Lezione del 20/12/2022

$A \in M_n(\mathbb{R}) \qquad A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$

Def. Si dice autovalore di A un numero reale $\lambda \in \mathbb{R}$ tale che $\exists\, \underline{v} \in \mathbb{R}^n$, $\underline{v} \neq 0$ per il quale

$$A\underline{v} = \lambda \underline{v}$$

$\underline{v}$ si chiama autovettore associato all'autovalore λ.

OSS. λ autovalore $\Leftrightarrow$ $(A - \lambda I)\underline{v} = 0$, $v \neq 0$

$$\underbrace{(A - \lambda I)\underline{x} = 0}_{\substack{\text{sistema lineare} \\ n \text{ eq. in } n \text{ incognite}}}$$

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (5-\lambda)(2-\lambda) - 4 = 10 - 5\lambda - 2\lambda + \lambda^2 - 4$$

$= \lambda^2 - 7\lambda + 6 = 0$: $\lambda_{1,2} = \frac{7 \pm 5}{2} \begin{matrix} 1 \\ 6 \end{matrix}$

$\Delta = 49 - 24 = 25$

$\lambda_1 = 1$, $\lambda_2 = 6$ autovalori della matrice.

λ autovalore: gli autovettori associati a λ sono le soluzioni (non nulle) del sistema lineare

$$(A - \lambda I)\underline{x} = 0$$

$\infty^{n - \rho(A - \lambda I)}$ soluzioni

$$A - I = \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \qquad |A - I| = 0$$

autovettori: soluzioni non nulle di

$$(A - I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} 4x + 4y = 0 \\ x + y = 0 \end{cases}$$

$$\iff x + y = 0 \iff x = -y$$

soluzioni sono $(x,y) = (-y,y) \quad \forall y \in \mathbb{R}$

$$= y(-1,1) \quad \text{"}$$

V_1 = autovettori associati a $\lambda = 1$

$$= \{ y(-1,1) : y \in \mathbb{R} \setminus \{0\} \}$$

$\lambda = 6$:
$$A - 6I = \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}$$

Autovettori:
$$(A - 6I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -x + 4y = 0 \\ x - 4y = 0 \end{cases} \Leftrightarrow x = 4y$$

$$(x,y) = (4y, y) = y(4,1) \quad : y \in \mathbb{R}$$

$$A = \begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & -15 \\ 0 & 2 & 8 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -3-\lambda & -15 \\ 0 & 2 & 8-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)\left[-(3+\lambda)(8-\lambda) + 30\right]$$

$$= (1-\lambda)\left[-24 + 3\lambda - 8\lambda + \lambda^2 + 30\right]$$

$$= (1-\lambda)\left[\lambda^2 - 5\lambda + 6\right] = 0$$

$$\Leftrightarrow \quad \lambda = 1 \quad \text{opp.} \quad \lambda^2 - 5\lambda + 6 = 0$$

$$\lambda_{1/2} = \frac{5 \pm 1}{2} \begin{matrix} \nearrow 2 \\ \searrow 3 \end{matrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$$

$$A - I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & -15 \\ 0 & 2 & 7 \end{pmatrix} \leftarrow$$

$$\rho(A - I) = 2$$

$$(A - I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -4y - 15z = 0 \\ 2y + 7z = 0 \\ x = t \end{cases}$$

$$\Leftrightarrow \quad 2\times\begin{cases} 4y + 15z = 0 \\ 2y + 7z = 0 \\ x = t \end{cases} -$$

$$\begin{cases} 4y + 15z = 0 \\ z = 0 \\ x = t \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ z = 0 \\ x = t \end{cases}$$

$$(x, y, z) = (t, 0, 0) = t\,(1, 0, 0), \quad t \in \mathbb{R}$$

Applicazioni lineari

$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ applicazione lineare se

$$L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$$

$$\forall v_1, v_2 \in \mathbb{R}^n, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

$$\begin{cases} L(v_1 + v_2) = L(v_1) + L(v_2) \\ L(\alpha v) = \alpha L(v) \qquad \forall v_1, v_2 \in \mathbb{R}^n \\ \qquad\qquad\qquad\qquad \forall \alpha \in \mathbb{R} \end{cases}$$

$$L(x, y) = (2x, x+y)$$

$$L(\underset{x}{1}, \underset{y}{1}) = (2, 1+1) = (2, 2)$$

$$L(3, 2) = (2 \cdot 3, 3+2) = (6, 5)$$

$v_1 = (x_1, y_1) \quad , \quad V_2 = (x_2, y_2)$

$$\alpha_1 v_1 + \alpha_2 v_2 = (\overbrace{\alpha_1 x_1 + \alpha_2 x_2}^{x}, \overbrace{\alpha_1 y_1 + \alpha_2 y_2}^{y})$$

$$L(\alpha_1 v_1 + \alpha_2 v_2) = (2(\alpha_1 x_1 + \alpha_2 x_2), \alpha_1 x_1 + \alpha_2 x_2 + \alpha_1 y_1 + \alpha_2 y_2)$$

$$= (2\alpha_1 x_1 + 2\alpha_2 x_2, \alpha_1 x_1 + \alpha_1 y_1 + \alpha_2 x_2 + \alpha_2 y_2)$$

$$= (2\alpha_1 x_1, \alpha_1 x_1 + \alpha_1 y_1) + (2\alpha_2 x_2, \alpha_2 x_2 + \alpha_2 y_2)$$

$$= \alpha_1 \underbrace{(2x_1, x_1 + y_1)}_{L(v_1)} + \alpha_2 \underbrace{(2x_2, x_2 + y_2)}_{L(v_2)}$$

$$= \alpha_1 L(v_1) + \alpha_2 L(v_2)$$

<u>Def.</u> (Nucleo di L)

$$N(L) = \mathrm{Ker}(L) = \{ \underline{v} \in \mathbb{R}^n : L(\underline{v}) = \underline{0} \}$$
$$= \{ (0,0) \}$$

$$L(x,y) = (2x, x+y) = (0,0)$$

$$\Longleftrightarrow \begin{cases} x = 0 \\ \underset{\overset{\shortparallel}{0}}{x} + y = 0 \end{cases} \Longleftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\Longleftrightarrow (x,y) = (0,0)$$

ES. $T(x,y) = (\underbrace{x-y}, 2x-2y)$

$N(L)$? $\quad T(x,y) = (0,0)$

$$\begin{cases} x = y \\ \cancel{2x = 2y} \end{cases} \Longleftrightarrow x = y$$

$$(x,y) \in N(L) \Longleftrightarrow x = y$$

$$N(L) = \{ \underset{= x(1,1)}{(x,x)} : x \in \mathbb{R} \} \neq \{ 0 \}$$

$$\dim \mathcal{N}(L) = 1$$

$$L : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \underline{e}_1 = (1,0)$$

$$\underline{e}_2 = (0,1)$$

$$L(1,0) = (a_{11}, a_{21})$$

$$L(0,1) = (a_{12} \ \ a_{22})$$

$$A = \begin{array}{c} \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \end{matrix}} \end{pmatrix} \\ \begin{matrix} L(1,0) & L(0,1) \end{matrix} \end{array}$$

matrice associata

$$L(x,y) = (2x, x+y)$$

$$L(1,0) = (2,1) \qquad L(0,1) = (0,1)$$

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (2x, x+y)$$
$$= L(x,y)$$

<u>In generale</u> se A matrice associata ad L,

scriveremo $L(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix}$

$$L(\underline{x}) = A\,\underline{x} \qquad A \text{ matrice associata ad } L$$

$\operatorname{Ker} L = N(L) = \{$ insieme soluzioni del sistema omogeneo $A\underline{x} = \underline{0}\}$

In particolare, $|A| \neq 0 \iff A\underline{x} = \underline{0}$
ha unica soluzione $\underline{x} = 0 \iff N(L) = \{\underline{0}\}$
$\iff L$ è iniettiva.

L suriettiva $\iff$ $\forall \underline{y} \in \mathbb{R}^n \ \exists \underline{x} \in \mathbb{R}^n$ t.c.

$$L\underline{x} = \underline{y}$$

$$\iff \quad A\underline{x} = \underline{y} \quad \text{ha sempre soluzione}$$

$$\rightsquigarrow$$

$$|A| \neq 0 \qquad \searrow \text{unica soluzione}$$

OSS. L è iniettiva $\iff$ L è suriettiva.

•) $L(x,y) = (x-y,\ x+y)$

$L(x,y) = (3x+3y,\ x+y)$

$L(x,y) = (x, 1)$

lineari
determinare
$N(L)$
e verificare
l'iniettività

Seconda regola di sostituzione

$$\int f(x)\,dx = \left[\int f(\varphi(t))\cdot\varphi'(t)\,dt\right]_{t=\varphi^{-1}(x)}$$

$$x=\varphi(t) \qquad dx=\varphi'(t)\,dt$$

Es.

$$\int \frac{\sqrt{x}}{1+x}\,dx = \qquad t=\sqrt{x}$$

$$x=t^2, \quad dx = 2t\,dt$$

$$= \text{sostituendo} = \int \frac{t}{(1+t^2)}\cdot 2t\,dt =$$

$$= 2\int \frac{t^2}{1+t^2}\,dt = 2\int \frac{(1+t^2)-1}{1+t^2}\,dt$$

$$= 2\int 1\,dt - 2\int \frac{dt}{1+t^2} = 2\left(t-\operatorname{arctg} t\right)_{t=\sqrt{x}} + c$$

$$= 2\left(\sqrt{x}-\operatorname{arctg}\sqrt{x}\right)+c$$

$$\int \sqrt{1-x^2}\,dx \qquad x = \sin t \iff t = \arcsin x$$

$$dx = \cos t\,dt$$

$$= \int \sqrt{\underbrace{1-\sin^2 t}_{\cos t}}\;\cos t\,dt = \int \cos^2 t\,dt$$

$$= \int \cos^2 \frac{(2t)}{2}\,dt = \int \frac{1+\cos 2t}{2}\,dt =$$

$$= \frac{1}{2}t + \frac{1}{2}\int \cos 2t\,dt =$$

$$= \frac{1}{2}t + \frac{1}{4}\int \cos(2t)\cdot 2\,dt =$$

$$= \frac{1}{2}t + \frac{1}{4}\sin 2t =$$

$$= \frac{1}{2}(t + \sin t\cos t) =$$

$$= \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + C$$

$$\cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2})$$

$$\int \frac{1}{1+e^x}\,dx \qquad e^x = t$$

$$x = \log t$$

$$dx = 1/t \, dt$$

$$= \int \frac{1}{1+t} \cdot \frac{1}{t}\,dt = \int \frac{dt}{t(t+1)}$$

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

$$\Leftrightarrow 1 = At + A + Bt = (A+B)t + A$$

$$\Leftrightarrow \begin{cases} A+B=0 \\ A=1 \end{cases} \Leftrightarrow B=-1$$

$$\int \frac{1}{t(t+1)} = \int \frac{1}{t}\,dt - \int \frac{dt}{t+1}$$

$$= \log \left| \frac{t}{t+1} \right| \Bigg|_{t=e^x} + c$$

$$= \log \frac{e^x}{e^x+1} + c \qquad , c \in \mathbb{R}$$

$$\int \frac{1+\operatorname{tg} x}{\operatorname{tg}^2 x - 4\operatorname{tg} x + 3}\,dx$$

$t = \operatorname{tg} x$

$x = \operatorname{arctg} t$

$dx = \frac{1}{1+t^2}\,dt$

$$= \int \frac{1+t}{t^2-4t+3} \cdot \frac{dt}{1+t^2}$$

fratti semplici!

$$\int \frac{1}{\sin x + \cos x}\, dx$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$t = \operatorname{tg} \frac{x}{2}$$

$$\frac{x}{2} = \operatorname{arctg} t$$

$$dx = \frac{2}{1+t^2}\, dt$$

$$= \int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$$

$$= 2 \int \frac{dt}{2t - t^2 + 1}$$

Formula di integrazione per parti

$f, g \in C^1([a,b])$

$$\int_a^b f(x) g'(x)\, dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f'(x) g(x)\, dx$$

$$\int_a^b f(x)\, dx = G(b) - G(a)$$

$$G'(x) = f(x)$$

$$\int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{1}{3}$$

$$\int_0^2 \sqrt{x}\, dx = \left[x\sqrt{x} \right]_0^2 = 2\sqrt{2}$$

$$\int_1^2 2x e^{x^2} dx = \left(e^{x^2} \right]_{x=1}^{x=2} = e^4 - e$$

$$\int 2x e^{x^2} dx = e^{x^2} + C$$

$$\int_0^{-1} \frac{x}{1+x^2} dx = -\frac{1}{2} \int_{-1}^{0} \frac{2x}{1+x^2} dx$$

$$= -\frac{1}{2} \left[\log(1+x^2) \right]_{x=-1}^{x=0}$$

$$= -\frac{1}{2} \left[-\log 2 \right] = \frac{\log 2}{2}$$

$$\int_{\frac{\pi}{2}}^{\pi} \sin x \, dx = -\left[\cos x\right]_{\frac{\pi}{2}}^{\pi} = 1$$

$$\int_0^1 \frac{x-1}{x+1} \, dx = \int_0^1 \frac{(x+1)-2}{x+1} \, dx =$$

$$= \int_0^1 dx - 2\int_0^1 \frac{dx}{x+1}$$

$$= 1 - 2\left[\log|x+1|\right]_0^1$$

$$= 1 - 2\log 2$$

$$\int_0^{\pi} x \sin x \, dx$$

$$\int x \sin x \, dx =$$

$$f(x) = x \qquad g'(x) = \sin x$$

$$f'(x) = 1 \qquad g(x) =$$

$$= -\cos x$$

$$= -x\cos x + \int \cos x \, dx =$$

$$= -x\cos x + \sin x + C$$

$$\int_0^{\pi} x \sin x \, dx = \left[\sin x - x\cos x\right]_0^{\pi}$$

$$= -\pi(-1) = \pi$$

Formula di sostituzione

$f : [a,b] \to \mathbb{R}$ continua ,

$\varphi : t \in [c,d] \to x = \varphi(t) \in [a,b]$

di classe $C^1([c,d])$ tale che

$\varphi'(t) \neq 0, \quad \forall t \in [a,b].$

Allora

$$\int_a^b f(x)\,dx = \int_c^d f(\varphi(t))\,|\varphi'(t)|\,dt$$

Ex.

$$\int_0^1 \frac{e^x}{e^{2x}+1}\,dx = \qquad \begin{array}{l} e^x = t \\ dx = \frac{dt}{t} \end{array}$$

$$= \int_1^e \frac{t}{1+t^2}\frac{dt}{t} = \left[\operatorname{arctg} t\right]_1^e =$$

$= \operatorname{arctg} e - \pi/4$

THE END!