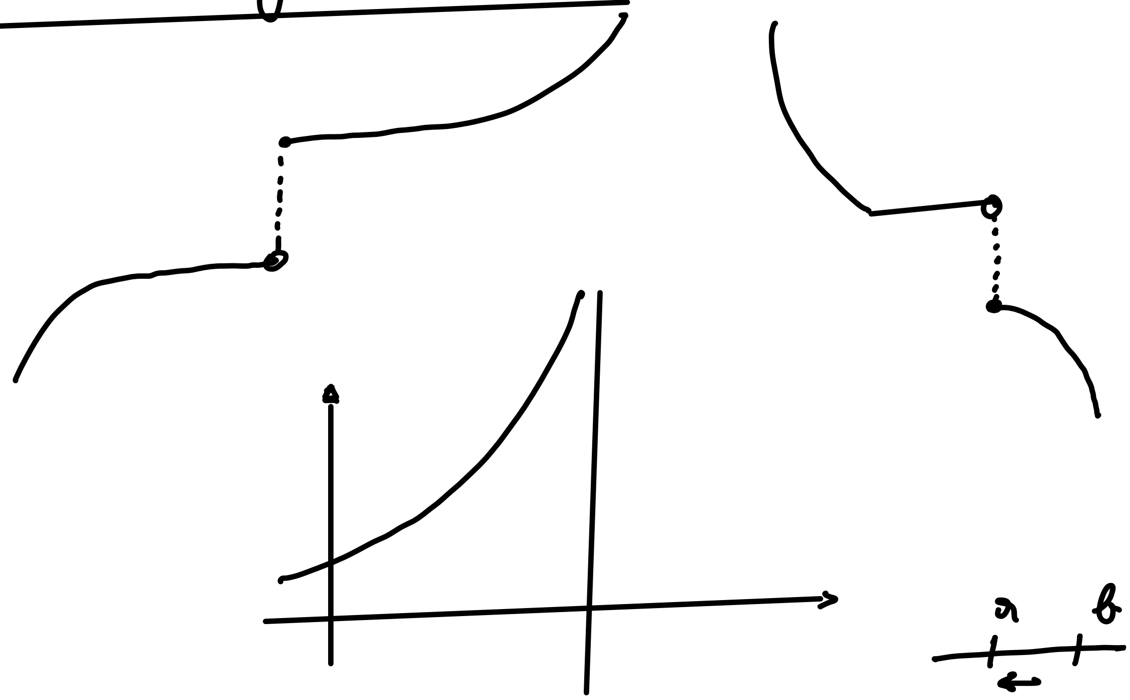
Lezione del 18/11/2022

Continuità delle funzioni monotòne

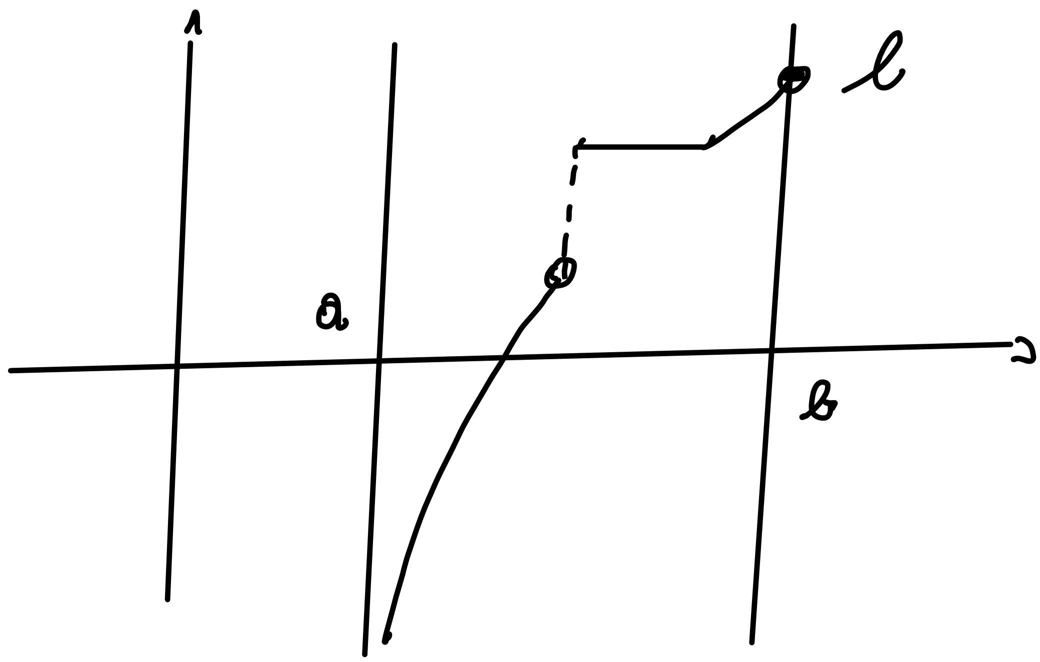


Teorema $f :]a,b[\to \mathbb{R}$ monotòna.

Allora, se f è crescente (risp. decrescente)

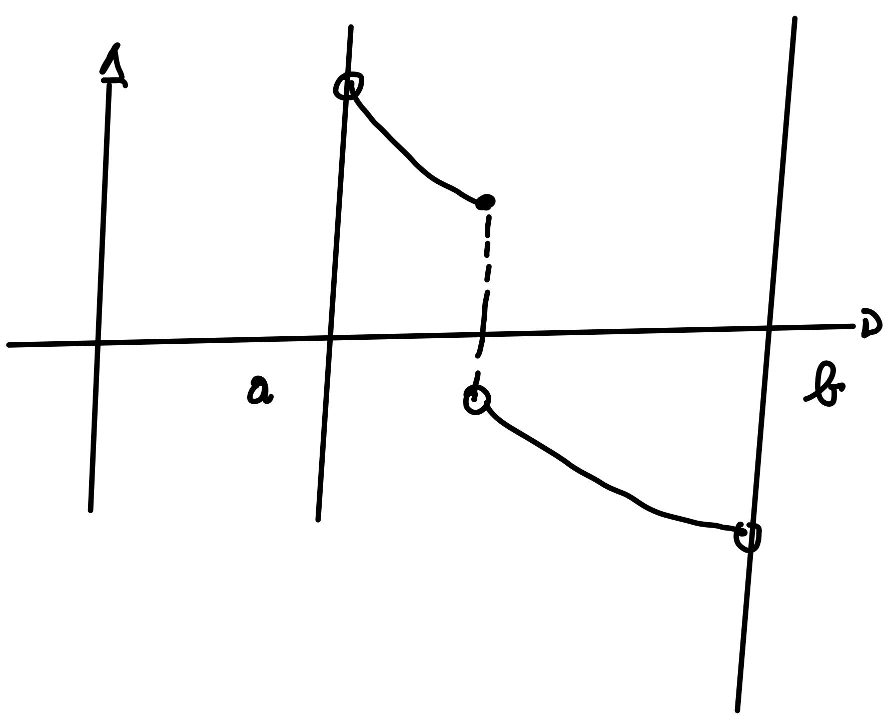
$$\lim_{x \to a^+} f(x) = \inf_{x \in]a,b[} f(x) \quad \left(\text{risp.} = \sup_{]a,b[} f(x)\right)$$

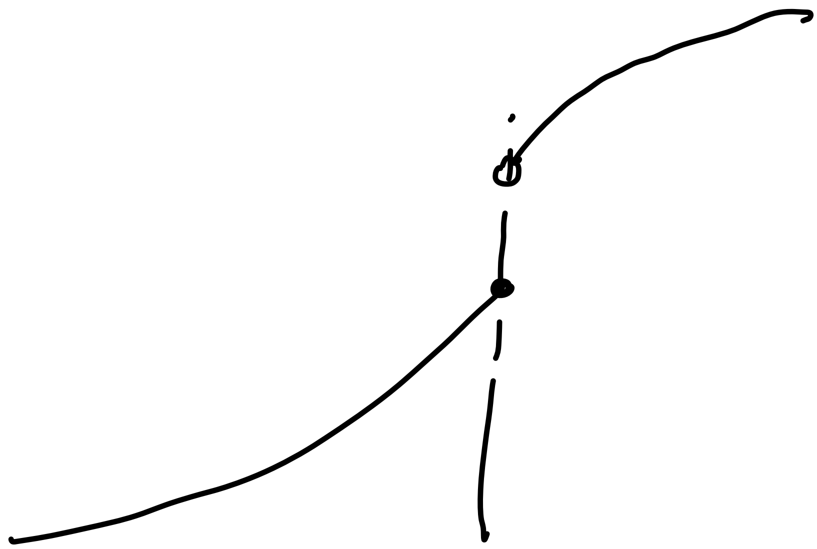
$$\lim_{x \to b^-} f(x) = \sup_{x \in]a,b[} f(x) \quad \left(\text{risp.} \inf_{]a,b[} f(x)\right)$$



$$\lim_{x \to a^+} f(x) = -\infty = \inf_{(a,b)} f(x)$$

$$\lim_{x \to b^-} f(x) = l = \sup_{(a,b)} f(x)$$





<u>Teorema</u> (continuità delle funzioni monotòne)

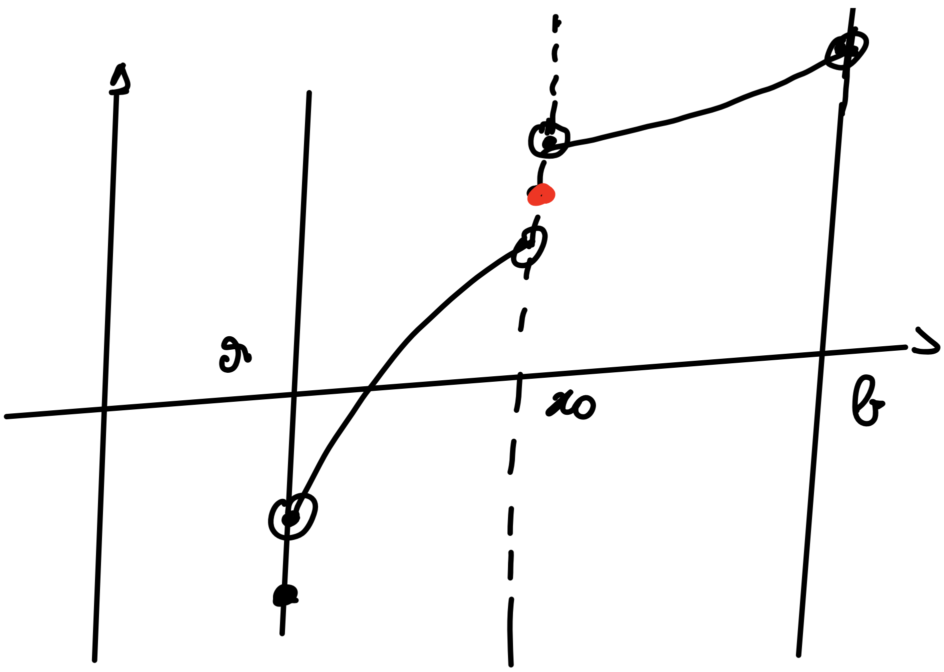
$f:[a,b] \to \mathbb{R}$ monotòna. <u>Allora</u>

se f è <u>crescente</u>:

$$f(a) \le \lim_{x\to a^+} f(x), \qquad \lim_{x \to b^-} f(x) \le f(b).$$

<u>Inoltre</u>, se $x_0 \in \,]a,b[$, si ha

$$f(a) \le \lim_{x\to x_0^-} f(x) \le f(x_0) \le \lim_{x\to x_0^+} f(x) \le f(b)$$

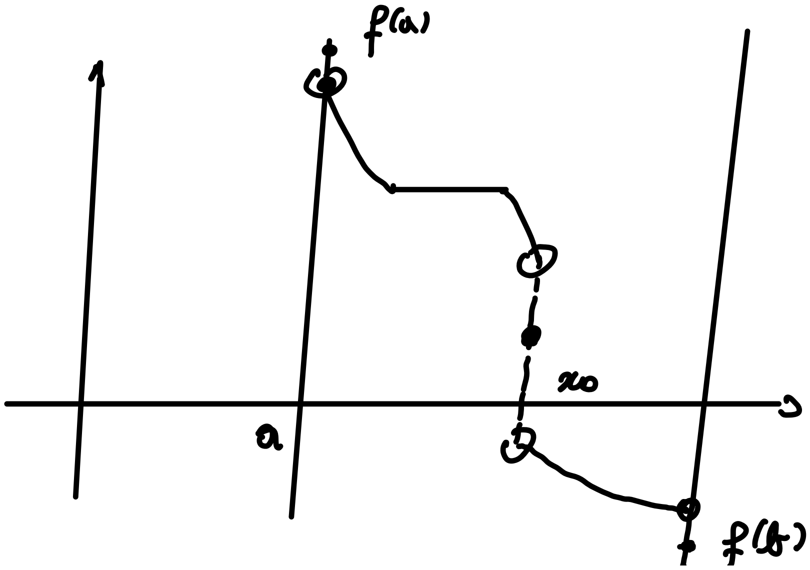


Se invece f è decrescente:

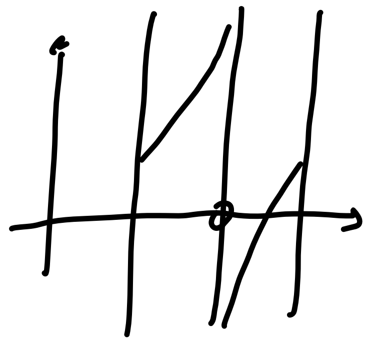
$$f(a) \geq \lim_{x\to a^+} f(x), \qquad f(b) \leq \lim_{x\to b^-} f(x)$$

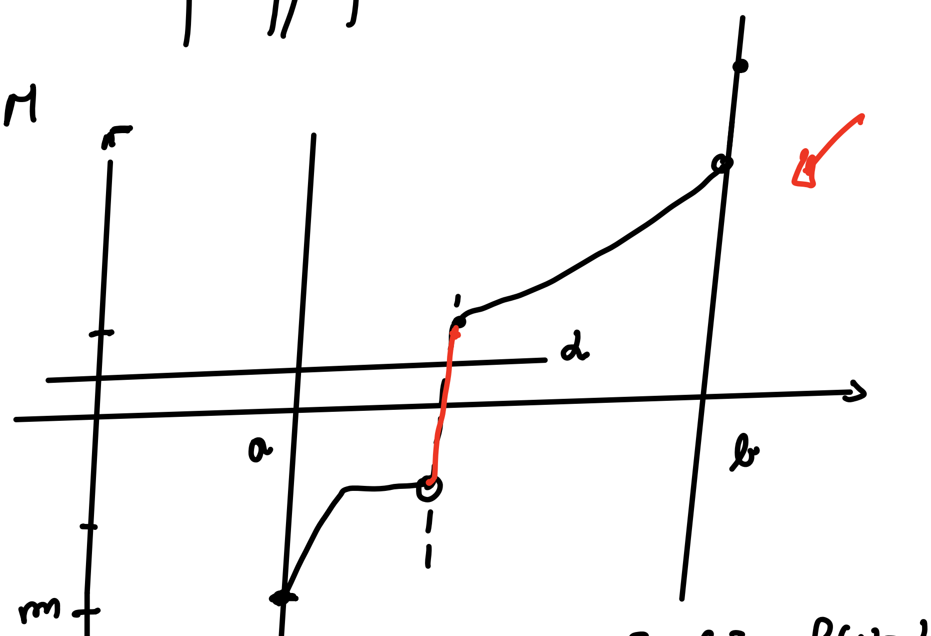
e se $x_0 \in]a,b[$, $\lim_{x\to x_0^-} f(x) \geq f(x_0) \geq \lim_{x\to x_0^+} f(x) \geq f(b^-)$

$f(a) \geq$



OSS. Dal teorema di pume si ha che se $f:[a,b]\to\mathbb{R}$ è monotona, negli estremi ci possono essere al più discontinuità eliminabili; nei punti interni si possono presentare al più discontinuità di I^a specie (salti)





$\nexists\, x \in [a,b] : f(x)=\alpha$

OSS. Se $f:[a,b]\to\mathbb{R}$ è continua, essa assume tutti i valori in $[m, M]$

$$f \text{ continua} \Rightarrow f([a,b]) = [m, M]$$

$\overset{?}{\Leftarrow}$

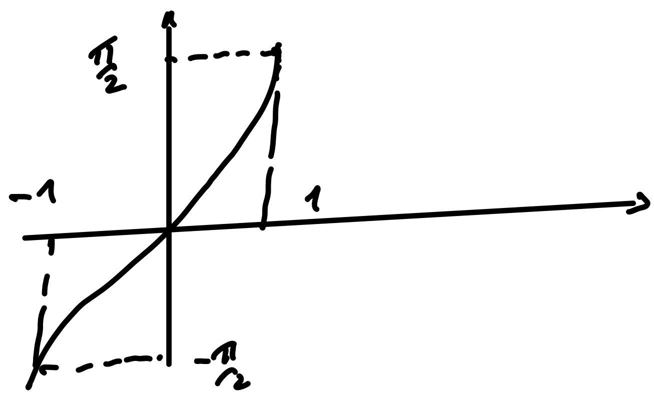
<u>Proposizione</u> Sia $f: [a,b] \to \mathbb{R}$ monotona.

Allora f è dotata di minimo m e massimo M.

Inoltre, se f assume tutti i valori in $[m, M]$,

f è continua in $[a,b]$.

$f(x) = \arcsin x \quad \forall x \in [-1, 1] \Rightarrow f$ è continua



$f(x) = \arccos x$, $f(x) = \operatorname{arctg} x$

Limiti

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} \qquad \text{f.i. } \left[\frac{\infty}{\infty}\right]$$

$$\lim_{x\to+\infty} \frac{x^2+3x+4}{x+1} \qquad \left[\text{f.i. } \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to+\infty} \frac{\cancel{x^2}\,\overbrace{\left(1+\underbrace{\tfrac{3}{x}}_{\to 0}+\underbrace{\tfrac{4}{x^2}}_{\to 0}\right)}^{\to 1}}{\cancel{x}\,\underbrace{\left(1+\tfrac{1}{x}\right)}_{\to 1}} = \lim_{x\to+\infty} x = +\infty$$

$$\lim_{x\to-\infty} \frac{x^3-4x^2+5x-6}{x^2+x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to-\infty} \frac{\cancel{x^3}\,\overbrace{\left(1-\tfrac{4}{x}+\tfrac{5}{x^2}-\tfrac{6}{x^3}\right)}^{\to 1}}{\cancel{x^2}\,\underbrace{\left(1+\tfrac{1}{x}\right)}_{\to 1}} = \lim_{x\to-\infty} x = -\infty$$

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$Q(x) = b_m x + b_{m-1} x + \cdots$$

$$+ \cdots + b_1 x + b_0$$

$$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$$

$$= \frac{a_n}{b_m} \lim_{x \to \pm\infty} x^{n-m}$$

$n > m$

$n < m$

$$= \begin{cases} \infty & n > m \\ 0 & n < m \\ \dfrac{a_n}{b_m} & n = m \end{cases}$$

$$\lim_{x \to -\infty} \frac{x^3 - x^2}{x^4 + 1} =$$

$$= \lim_{x \to -\infty} \frac{\cancel{x^3} \overbrace{\left(1 - \frac{1}{x}\right)}^{\to 1}}{x^4 \underbrace{\left(1 + \frac{1}{x^4}\right)}_{\to 1}} =$$

$$= \lim_{x \to -\infty} \frac{1}{x} = 0$$

$$\frac{a}{\infty} = 0 \qquad \forall a \in \mathbb{R}$$

$$\lim_{x \to +\infty} \frac{3x^3 - 7x^2}{4x^3 + x} = \frac{3}{4}$$

$$\lim_{x\to+\infty} \frac{4x^4-3x^2+x-2}{1-3x^2+x^3-x^4} = \frac{4}{-1} = -4$$

ES. $$\lim_{x\to 1} \frac{x^3-x^2-x+1}{x^2+1} \cdot \frac{x-x^2}{1-2x+2x^3-x^4}$$

$$\left[f.i.\ \frac{0}{0} \right]$$

$$\downarrow$$

$$= \lim_{x\to 1} \frac{x^3-x^2-x+1}{x^2+1} \cdot \frac{x^2-x}{x^4-2x^3+2x-1}$$

$$= \frac{1}{2} \lim_{x\to 1} \frac{(x^3-x^2-x+1)\cdot x(x-1)}{\underbrace{x^4-2x^3+2x-1}} \quad \left[\frac{0}{0}\right]$$

$P(x) = x^4 - 2x^3 + 2x - 1$ ha radice $x_0 = 1$

RUFFINI:

$$
\begin{array}{c|cccc|c}
 & 1 & -2 & 0 & 2 & -1 \\
1 & & 1 & -1 & -1 & 1 \\
\hline
 & 1 & -1 & -1 & 1 & \|
\end{array}
$$

$$P(x) = x^4 - 2x^3 + 2x - 1 = (x^3 - x^2 - x + 1)(x-1)$$

$$\lim_{x \to 1} \dots =$$

$$= \frac{1}{2} \lim_{x \to 1} \frac{\cancel{(x^3 - x^2 - x + 1)}\, x \cdot \cancel{(x-1)}}{\cancel{(x^3 - x^2 - x + 1)}\cancel{(x-1)}}$$

$$= \frac{1}{2} \lim_{x \to 1} x = \frac{1}{2}$$

$$\lim_{x \to x_0} \frac{P(x)}{Q(x)} \qquad Q(x_0) = 0$$

si applica la regola di Ruffini per scomporre $Q(x)$ in fattori primi:

$$Q(x) = (x - x_0)\, R(x)$$

$$\lim_{x \to -2} \frac{\sqrt{x^2 - 8x + 5} - 5}{x+2} \qquad \left[\frac{0}{0}\right]$$

$$\sqrt{x^2 - 8x + 5} + 5$$

$$= \lim_{x \to -2} \frac{\left(\sqrt{x^2-8x+5}-5\right)\left(\sqrt{x^2-8x+5}+5\right)}{(x+2)\left(\sqrt{x^2-8x+5}+5\right)}$$

$$= \lim_{x \to -2} \frac{x^2-8x+5-25}{(x+2)\left(\sqrt{x^2-8x+5}+5\right)}$$

$$= \lim_{x \to -2} \frac{x^2-8x-20}{(x+2)\underbrace{\left(\sqrt{x^2-8x+5}+5\right)}_{5+5=10}}$$

$$= \frac{1}{10} \lim_{x \to -2} \frac{x^2-8x-20}{x+2} \quad \text{f.i.} \left[\tfrac{0}{0}\right]$$

$$x^2-8x-20=0 \qquad \frac{\Delta}{4} = 16+20 = 36$$

$$x_{1/2} = 4 \pm 6 \quad ; \quad x_1 = -2$$
$$x_2 = 10$$

$$x^2 - 8x - 20 = (x+2)(x-10).$$

$$= \frac{1}{10} \lim_{x \to -2} \frac{\cancel{(x+2)}(x-10)}{\cancel{x+2}} =$$

$$= -\frac{12}{10} = -\frac{6}{5}.$$

$$\lim_{x \to 0} \frac{\sin 5x - \sin 3x}{4x} \qquad \left[\frac{0}{0}\right]$$

$$= \frac{1}{4} \lim_{x \to 0} \frac{\sin 5x - \sin 3x}{x} =$$

$$= \frac{1}{4}\left[\underbrace{\lim_{x\to 0}\frac{\sin 5x}{x}}_{?} - \underbrace{\lim_{x\to 0}\frac{\sin 3x}{x}}_{?}\right] \quad (*)$$

$$\lim_{x\to 0}\frac{\sin(x)}{(x)} = 1$$

$$\lim_{x\to 0}\frac{\sin 5x}{x} = \lim_{x\to 0} 5\left(\frac{\sin 5x}{5x}\right) =$$

$$= 5\lim_{x\to 0}\frac{\sin 5x}{5x} = 5\lim_{y\to 0}\frac{\sin y}{y} = 5$$

$$y = 5x : \ x\to 0 \ , \ y \to 0$$

$$(*) = \frac{1}{4}\left[5\underbrace{\lim_{x\to 0}\frac{\sin 5x}{5x}}_{=1} - 3\underbrace{\lim_{x\to 0}\frac{\sin 3x}{3x}}_{=1}\right]$$

$$= \frac{1}{4}(5-3) = \frac{1}{2}$$

$$\lim_{x \to \frac{\pi}{4}} \frac{\cos 2x}{1-\sqrt{2}\sin x} =$$

$\cos 2x = \cos^2 x - \sin^2 x$

$\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cos^2 x - \sin^2 x}{1-\sqrt{2}\sin x} =$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{1-2\sin^2 x}{1-\sqrt{2}\sin x} = \lim_{x \to \frac{\pi}{6}}$$

$(1-2\sin^2 x) = (1-\sqrt{2}\sin x)(1+\sqrt{2}\sin x)$

$2 = (\sqrt{2})^2$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cancel{(1-\sqrt{2}\sin x)}(1+\sqrt{2}\sin x)}{\cancel{1-\sqrt{2}\sin x}} =$$

$$= 1 + \sqrt{2} \cdot \frac{\sqrt{2}}{\cancel{2}} = 2$$

$$\lim_{x \to 0} \frac{x \sin x}{1-\cos^2 x} = \lim_{x \to 0} \frac{x \cancel{\sin x}}{\sin^{\cancel{2}} x} =$$

$$= \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}} = 1$$

$$\lim_{x \to \pi} \frac{\sin x}{1-\left(\frac{x}{\pi}\right)^2}$$

f.i. $\left[\frac{0}{0}\right]$

$$= \lim_{x \to \pi} \frac{\sin x}{\frac{\pi^2 - x^2}{\pi^2}} =$$

$$= \pi^2 \lim_{x \to \pi} \frac{\sin x}{\pi^2 - x^2} =$$

$$= \pi^2 \lim_{x \to \pi} \frac{\sin x}{(\pi - x)\underbrace{(\pi + x)}_{\to 2\pi}} =$$

$$= \frac{\pi^{\cancel{2}}}{2\cancel{\pi}} \lim_{x \to \pi} \frac{\sin x}{\pi - x} = \frac{\pi}{2} \overbrace{\lim_{y \to 0} \frac{\sin y}{y}}^{=1}$$

$$= \frac{\pi}{2}$$

$y = \pi - x$, $\quad x = \pi - y$:

$$\sin x = \sin(\pi - y) = \sin y$$

$$x \to \pi, \quad y \to 0$$

$$\lim_{x\to \pi/4} \frac{\cos x - \sin x}{x - \pi/4} \qquad \text{f.i. } [0/0]$$

$y = x - \pi/4, \quad x = y + \pi/4 \quad : x \to \pi/4, y \to 0$

$$\cos x = \cos(y + \pi/4) = \cos y \cos \pi/4 - \sin y \sin \pi/4$$

$$= \frac{\sqrt{2}}{2} \cos y - \frac{\sqrt{2}}{2} \sin y$$

$$\sin x = \sin(y + \pi/4) = \sin y \cos \pi/4 + \sin \pi/4 \cos y$$

$$= \frac{\sqrt{2}}{2} \sin y + \frac{\sqrt{2}}{2} \cos y$$

$$\cos x - \sin x = -\sqrt{2} \sin y$$

$$\lim_{x\to \pi/4} \dots = \lim_{y\to 0} -\frac{\sqrt{2} \sin y}{y} = -\sqrt{2}$$

$$\lim_{x \to 4} \frac{x^2-3x-4}{x^3-4x^2-16x+64} \overset{(*)}{=} \text{f.i.} \left[\frac{0}{0}\right]$$

$$x^2-3x-4=0 \qquad \Delta = 9+16=25$$

$$x_{1/2} = \frac{3 \pm 5}{2} \begin{cases} -1 \\ 4 \end{cases}$$

$$x^2-3x-4 = (x+1)(x-4)$$

	1	-4	-16	64
4		4	0	-64
	1	0	-16	//

$$x^3-4x^2-16x+64 = (x^2-16)(x-4)$$

$$= (x-4)(x+4)(x-4) = (x-4)^2(x+4)$$

$$\overset{*}{=} \lim_{x \to 4} \frac{(x+1)\cancel{(x-4)}}{(x-4)^{\cancel{2}}(x+4)} =$$

$$= \frac{5}{8} \lim_{x \to 4} \frac{1}{x-4} \quad \underline{\underline{\text{non esiste}}}$$

$$\lim_{y \to 0} \frac{1}{y} \quad \underline{\underline{\text{non esiste}}} \; !$$

<u>ASSEGNO</u>

$$\lim_{x \to 2} \frac{x^3-2x^2-4x+8}{x^3+x^2-2x-8} \qquad [R.\ 0]$$

$$\lim_{x \to 0} \frac{\arcsin x}{x}, \quad \frac{\arccos x}{x}, \quad \frac{\operatorname{arctg} x}{x}$$

$$\lim_{x \to 0} \frac{\arcsin x}{(1-\cos x)^{1/2}}$$

$$\lim_{x \to \pi} \frac{\sin(x-\pi)}{\sin 2x}$$ [R. $\frac{1}{2}$]

$$\lim_{x \to 0} \frac{\log(1+x)}{\sin x}$$