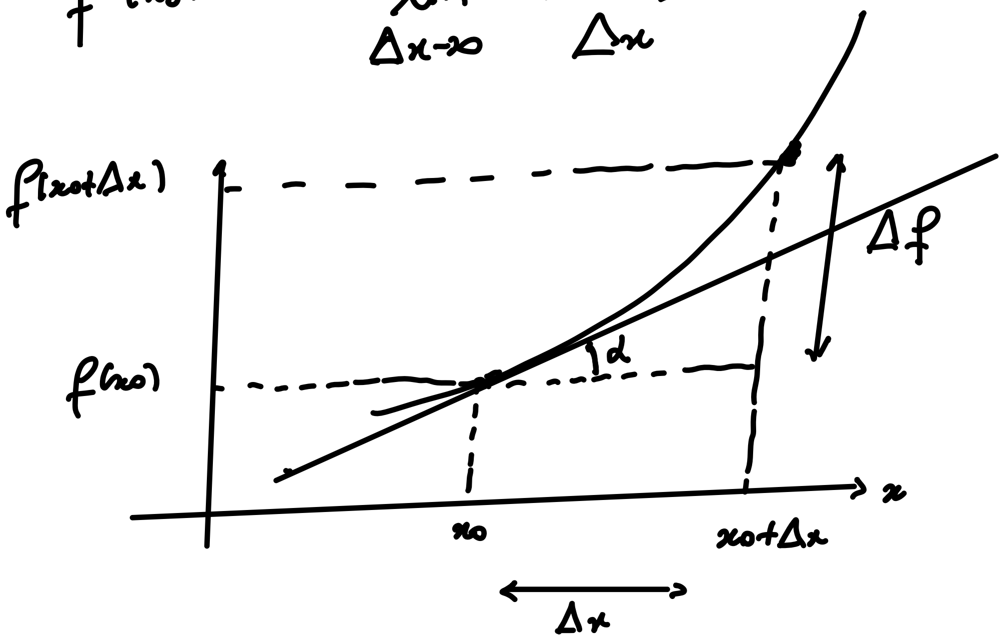
$$x = x_0 + h$$

$$x \to x_0,\quad h \to 0$$

$$= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$h = \Delta x$ incr. di x $\qquad \Delta f = f(x_0+\Delta x) - f(x_0) \equiv$ incremento di f

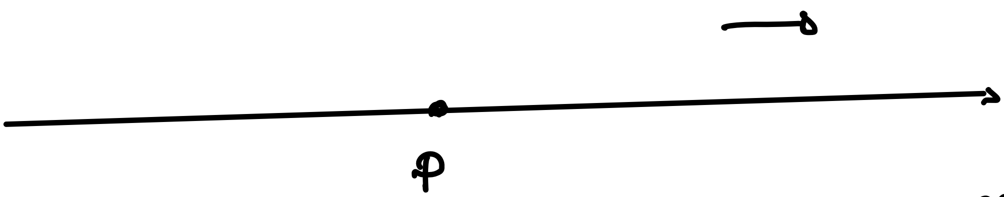
$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$



$$tg\alpha = f'(x_0)$$

Alcune interpretazioni della derivata

1) $s = s(t)$



$$v(t) = \frac{ds}{dt}(t)$$ velocità di P all'istante t

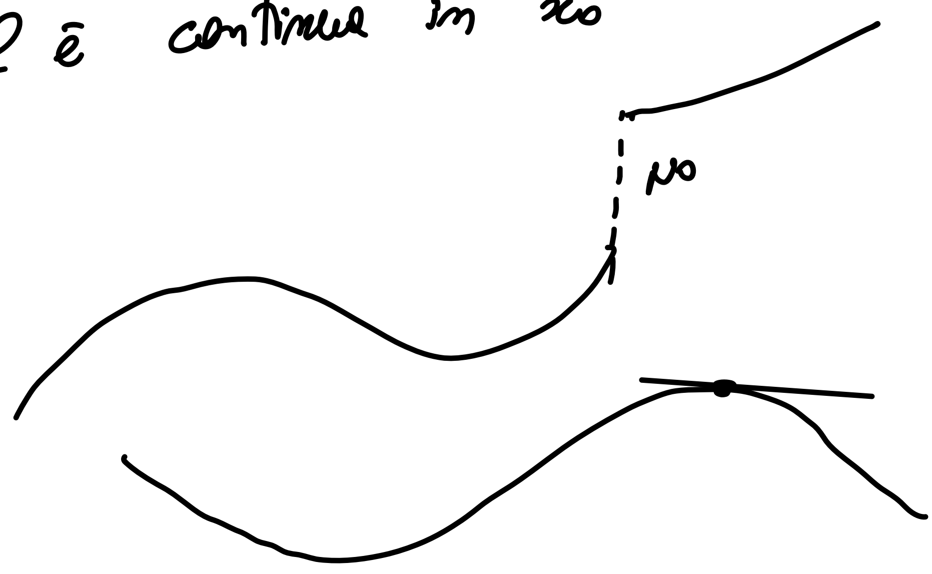
$$a(t) = \frac{dv}{dt}(t) = \frac{d^2 s}{dt^2}(t)$$

2) $C(p)$ costo di produzione di una quantità p di un certo bene

$$\frac{C(p+\Delta p) - C(p)}{\Delta p}$$ costo marginale medio di produzione

$$\lim_{\Delta p \to 0} \frac{C(p+\Delta p) - C(p)}{\Delta p} = C'(p)$$

<u>Proprietà</u> Se f è derivabile in $x_0 \in I$, allora f è continua in x_0



Dim. $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$

$\lim_{x \to x_0} f(x) = f(x_0)$ (TESI)

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left([f(x) - f(x_0)] + f(x_0) \right)$$

$$= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) + f(x_0) \right)$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) + f(x_0)$$

$$= \overbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}^{f'(x_0)} \cdot \overbrace{\lim_{x \to x_0} (x - x_0)}^{0} +$$

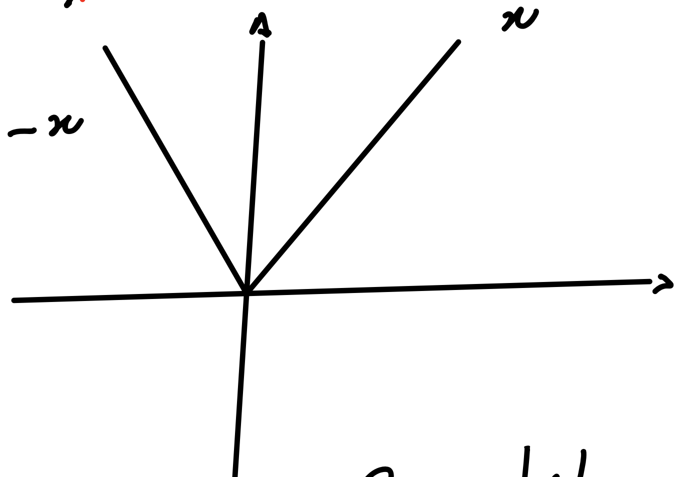
$$+ f(x_0) = 0 + f(x_0) = f(x_0)$$

c.v.d.

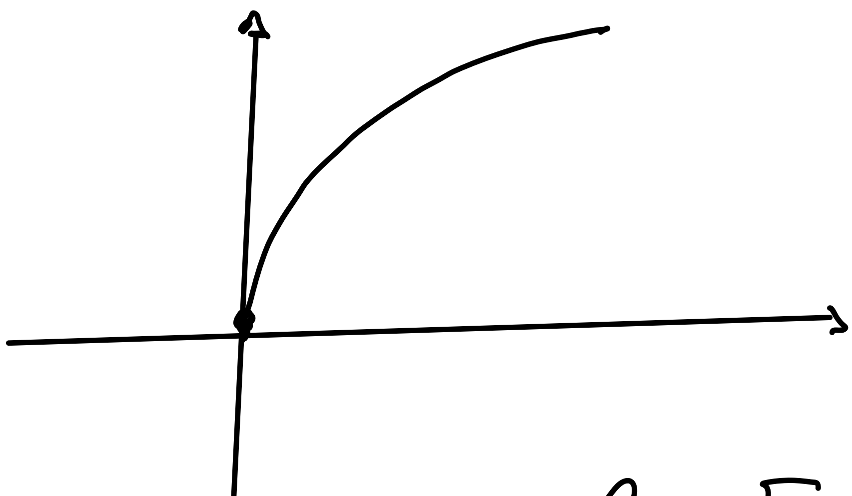
f derivabile $\Rightarrow$ f continua

$\not\Leftarrow$

$f(x) = |x|$



$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{|x|}{x}$$ NON ESISTE

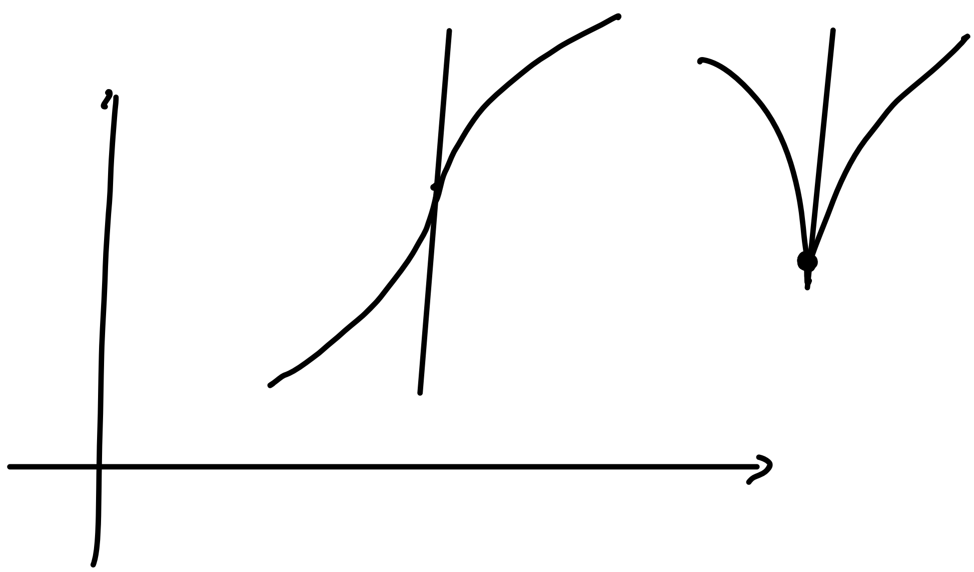


$f(x) = \sqrt{x}$
$x \geq 0$

$$\lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{\sqrt{x}}{x} \quad \left[\frac{0}{0}\right]$$

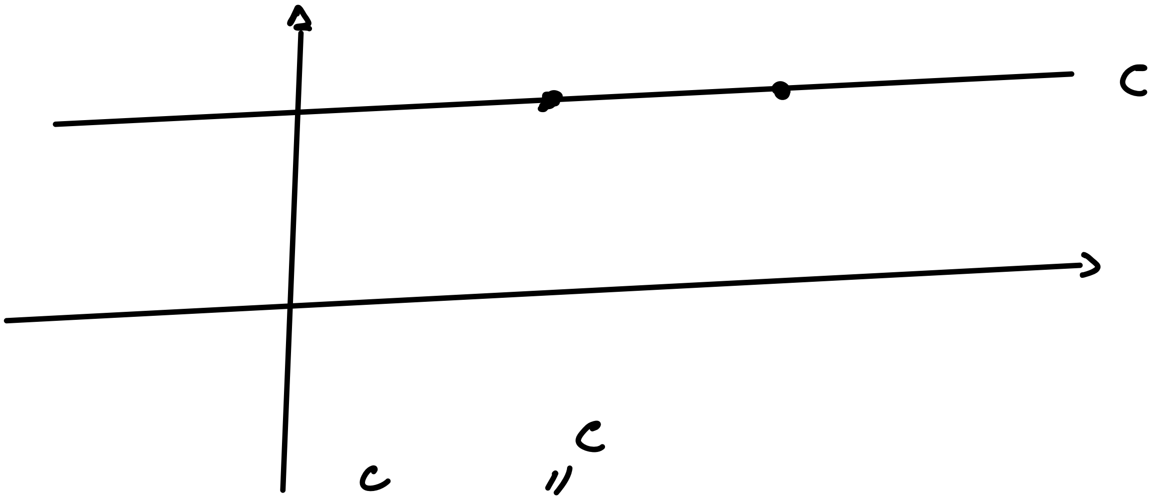
$$= \lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{x^2}} = \lim_{x \to 0^+} \sqrt{\frac{\cancel{x}}{x^{\cancel{2}}}} =$$

$$= \lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty$$



Derivate delle funzioni elementari

1) $f(x) = c \quad , c \in \mathbb{R}$

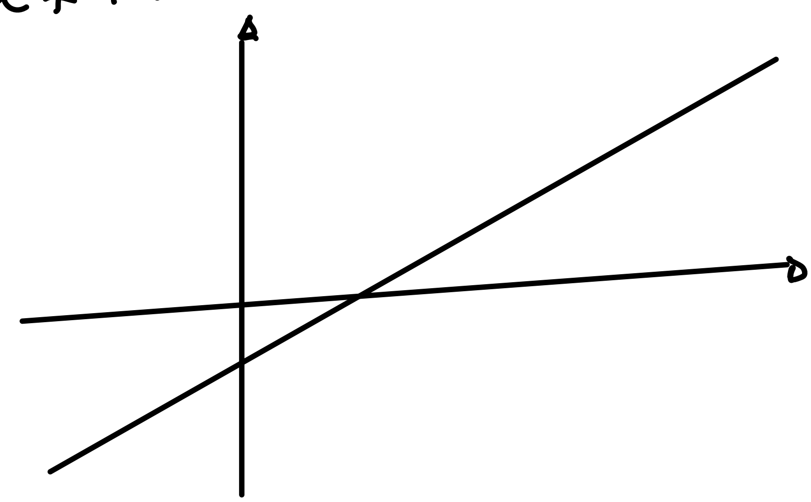


$$\frac{\overset{c}{\overset{\shortparallel}{f(x+h)}} - \overset{c}{\overset{\shortparallel}{f(x)}}}{h} = 0$$

$Df = 0$ quando f è costante

$f'(x) = m$

$f(x) = mx + n$



$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{m(x+h) \overset{+n}{} - mx - n}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{mx} + mh \overset{\cancel{+n}}{} - \cancel{mx} - \cancel{n}}{h} = m$$

$f = 4x - 5 \qquad f'(x) = 4$

2) $f(x) = x^2$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} =$$

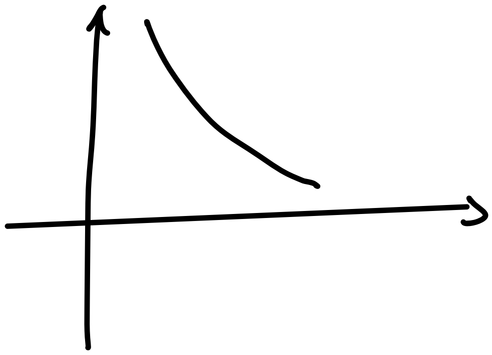
$$= \lim_{h \to 0} \underbrace{\frac{2\cancel{h}x}{\cancel{h}}}_{2x} + \underbrace{\frac{h^{\cancel{2}}}{\cancel{h}}}_{0} = 2x$$

$$D x^2 = 2x \qquad Dx = 1$$

$$D x^3 = 3x^2$$

$$D x^n = n x^{n-1} \qquad \forall n \in \mathbb{N}, \forall x \in \mathbb{R}$$

$$\boxed{D x^\alpha = \alpha x^{\alpha - 1} \qquad \forall x > 0, \forall \alpha \in \mathbb{R}}$$



$$D\sqrt{x} = D x^{\frac{1}{2}} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$D x^{\pi} = \pi x^{\pi - 1}$$

3) $0 < a \neq 1$

$$\boxed{\begin{aligned} D a^x &= a^x \log a \\ D e^x &= e^x \log e = e^x \end{aligned}}$$

4) $f(x) = \log_a x$ $\quad \forall x > 0$

$0 < a \neq 1$

$$D \log_a x = \frac{1}{x} \log_a e$$

$$\boxed{D \log x = \frac{1}{x}}$$

5) $f(x) = \sin x, \cos x$

$$\boxed{\begin{aligned} D(\sin x) &= \cos x \\ D(\cos x) &= -\sin x \end{aligned}}$$

$\sin x \quad \cos x \quad -\sin x \quad -\cos x \quad \sin x$

Operazioni con le derivate

f, g , $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$

derivabili.

Allora:

1) $(c f)'(x) = c f'(x)$, $\forall x \in I$
$\forall c \in \mathbb{R}$

$$D(2x^2) = 2 D(x^2) = 2 \cdot 2x = 4x$$

$$D(\pi x^\alpha) = \pi \alpha x^{\alpha - 1}$$

2) $(f \pm g)'(x) = f'(x) \pm g'(x)$

$$D(\sin x + \log x) = \cos x + \frac{1}{x}$$

3) $(f \cdot g)'(x) \neq f'(x)\, g'(x)$

$$(f \cdot g)'(x) = f'(x)\, g(x) + f(x)\, g'(x)$$

$$D(e^x (1-x^2)) = e^x (1-x^2) + e^x(-2x)$$

$$= e^x (1-x^2) - 2x e^x$$

$$D(\sin x \cos x) = D(\sin x) \cos x + \sin x\, D(\cos x) = \cos^2 x - \sin^2 x$$

$$D\left(a^x \log_b x\right) =$$

$$= \left(a^x \log a\right) \log_b x + a^x \frac{1}{x} \log_b e$$

4) $g(x) \neq 0$ $\qquad \dfrac{f}{g}$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}$$

$$D\,\frac{1}{x} = D\left(x^{-1}\right) = -1\,x^{-2}$$

$$= -\frac{1}{x^2}$$

$$= \frac{0 \cdot x - 1 \cdot 1}{x^2} = -1/x^2$$

$$D\,\mathrm{tg}x = D\,\frac{\sin x}{\cos x} =$$

$$= \frac{D(\sin x)\cos x - \sin x\, D(\cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= 1 + \mathrm{tg}^2 x$$

__ES.__ $D\left(\frac{\cos x}{x^2+1}\right)$

$$= \frac{-[\sin x](x^2+1) - 2x\cos x}{(x^2+1)^2}$$

$$= -\frac{(x^2+1)\sin x + 2x\cos x}{(x^2+1)^2}$$

$$D\left(\frac{x^3-1}{x^4-1}\right) =$$

$$= \frac{3x^2(x^4-1) - (x^3-1)\cdot 4x^3}{(x^4-1)^2}$$

$$= \frac{3x^6 - 3x^2 - 4x^6 + 4x^3}{(x^4-1)^2}$$

$$= \frac{-x^6 + 4x^3 - 3x^2}{(x^4-1)^2}$$

$$D(\cos^2 x) = D(\cos x \cdot \cos x) =$$

$$= -\sin x \cos x + \cos x(-\sin x) =$$

$$= -2 \sin x \cos x$$

$$D\left(\frac{3}{\sin x}\right) = \frac{-3\cos x}{\sin^2 x}$$

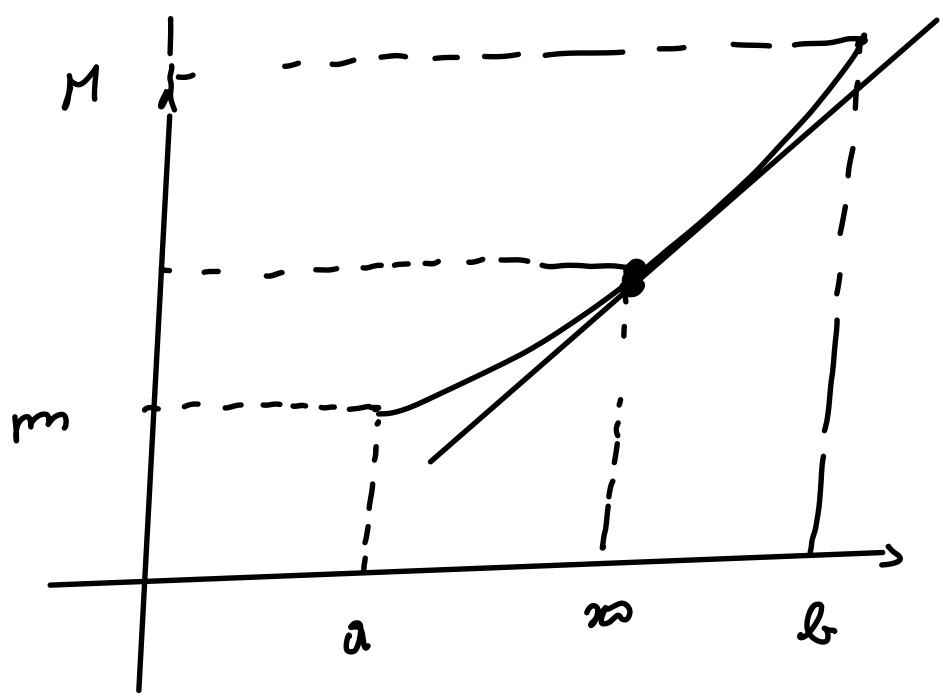
$$= -3 \frac{\cos x}{\sin^2 x}$$

$$D\left(2^x \log_3 x + \frac{\sin x}{x}\right)$$

Teorema di derivazione delle funzioni inverse

$f : [a, b] \to [m, M]$

$f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ strettamente monotona



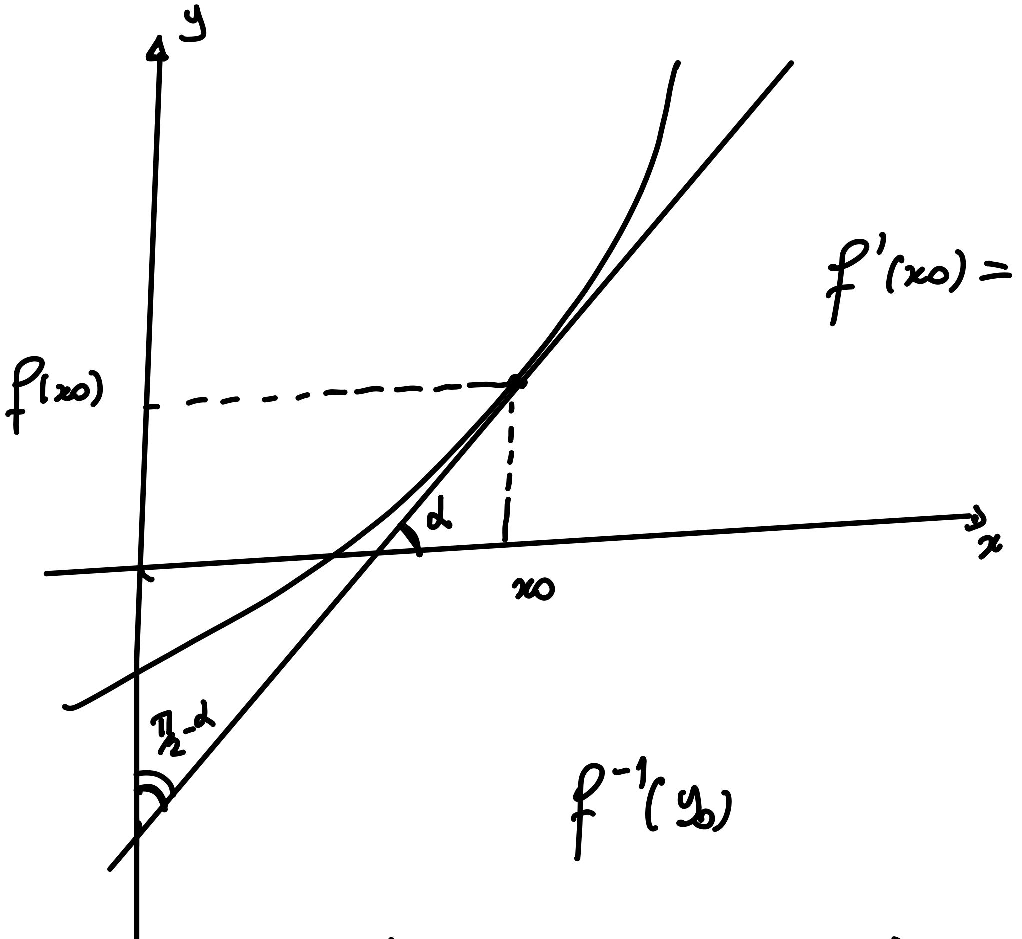
Allora, se f è derivabile in $x_0 \in]a, b[$

e $f'(x_0) \neq 0$, allora $f^{-1} : [m, M] \to [a, b]$

è derivabile in $y_0 = f(x_0)$ e si ha

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

(regola di derivazione delle funzioni inverse)



$f'(x_0) = \operatorname{tg}\alpha$

$f^{-1}(y_0)$

$$(f^{-1})'(y_0) = \operatorname{tg}\left(\frac{\pi}{2}-\alpha\right)$$

$$= \frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\cos\left(\frac{\pi}{2}-\alpha\right)}$$

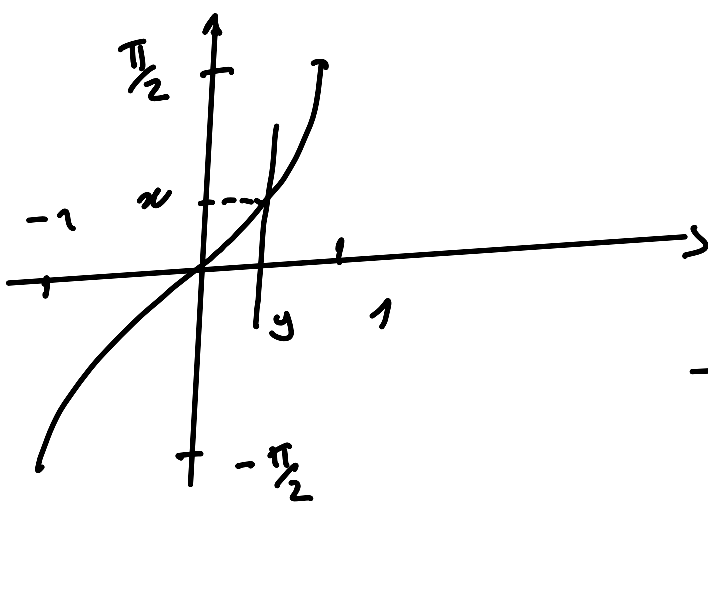
$$= \frac{\cos\alpha}{\sin\alpha} = \frac{1}{tg\alpha}$$

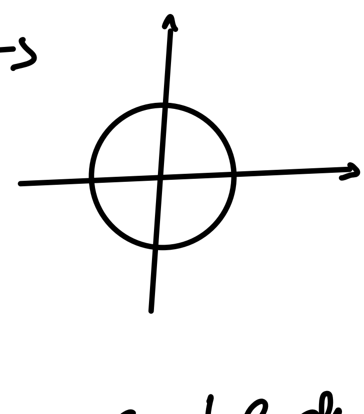
$$= \frac{1}{f'(x_0)}$$

ES.

$f(x) = \sin x \qquad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$f^{-1}(y) = \arcsin y \qquad \forall y \in [-1, 1]$





Sia $y \in]-1,1[$ e sia $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ tale che

$\sin x = y$

$f'(x) = \cos x \neq 0$: applicando il teorema di derivazione delle funzioni inverse, si ha

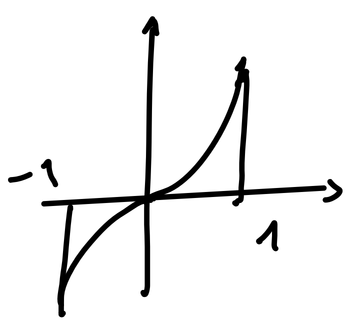
$$(f^{-1})'(y) = \frac{1}{f'(\underbrace{f^{-1}(y)})} = \frac{1}{\cos(\arcsin y)} =$$

$$\cos x = \pm\sqrt{1-\sin^2 x}$$

$$= \frac{1}{\pm\sqrt{1-\sin^2(\arcsin y)}} = \frac{1}{\sqrt{1-y^2}}$$

Quindi $\quad D \arcsin y = \dfrac{1}{\sqrt{1-y^2}}$

$\forall y \in \,]-1, 1[$



$D \arccos y = -\dfrac{1}{\sqrt{1-y^2}} \quad , \forall y \in]-1,1[$

$f(x) = \operatorname{tg} x \qquad f^{-1}(y) = \operatorname{arctg} y \qquad \forall y \in \mathbb{R}$

$\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$

Sia $y \in \mathbb{R}$ e sia $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ tale che

$$\operatorname{tg} x = y$$

$$f'(x) = \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x$$

$$(f^{-1})'(y) = \frac{1}{f'(\operatorname{arctg} y)} = \frac{1}{1 + \operatorname{tg}^2(\operatorname{arctg} y)}$$

$$= \frac{1}{1+y^2}$$

$$D \operatorname{arctg} y = \frac{1}{1+y^2} \qquad \forall y \in \mathbb{R}$$

Teorema di derivazione delle funzioni composte

$\sqrt{\sin x}$ $\qquad$ $\log(\sin x)$

$\operatorname{arctg}(2x)$ $\qquad$ $\dfrac{1}{\log\sqrt{1+\operatorname{tg}^2 x}}$ $\qquad$ $f=f(x)$
$g=g(y)$

Sia $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ derivabile in I

e $g : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ " in J.

Supponiamo che $f(I) \subseteq J$:

$$
\begin{tikzcd}
I \arrow[r, "f"'] \arrow[rr, bend right, "g \circ f"'] & J \arrow[r, "g"'] & \mathbb{R}
\end{tikzcd}
$$

allora, $g \circ f : I \to \mathbb{R}$

$$(g \circ f)(x) = g(f(x)) \quad \text{è}$$

derivabile e inoltre

$$\boxed{(g \circ f)'(x) = g'(f(x)) \cdot f'(x)}$$

$W = g \circ f \qquad f = f(x), \quad g = g(y)$

$$\frac{dW}{dx} = \frac{dg}{dy}(f(x)) \cdot \frac{df}{dx} =$$

$$= \frac{dW}{dy} \cdot \frac{dy}{dx}$$

ES. $f(x) = (\sin x)^3$

$f_1(x) = \sin x$

$g_1(y) = y^3$

$$D(\sin x)^3 = 3y^2\Big|_{y=\sin x} \cdot \cos x = 3\sin^2 x \cos x$$

$D \log_a |x|$ $\qquad$ $\log_a |x| = \begin{cases} \log_a x, & x > 0 \\ \log_a(-x), & x < 0 \end{cases}$

se $x > 0$, $D \log_a |x| = \begin{cases} \frac{1}{x} \log_a e & , x > 0 \\ \text{''} \quad \text{''} & x < 0 \end{cases}$

$$D \log_a (-x) = \frac{1}{(-x)} \cdot \log_a e \cdot (-1)$$

$$= \frac{1}{x} \log_a e$$

$$D \log_a |x| = \frac{1}{x} \log_a e \qquad \forall x \neq 0$$

n.p. $\boxed{D \log |x| = \frac{1}{x}}$

3) $D\, 6^{\cos x} = 6^{\cos x} \log 6 \cdot (-\sin x) =$

$$= -6^{\cos x} \log 6 \cdot \sin x$$

4) $D\left(\sqrt{tg x}\right) = \frac{1}{2\sqrt{tg x}} \cdot \frac{1}{\cos^2 x}$

$D\sqrt{t} = \frac{1}{2\sqrt{t}}$

5) $D \log_2 \left(\left| \cot g x \right| \right) = $ $\quad // -\frac{1}{\sin^2 x}$

$= \frac{1}{\cot g x} \log_2 e \cdot \frac{-\sin x \cdot \sin x - \cos x \cdot \cos^2 x}{\sin^2 x}$

6) $D \cos(5x) = -\sin(5x) \cdot 5 = -5 \sin 5x$

7) $D \log\left(\sqrt{1+\sin x}\right) = \frac{1}{\sqrt{1+\sin x}} \cdot \frac{1}{2\sqrt{1+\sin x}} \cdot \cos x$

DA FARE

1) $D x^x$

$x^x = e^{\log x^x} = e^{x \log x}$

2) $D \cos \log(x^2+1)$

3) $D \sqrt{\operatorname{tg} x \cdot \log(x)}$

4) $D\left(\sqrt{5^x-1} - \sqrt{5^x+1}\right)$

5) $D(\underbrace{2 + \sin^2 x}_{f(x)})^{\sqrt{\log x}\,\} g(x)}$

$$f(x)^{g(x)} = e^{g(x) \log f(x)}$$

6) $D \arcsin \log x$, $D \operatorname{arctg} \frac{x}{x+1}$

7) $D\left(\operatorname{tg} \sqrt{1+e^{x^5}}\right)$