Lezione del 02/12/22

$\frac{0}{0}$ $\frac{\infty}{\infty}$ $\lim_{x \to x_0} \frac{f(x)}{g(x)}$

Il teorema di de l'Hospital

$I \subseteq \mathbb{R}$ intervallo, x_0 di accumulazione per I (al finito o no) $f(x)$, $g(x)$ funzioni definite in I, il punto x_0 al più escluso, e derivabili in $I \setminus \{x_0\}$. Supponiamo inoltre che $g(x) \neq 0$, $g'(x) \neq 0$ $\forall x \in I \setminus \{x_0\}$.

Consideriamo i rapporti $\frac{f(x)}{g(x)}$ $\frac{f'(x)}{g'(x)}$

si supponga che il limite del primo rapporto, per $x \to x_0$, si presenti nella forma indeterminata $\frac{0}{0}$ o $\frac{\infty}{\infty}$.

Allora, <ins>se</ins> esiste $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$, esiste anche

$\lim_{x \to x_0} \frac{f(x)}{g(x)}$ e

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

1) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$ f.i. $\left[\frac{0}{0}\right]$

$\parallel H$

$$\lim_{x\to 0} \frac{1-\cos x}{3x^2} = \frac{1}{3} \lim_{x\to 0} \overbrace{\frac{1-\cos x}{x^2}}^{\to \frac{1}{2}} = \frac{1}{6}$$

2) $\lim_{x\to 0} \frac{x + x^2 \cos 1/x}{x}$ f.i. $[0/0]$

prodotto di una funzione infinitesima per una limitata $\Rightarrow$ tende a 0.

$$= \lim_{x\to 0} \left(\underset{\downarrow\, 1}{1} + \underset{\downarrow\, 0}{x \cos 1/x}\right) = 1$$

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \left(1 + 2x \cos 1/x + \cancel{x^2} \left(-\sin \tfrac{1}{x}\right)\left(-\tfrac{1}{\cancel{x^2}}\right)\right)$$

$$= \lim_{x\to 0} \left(1 + \underbrace{2x \cos 1/x}_{\downarrow} + \underbrace{\sin 1/x}_{\text{non esiste}}\right)$$ non esiste!

OSS. Se $\nexists \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ non si può concludere che $\nexists \lim_{x\to x_0} \frac{f(x)}{g(x)}$

3) $\lim_{x\to 0} \frac{x - tg x}{x^2}$ $\left[\frac{0}{0}\right]$

$\frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$
$= 1 + tg^2 x)$

$\overset{H}{=} \lim_{x\to 0} \frac{1 - (1 + tg^2 x)}{2x}$

$= -\lim_{x\to 0} \frac{tg^2 x}{2x} = -\frac{1}{2}\lim_{x\to 0} \underbrace{(tg x)}_{\to 0} \underbrace{\left(\frac{tg x}{x}\right)}_{\to 1} = 0$

$\lim_{x\to 0} \frac{tg x}{x} = 1$

4) $\lim_{x\to +\infty} \frac{2^x}{x^2} = +\infty$ f.i.c. $\left[\frac{+\infty}{+\infty}\right]$

$\left[\frac{\infty}{\infty}\right]$

$\overset{H}{=} \lim_{x\to +\infty} \frac{2^x \log 2}{2x} = \frac{\log 2}{2} \lim_{x\to +\infty} \frac{2^x}{x}$

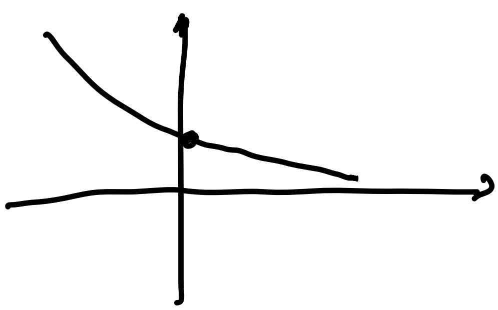
$= \frac{\log 2}{2} \underbrace{\lim_{x\to +\infty}}_{\to +\infty} (2^x \log 2) = +\infty$

5) $\lim_{x \to +\infty} \frac{\log x}{x^3} = 0$

$\overset{H}{=} \lim_{x \to +\infty} \frac{1/x}{3x^2} = \lim_{x \to +\infty} \frac{1}{3x^3} = 0$

a) $\lim_{x \to +\infty} \frac{a^x}{x^\alpha} = +\infty$ $\quad \forall a > 1$, $\forall \alpha > 0$

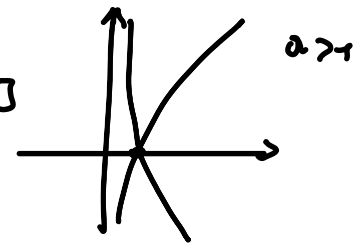
b) $\lim_{x \to -\infty} \frac{a^x}{|x|^\alpha} = +\infty$ $\quad \forall a \in (0,1)$, $\forall \alpha > 0$



c) $\lim_{x \to +\infty} \frac{\log_a x}{x^\alpha} = 0$ $\quad \forall a > 0, a \neq 1$, $\forall \alpha > 0$

d) $\lim_{x \to 0^+} x^\alpha \log_a x = 0$ $\quad \forall \alpha > 0$, $\forall a > 0, a \neq 1$

d) $\lim_{x\to 0^+} x^{\alpha} \underbrace{\log_a x}_{\to -\infty}$ $\quad [0\cdot \pm\infty]$



$\left[\frac{0}{0}\right] \quad \left[\frac{\infty}{\infty}\right]$

$$= \lim_{x\to 0^+} x^{\alpha} \log\left(\frac{1}{x}\right)^{-1} =$$

$$= -\lim_{x\to 0^+} x^{\alpha} \log\frac{1}{x} = -\lim_{x\to 0^+} \frac{\log\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)^{\alpha}}$$

$t = \frac{1}{x}$: $\quad x\to 0^+, \quad t\to +\infty$

$$= -\lim_{t\to +\infty} \frac{\log t}{t^{\alpha}} = 0$$

Quindi si ha la (d).

e) $\lim_{x\to 0} |x|^{\alpha} \log_a |x| = 0 \qquad \forall a>0, a\neq 1, \quad \forall \alpha>0$

$t = |x|$

6) $\lim_{x \to 0} \frac{e^{-1/x^2}}{x}$

$= \lim_{x \to 0} e^{-1/x^2} \left(3x^{-3} \right) =$

$= 3 \lim_{x \to 0} \frac{e^{-1/x^2}}{x^3} \ldots$

Per sostituzione: $t = 1/x^2 \Leftrightarrow$

$\Leftrightarrow x^2 = 1/t \Leftrightarrow x = \pm \frac{1}{\sqrt{t}}$

$\lim_{x \to 0^+} \frac{e^{-1/x^2}}{x} = \lim_{t \to +\infty} \frac{e^{-t}}{1/\sqrt{t}}$

$$= \lim_{t \to +\infty} \frac{\sqrt{t}}{e^t} = 0$$

7) $\lim_{x \to 0^+} x^x \quad [f.i. \; 0^0]$

$$= \lim_{x \to 0^+} e^{\log x^x} = \lim_{x \to 0^+} e^{x \log x}$$

$$= e^{\lim_{x \to 0^+} \underbrace{x \log x}_{\to 0}} = e^0 = 1$$

8) $\lim_{x \to 0} \frac{\sin^3 x}{x - \operatorname{arctg} x} \quad \left[\frac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{\sin^3 x}{x^3} \; \frac{x^3}{x - \operatorname{arctg} x} =$$

$$= \lim_{x \to 0} \frac{x^3}{x - \operatorname{arctg} x} \overset{H}{=}$$

$$= \lim_{x \to 0} \frac{3x^2}{1 - \frac{1}{1+x^2}} = \lim_{x \to 0} \frac{3x^2}{\frac{\cancel{1}+x^2-\cancel{1}}{1+x^2}}$$

$$= 3 \lim_{x \to 0} \cancel{x^2} \cdot \frac{(1+x^2)}{\cancel{x^2}} = 3$$

DA SVOLGERE :

$$\lim_{x \to +\infty} \frac{\log(e^x - 1)}{x}$$

$$\lim_{x \to 0} \frac{x(\cos x - 1)}{\sin x - x}$$

9) $\lim_{x\to 1} (1-x)\, tg\, \frac{\pi x}{2}$ $\qquad [0 \cdot \infty]$

$$= \lim_{x\to 1} \frac{1-x}{\cancel{\frac{1}{tg\left(\frac{\pi}{2}x\right)}}} = \lim_{x\to 1} \frac{1-x}{cotg\left(\frac{\pi}{2}x\right)} \quad \left[\frac{0}{0}\right]$$

$$\overset{H}{=} \lim_{x\to 1} \frac{-1}{-\cancel{\frac{1}{sin^2\left(\frac{\pi}{2}x\right)}}} \cdot \frac{\pi}{2} = \frac{2}{\pi} \lim_{x\to 1} \left(\underbrace{sin^2 \frac{\pi}{2} x}_{\to 1}\right) =$$

$$= \frac{2}{\pi}$$

$$D\, cotg\, x = D\left(\frac{cos\, x}{sin\, x}\right) = \frac{-sin^2 x - cos^2 x}{sin^2 x} =$$

$$= -\cancel{\frac{1}{sin^2 x}}$$

10) $\lim_{x\to 0} \left(\frac{1}{sin\, x} - \frac{1}{x}\right)$ $\qquad [+\infty - \infty]$

$$= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \qquad \left[\frac{0}{0}\right]$$

$$\overset{H}{=} \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{x\left(\underbrace{\frac{\sin x}{x}}_{\to 1} + \underbrace{\cos x}_{\to 1}\right)} = \frac{1}{2} \underbrace{\lim_{x \to 0} \frac{1 - \cos x}{x}}_{=0} = 0$$

$$\frac{\sin x}{x} + \cos x \to 2$$

$$\lim_{x \to \pi/2} \frac{\pi - 2x}{\cos x} \qquad ; \qquad \lim_{x \to 0} \frac{x - \operatorname{arctg} x}{\arcsin x - x} \quad [2]$$

$[2]$

Ordine di infinitesimo, ordine di infinito

$f(x)$ infinitesimo in $x_0 \in \overline{\mathbb{R}}$ se

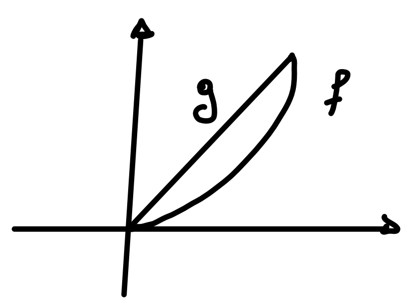
$$\lim_{x \to x_0} f(x) = 0$$

$f(x), g(x)$ definite in $I \subseteq \mathbb{R}$, escluso al più x_0, di accumulazione per I. Si supponga che

$$g(x) \neq 0 \quad \text{in} \quad I \setminus \{x_0\}.$$

Supponiamo che $\displaystyle\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0.$

$f = x^2$
$g = x$



1. $$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

$f(x)$ è un infinitesimo di <u>ordine</u> superiore rispetto a $g(x)$ in x_0

$f = o(g)$

o piccolo di g

$x^2 = o(x)$ $\qquad$ $x^3 = o(x^2)$

2. $\displaystyle\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| = +\infty$

$f(x)$ è un infinitesimo di ordine inferiore a $g(x)$ in x_0

$$\lim_{x \to 0^+} \frac{x}{x^2} = \lim_{x \to 0^+} \frac{1}{x} = +\infty$$

3.) $\displaystyle\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| = l > 0 \, , \; l < +\infty$

$f(x), g(x)$ infinitesimi dello stesso ordine o confrontabili in x_0

$f(x) = \sin x$ $\qquad$ $g(x) = x$ $\quad , x_0 = 0$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 = l$$

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

4.) $\lim_{x\to x_0} \left| \frac{f(x)}{g(x)} \right|$ non esiste : $f(x), g(x)$

non sono confrontabili

$$x \to 0$$

$$f(x) = x \sin \frac{1}{x} \to 0$$

$$g(x) = x$$

$\frac{f(x)}{g(x)} = \sin \frac{1}{x}$ non esiste

$$\lim_{x\to 0} \frac{x^3 + \sin x}{x^2 + tg^3 x} \quad \left[\frac{0}{0}\right]$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{x^3}{\sin x} =$$

$$= \lim_{x\to 0} \underbrace{\left(\frac{x}{\sin x}\right)}_{\to 1} \cdot \underbrace{x^2}_{\to 0} = 0$$

$$x^3 = o(\sin x)$$

$$\lim_{x\to 0} \frac{\operatorname{tg} x}{x} = 1 \quad ; \quad \lim_{x\to 0} \frac{\operatorname{tg}^3 x}{x^2} =$$

$$\operatorname{tg}^3 x = o(x^2) \qquad = \lim_{x\to 0} \underbrace{\left(\frac{\operatorname{tg}^2 x}{x^2}\right)}_{\to 1} \cdot \operatorname{tg} x = 0$$

$$\lim_{x\to 0^+} \frac{x^3 + \sin x}{x^2 + \operatorname{tg}^3 x} =$$

$$= \lim_{x\to 0^+} \frac{\sin x \overbrace{\left(\frac{x^3}{\sin x} + 1\right)}^{\to 1}}{x^2 \underbrace{\left(1 + \frac{\operatorname{tg}^3 x}{x^2}\right)}_{\to 1}} =$$

$$= \lim_{x\to 0^+} \frac{\sin x}{x^2} = \lim_{x\to 0^+} \overbrace{\frac{\sin x}{x}}^{\to 1} \cdot \left(\frac{1}{x}\right)$$

$$= +\infty$$

Proposizione $f_1(x), \dots, f_m(x)$

$g_1(x), \dots, g_m(x)$

infinitesime in x_0. Supponiamo che

$f_i(x)$ di ordine inferiore

a $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_m$

$g_i(x)$ di ordine inferiore

a $g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_m$:

$$\lim_{x \to x_0} \frac{f_1(x) + \dots + f_m(x)}{g_1(x) + \dots + g_m(x)} =$$

$$= \lim_{x \to x_0} \frac{f_i(x)}{g_i(x)}$$

$$\lim_{x \to x_0} |f(x)| = \lim_{x \to x_0} |g(x)| = +\infty$$

<u>**ES.**</u> $x_0 = +\infty$ $\quad f(x) = e^x, \quad g(x) = x^\alpha$.

$x_0 = 0$ $\quad f(x) = e^{1/x^2} \quad g(x) = 1/x^\alpha$

Si possono presentare quattro situazioni:

1) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$: $g(x)$ è un infinito in x_0 di ordine superiore a $f(x)$;

2) $\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| = +\infty$: $f(x)$ è un infinito in x_0 di ordine superiore a $g(x)$;

3) $\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| = \ell > 0$

$f(x)$, $g(x)$ infiniti confrontabili in x_0 di stesso ordine

1) $\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right|$ non esiste, $f(x)$, $g(x)$ non sono confrontabili.

<u>ES.</u> $\lim_{x \to +\infty} \frac{e^x}{x^\alpha} = +\infty$

$\lim_{x \to +\infty} \frac{\log_a x}{x^\alpha} = 0$

<u>Proposizione 2</u> $\lim_{x \to x_0} \frac{f_1(x) + f_2(x)}{g_1(x) + g_2(x)}$

$\lim_{x \to x_0} |f_1(x)| = \lim_{x \to x_0} |g_1(x)| = +\infty$

<u>Se</u> $\lim_{x \to x_0} \frac{f_2(x)}{f_1(x)} = 0$ e $\lim_{x \to x_0} \frac{g_2(x)}{g_1(x)} = 0$

Allora

$$\lim_{x \to x_0} \frac{f_1(x) + f_2(x)}{g_1(x) + g_2(x)} =$$

$$= \lim_{x \to x_0} \frac{f_1(x)}{g_1(x)}$$

ES.

$$\lim_{x \to +\infty} \frac{x^2 + e^x}{\sqrt{x} + \log x}$$

$$= \lim_{x \to +\infty} \frac{e^x}{\sqrt{x}} = +\infty$$

$$\lim_{x \to x_0} f(x) = 0 \qquad x_0 \in \mathbb{R}$$

$|x - x_0|$ infinitesimo del primo ordine

$$\lim_{x \to x_0} \frac{|f(x)|}{|x - x_0|^\alpha} = \ell > 0$$

$\alpha > 0$

Def. Il numero $\alpha > 0$ tale che valga la ⊙ si chiama ordine di infinitesimo di $f(x)$ in x_0

$$\alpha = \text{ord}(f)$$

$f(x) = \sin x \qquad x_0 = 0$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \underline{\alpha = 1}$$

$$\lim_{x \to 0} \left| \frac{\sin x}{x} \right| = \lim_{x \to 0} \frac{|\sin x|}{|x|^1} = 1$$

$f(x) = 1 - \cos x \qquad \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$

$\alpha = 2$

$f(x) = \text{tg}^4 x \quad , x_0 = 0$

$\text{ord} f = 4 \qquad \lim_{x \to 0} \frac{\text{tg}^4 x}{x^4} = 1$

Se $x_0 = \pm\infty$ $\quad \cancel{\frac{1}{|x|}}$

$$\alpha > 0 : \quad \lim_{x \to x_0} \frac{|f(x)|}{1/|x|^\alpha} = l \neq 0$$

$$\lim_{x \to x_0} |x|^\alpha |f(x)| = l \neq 0$$

$f(x) = \frac{1}{x^2}$: $\quad \lim_{x \to +\infty} x^\alpha \cdot \frac{1}{x^2} = 1$

$x_0 = +\infty$

$\alpha = 2$

$\mathrm{ord}\left(1/x^2\right) = 2$

$f \qquad g$ infinitesime in x_0

$\alpha = \operatorname{ord}(f), \qquad \beta = \operatorname{ord}(g)$

<u>Se</u> $\alpha > \beta$ allora f è di ordine superiore a g in x_0.

<u>ES.</u>

$$\lim_{x \to 0} \frac{\overset{1}{\sin x} + \overbrace{(e^x - 1)^3}^{3} + \overbrace{(\log(1+x))^2}^{2}}{\underbrace{\operatorname{tg} x}_{1} + \underbrace{(1 - \cos x)}_{2}} =$$

$$= \lim_{x \to 0} \frac{\sin x}{\operatorname{tg} x} = \dots$$