Lezione del 16/11/2022

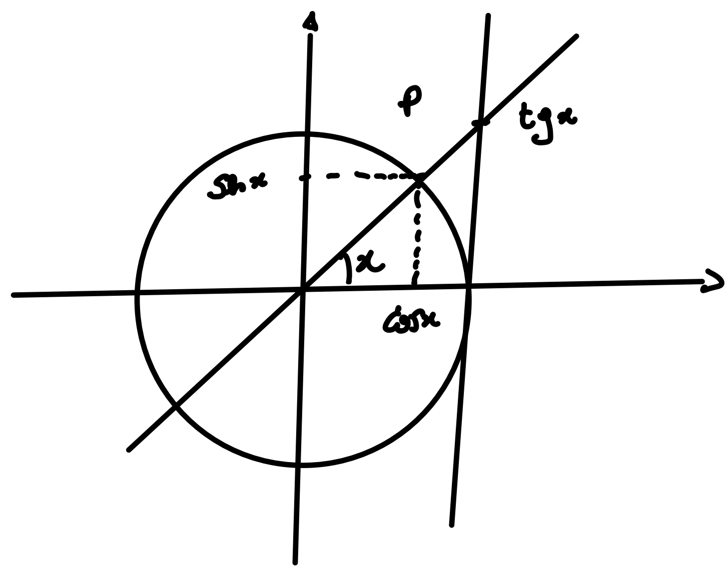
Limiti notevoli

1. $\lim_{x\to 0} \frac{\sin x}{x}$ f.i. $\left[\frac{0}{0}\right]$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Confronto

$$\begin{array}{ccc} f(x) & \le h(x) \le & g(x) \\ \downarrow & \downarrow & \downarrow \; {\scriptstyle x\to x_0} \\ l & l & l \end{array}$$



Se $x \in (0, \frac{\pi}{2})$:

$$\sin x < x < \operatorname{tg} x = \frac{\sin x}{\cos x}$$

dividiamo per $\sin x > 0$ $\quad 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$

(o) $\cos x < \dfrac{\sin x}{x} < 1 \quad , \forall x \in \underset{\sim}{(0}, \frac{\pi}{2})$

Se $x \in (-\frac{\pi}{2}, 0)$, allora $0 < -x < \frac{\pi}{2}$: sostituendo

in (o)

$$\cos(-x) < \frac{\sin(-x)}{-x} < 1$$

$$\begin{aligned} &\parallel \\ &\cos x < \underbrace{\frac{-\sin x}{-x}}_{= \sin x / x} < 1 \end{aligned}$$

In def. $\cos x < \dfrac{\sin x}{x} < 1 \quad \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $x \neq 0$

Poiché $\lim\limits_{x \to 0} \cos x = \cos 0 = 1$

$$\begin{array}{ccccc} \cos x & < & \dfrac{\sin x}{x} & < & 1 \\ \downarrow & & \downarrow & & \downarrow \quad \text{per } x \to 0 \\ 1 & & 1 & & 1 \end{array}$$

dal teorema dei carabinieri.

$a^2 - b^2 = (a+b)(a-b)$

$\sin^2 x + \cos^2 x = 1$

2.) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} \quad [\text{f.i. } \frac{0}{0}]$

$$= \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{(1 + \cos x)x} =$$

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x \to 0} \frac{\sin^2 x}{x\underbrace{(1+\cos x)}_{\substack{\downarrow \\ 1+1=2}}} \stackrel{(*)}{=}$$

$$\left(\lim_{x \to x_0} \alpha f(x) = \alpha \lim_{x \to x_0} f(x)\right)$$

$\alpha \in \mathbb{R}$

$$=^{*} = \frac{1}{2} \lim_{x \to 0} \frac{\sin^2 x}{x} = \frac{1}{2} \lim_{x \to 0} \underbrace{\frac{\sin x}{x}}_{\nearrow 1} \cdot \underbrace{\sin x}_{\nearrow 0}$$

$$= 0$$

$$\lim_{x \to 0} \frac{1-\cos x}{x} = 0$$

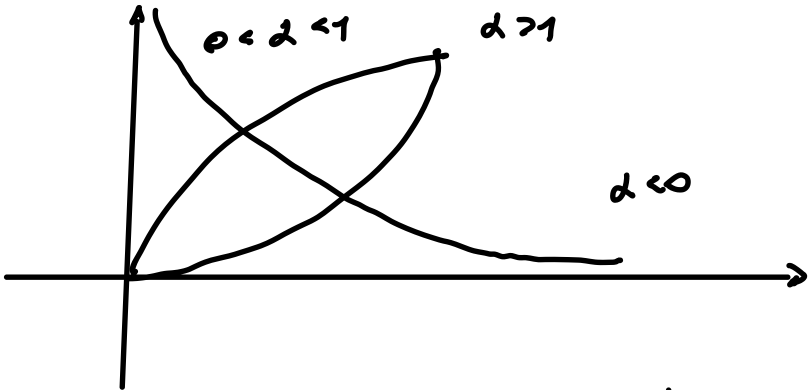
3.) $$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin^2 x}{x^2(1+\cos x)} =$$

$$= \frac{1}{2} \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 =$$

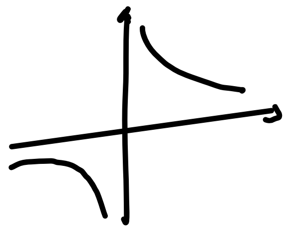
$$= \frac{1}{2} \left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2 = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

4.)



$$\lim_{x\to+\infty} x^{\alpha} = \begin{cases} +\infty & \text{se} \quad \alpha > 0 \\ 0 & \text{se} \quad \alpha < 0 \end{cases}$$



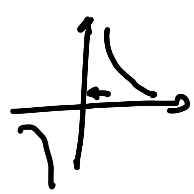
5.) $\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^{x}$

$\frac{1}{x} \xrightarrow{x\to\pm\infty} \frac{1}{\infty} = 0$

f.i. $\underbrace{1}^{\pm\infty}$

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^{x} = e \quad \text{Numero di Nepero}$$

$e = 2,781828\ldots$



6.) $\lim_{x\to0}(1+x)^{1/x}$

$x\to0^+ \quad 1/x \to +\infty \quad ; \ x\to0^- \ , \ 1/x \to -\infty$

$$\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} \left(1+\frac{1}{(1/x)}\right)^{1/x} =$$

$y = 1/x$: per $x \to 0^+$, $y = 1/x \to +\infty$

$$= (\text{limiti per sostituzione}) = \lim_{y\to +\infty} \left(1+\frac{1}{y}\right)^{y} = e$$

$$= \lim_{x\to 0^-} (1+x)^{1/x}$$

Allora $\underline{\underline{\lim_{x\to 0} (1+x)^{1/x} = e}}$

$\log_a x^\alpha = \alpha \log_a x$

7) $$\lim_{x\to 0} \frac{\log_a (1+x)}{x} = \log_a e \quad \text{f.i. } \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{1}{x} \log_a (1+x) = \lim_{x\to 0} \log_a (1+x)^{1/x}$$

$$= \log_a \Big(\underbrace{\lim_{x\to 0} (1+x)^{1/x}}_{= e} \Big) = \log_a e$$

$a = e$ $\qquad \lim_{x \to 0} \frac{\log(1+x)}{x} = \log_e e = 1$

8) $\lim_{x \to 0} \frac{a^x - 1}{x}$ $\qquad [\text{f.i. } \frac{0}{0}]$

$y = a^x - 1 \iff a^x = 1+y$

$x \to 0 \Rightarrow y \to 0 \qquad \iff \quad x = \log_a(1+y)$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log_a(1+y)} =$$

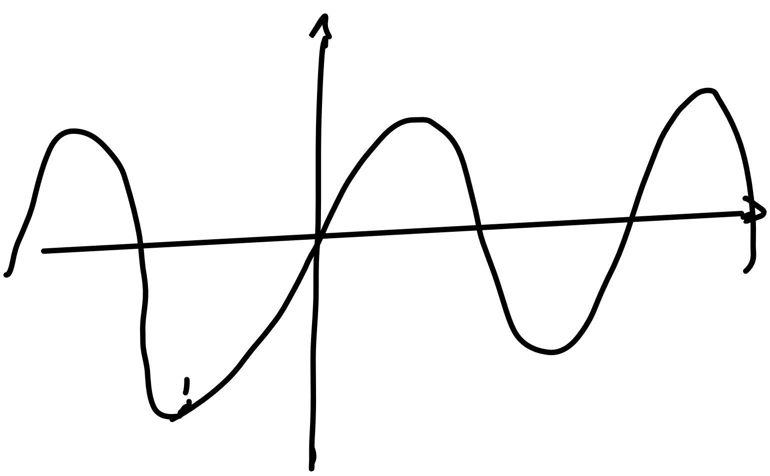
$$= \lim_{y \to 0} \frac{1}{\underbrace{\left(\frac{\log_a(1+y)}{y}\right)}_{\to \log_a e}} = \frac{1}{\log_a e} = \log_e a$$

In particolare: $\lim_{x\to 0} \frac{e^x - 1}{x} = \log e = 1$

g) $\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \qquad \left[\frac{0}{0}\right]$

$\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = 1/2$ $\qquad \alpha = 1/2$

Limiti che non esistono:



$\lim_{x\to \pm\infty} \sin x$ non esiste

$$\lim_{x \to \pm\infty} \cos x \quad \text{non esiste}$$

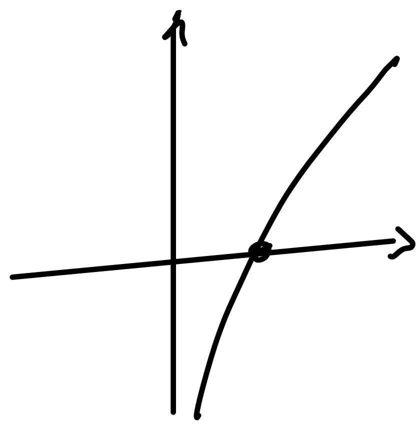
$$\lim_{x \to 0} \sin \frac{1}{x} \qquad y = \frac{1}{x}$$

$x \to 0^+$, $y \to +\infty$

non esiste

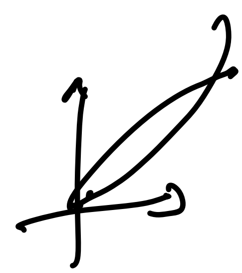
Gerarchia degli infiniti

$$\lim_{x \to +\infty} \log_a x = +\infty$$

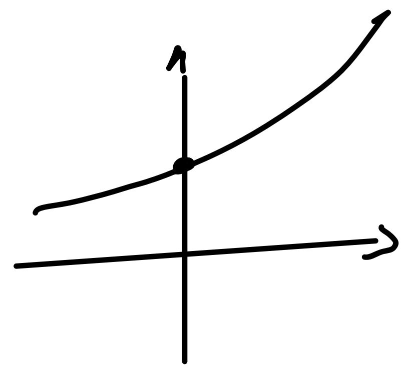


$a > 1$

$$\lim_{x \to +\infty} x^{\alpha} = +\infty \quad , \ \alpha > 0$$



$$\lim_{x \to +\infty} b^x = +\infty$$



Si dimostra che

$$\lim_{x \to +\infty} \frac{x^\alpha}{\log_a x} = +\infty$$

$$\lim_{x \to +\infty} \frac{b^x}{x^\alpha} = +\infty$$

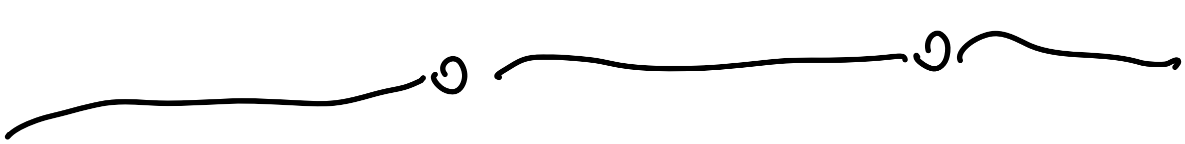
$$\log_a x \ , \quad x^\alpha \ , \quad b^x$$

$$x \to +\infty$$

$$\lim_{x\to+\infty} \frac{\log_a x}{b^x} =$$

$$= \lim_{x\to+\infty} \frac{1}{\dfrac{b^x}{\log_a x} \to +\infty} = 0$$

$$\frac{1}{\pm\infty} = 0$$



Discontinuità

$f: A \subseteq \mathbb{R} \to \mathbb{R}$, $x_0 \in A$. Si dice che f è discontinua in x_0 se non è continua in x_0 $\iff$

o $\lim_{x \to x_0} f(x)$ non esiste oppure

$$\lim_{x \to x_0} f(x) = \ell \neq f(x_0)$$

Classificazione delle discontinuità

(1) Si dice che x_0 è una discontinuità di I^a specie (o salto) se

$$\lim_{x \to x_0^-} f(x) \quad e \quad \lim_{x \to x_0^+} f(x)$$

esistono finiti ma non coincidono

$$\lim_{x \to x_0^-} f(x) = f(x_0^-)$$

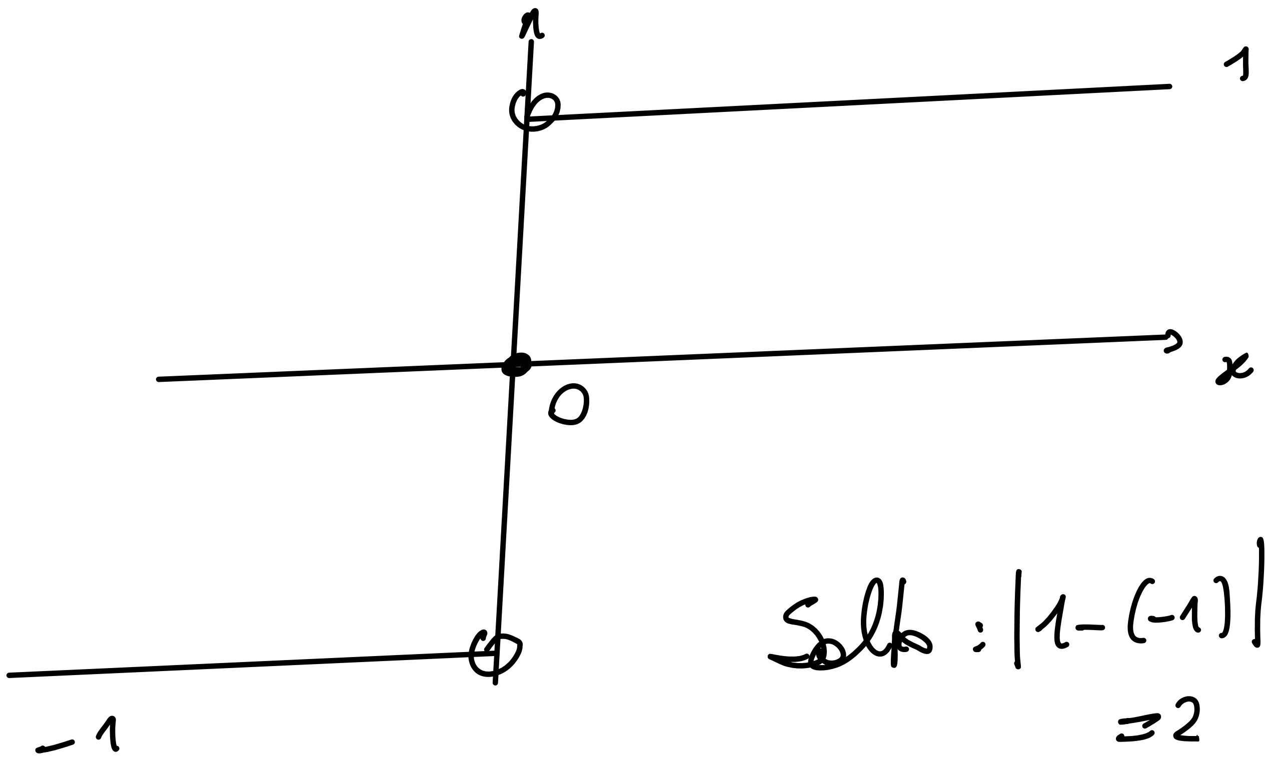
$$\lim_{x \to x_0^+} f(x) = f(x_0^+)$$

$$|f(x_0^+) - f(x_0^-)| \quad \text{salto}$$

di discontinuità.

ES:

$$f(x) = \begin{cases} 1 & \text{se } x > 0 \\ 0 & \text{se } x = 0 \\ -1 & \text{se } x < 0 \end{cases}$$



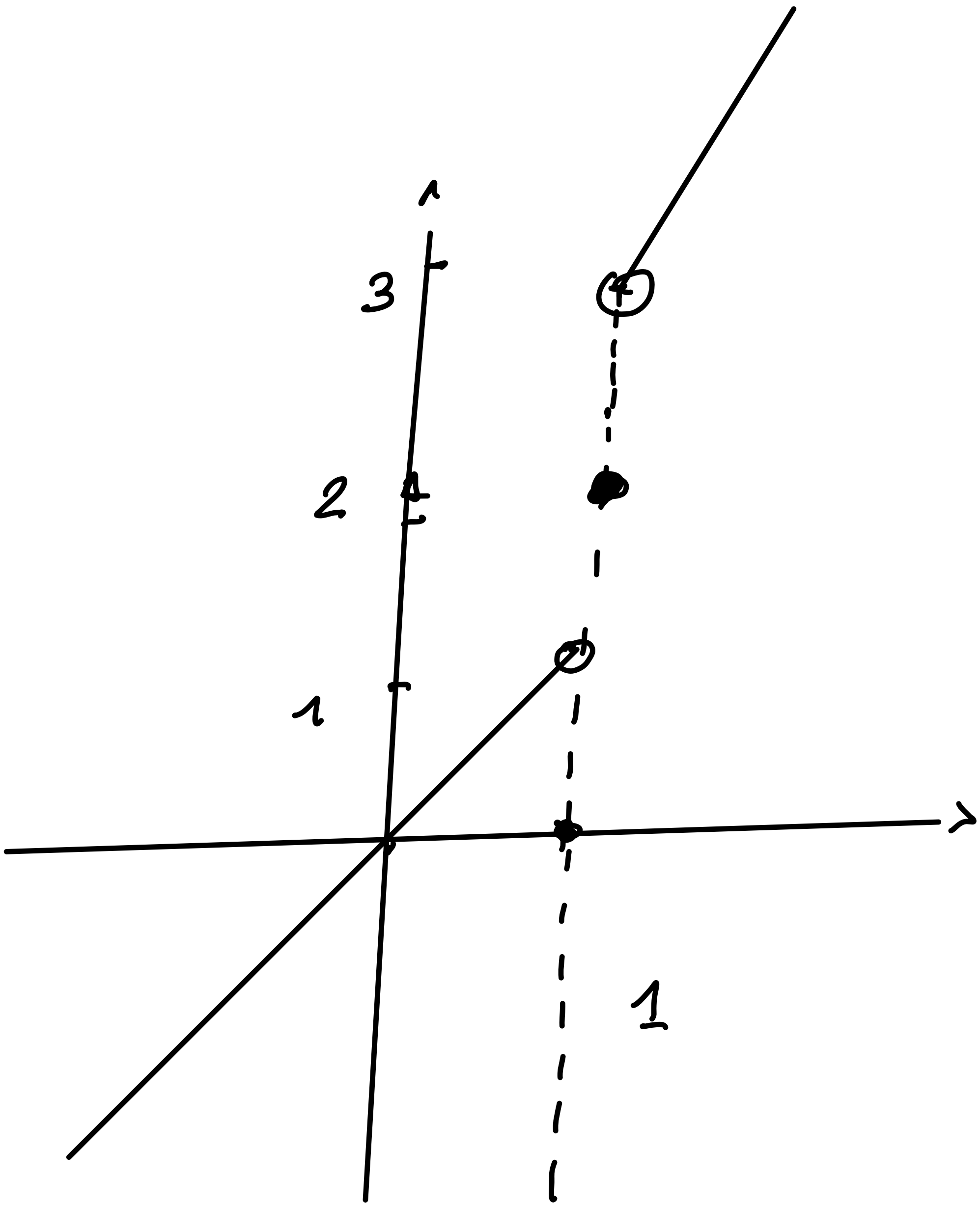
Salto: $|1-(-1)| = 2$

$\lim_{x \to 0} f(x)$ non esiste

$$\lim_{x \to 0^-} f(x) = -1 \quad , \quad \lim_{x \to 0^+} f(x) = 1$$

$$\neq$$

$$f(x) = \begin{cases} x & \text{se } x < 1 \\ 2 & \text{se } x = 1 \\ 2x+1 & \text{se } x > 1 \end{cases}$$



$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x+1) = 3$$

$$2+1$$

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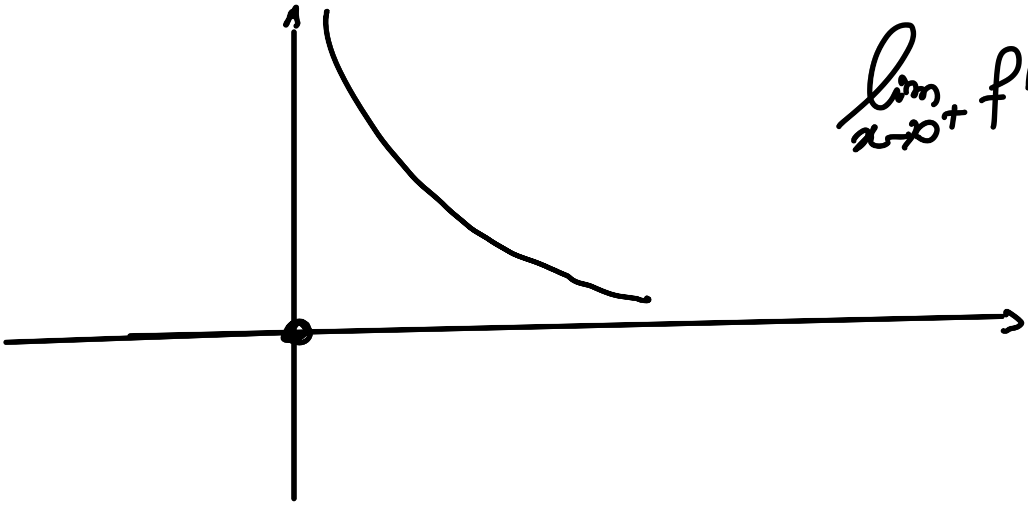
② Si dice che x_0 è una discontinuità di (II^a) specie se almeno uno dei due limiti

$$\lim_{x\to x_0^-} f(x) \qquad \lim_{x\to x_0^+} f(x)$$

non esiste, oppure è infinito

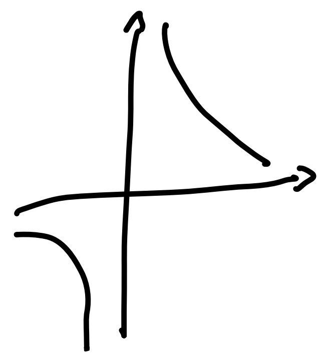
Es. $f(x) = \begin{cases} 1/x & \text{se } x > 0 \\ 0 & \text{se } x \le 0 \end{cases}$



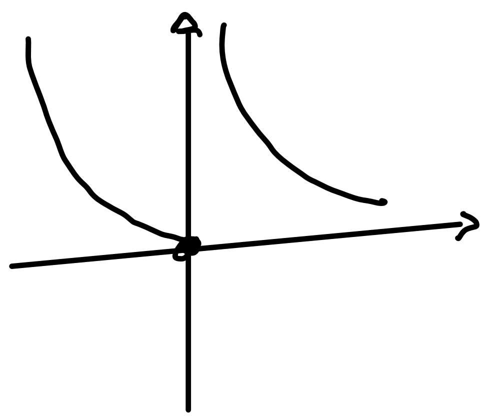
$$\lim_{x \to 0^+} f(x) = +\infty$$

$f(x) = 1/x$

$x = 0$



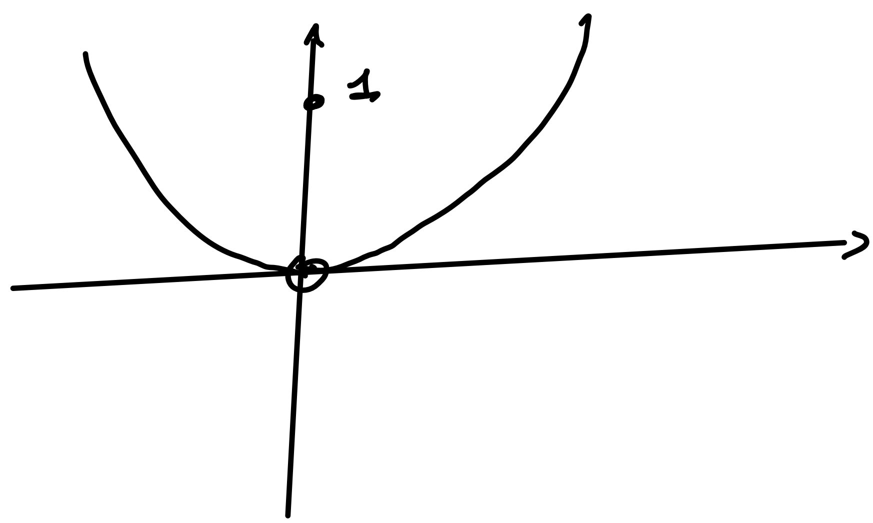
$$f(x) = \begin{cases} 1/x & \text{si } x > 0 \\ x^2 & \text{si } x \leq 0 \end{cases}$$



③ Si dice che x_0 è una discontinuità di III^a specie se

$$\exists \lim_{x \to x_0} f(x) = l \neq f(x_0)$$

ES. $f(x) = \begin{cases} x^2 & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases}$ $\qquad f(0) = 1$



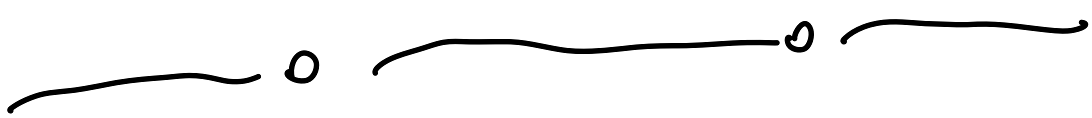
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0 \neq 1 = f(0)$$

ES $\bar{f}(x) = \begin{cases} x^2 & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases} = x^2$

Discontinuità "eliminabili", nel senso che si può "eliminare" la discontinuità ponendo

$$\bar{f}(x) = \begin{cases} f(x) & \text{se } x \neq x_0 \\ \ell & \text{se } x = x_0 \end{cases}$$

$\bar{f}$ è continua in x_0



Ci possono essere delle situazioni in cui $f(x)$ <u>non</u> è definita in x_0, ma si ha

$$\exists \lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$$

Si definisce

$$\bar{f}(x) = \begin{cases} f(x) & \text{se } x \neq x_0 \\ \ell & \text{se } x = x_0 \end{cases}$$

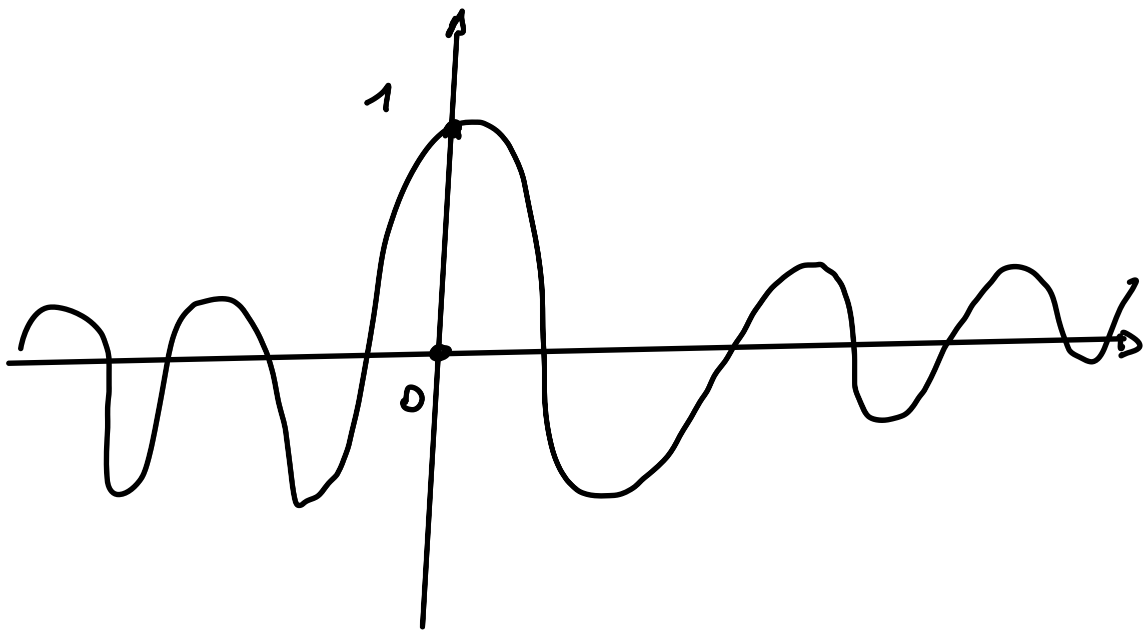
$\bar{f}$ è <u>continua</u> in x_0

"prolungamento per continuità di f in x_0"

<u>ES</u>. $\Rightarrow$ $f(x) = \dfrac{\sin x}{x}$ $\quad x \neq 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\bar{f}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases}$$



$$\lim_{x\to\pm\infty} \frac{\sin x}{x} = \lim_{x\to\pm\infty} \frac{1}{x}\cdot \sin x = 0$$

$\frac{1}{x} \to 0$

$\sin x \to |\sin x| \leq 1$

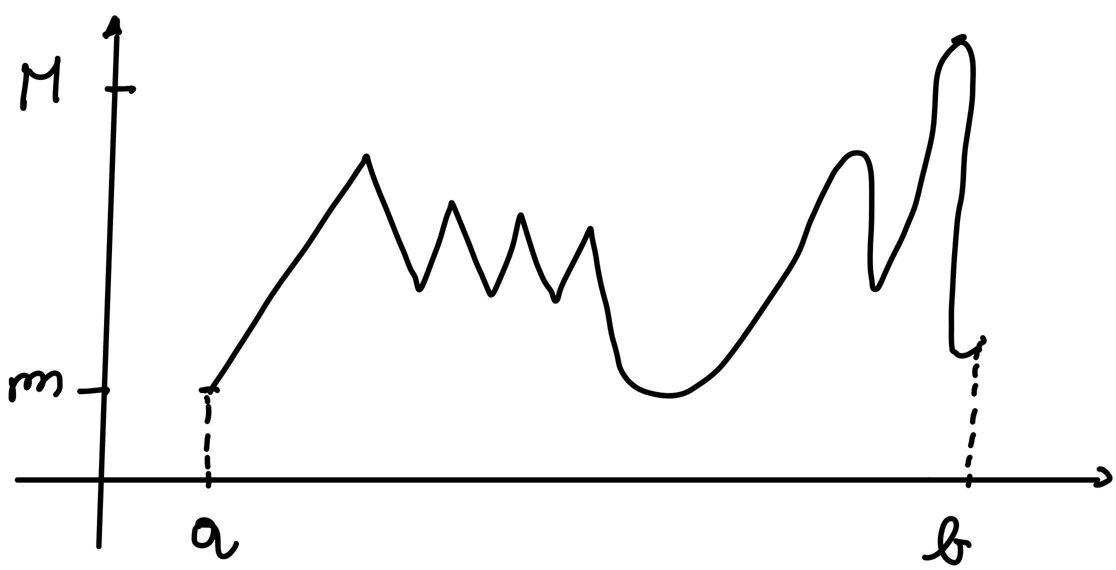
Prop. Se $\lim_{x\to x_0} f(x) = 0$

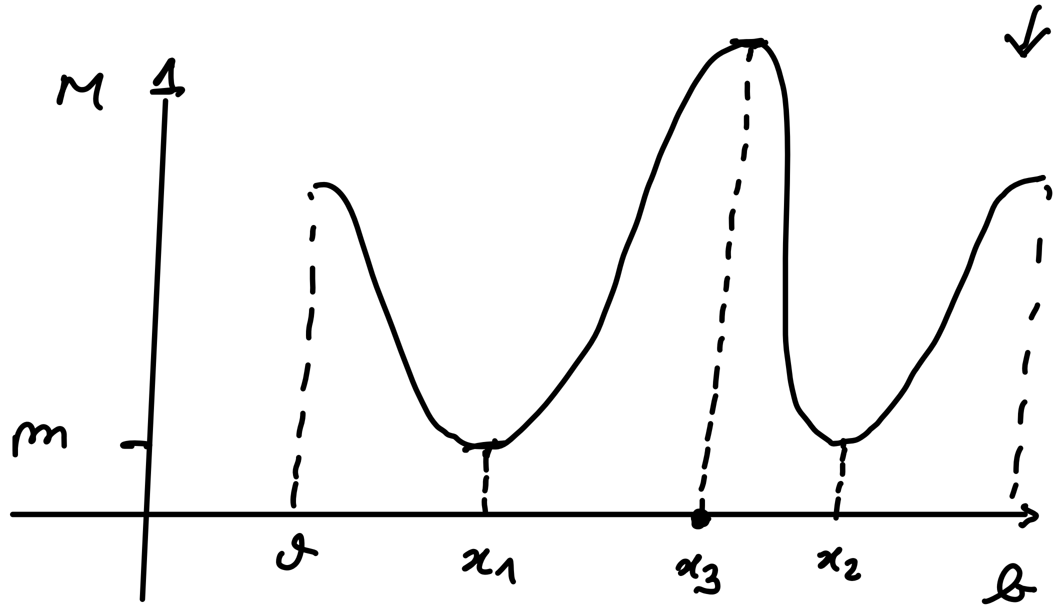
e $g(x)$ è limitata, ossia

$|g(x)| \leq M$

$$\Rightarrow \lim_{x\to x_0} f(x)g(x) = 0$$

$f = f(x)$ continua in $[a,b]$





$m = \min_{[a,b]} f(x)$ $\max_{[a,b]} f(x) = M$

Teorema di Weierstrass

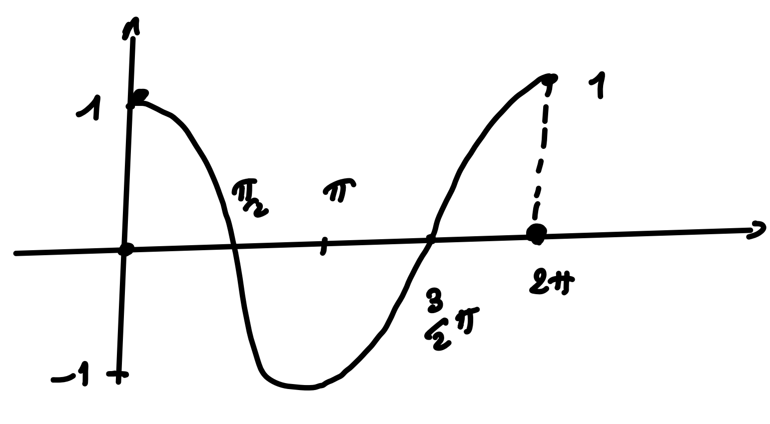
Sia $f=f(x)$, $f:[a,b]\longrightarrow\mathbb{R}$ una funzione continua. Allora f è dotata di minimo e massimo, ossia $\exists\ \bar{x}, \bar{\bar{x}} \in [a,b]$ tale che

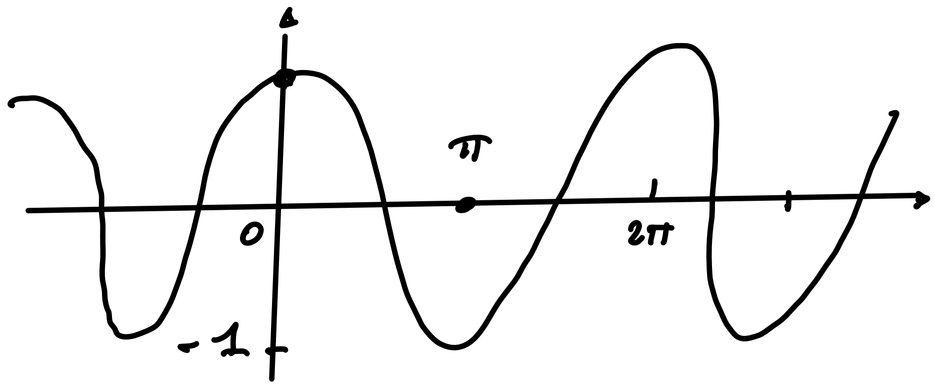
$$m = f(\bar{x}) \le f(x) \le f(\bar{\bar{x}}) = M \quad , \forall x\in[a,b]$$

$\bar{x}$ = punto di minimo assoluto

$\bar{\bar{x}}$ = " " massimo assoluto

ES. $f(x) = \cos x \ , \ x \in [0, 2\pi]$

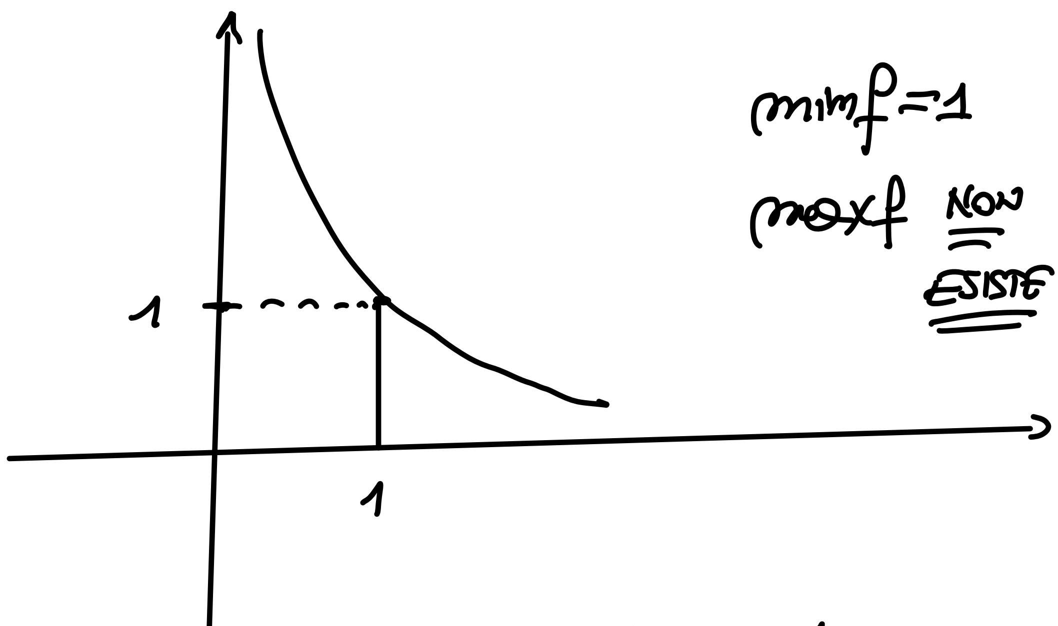




Punti di minimo assoluto $x = \pi + 2k\pi$, $k \in \mathbb{Z}$

" massimo " $x = 2k\pi$, $k \in \mathbb{Z}$

ES.

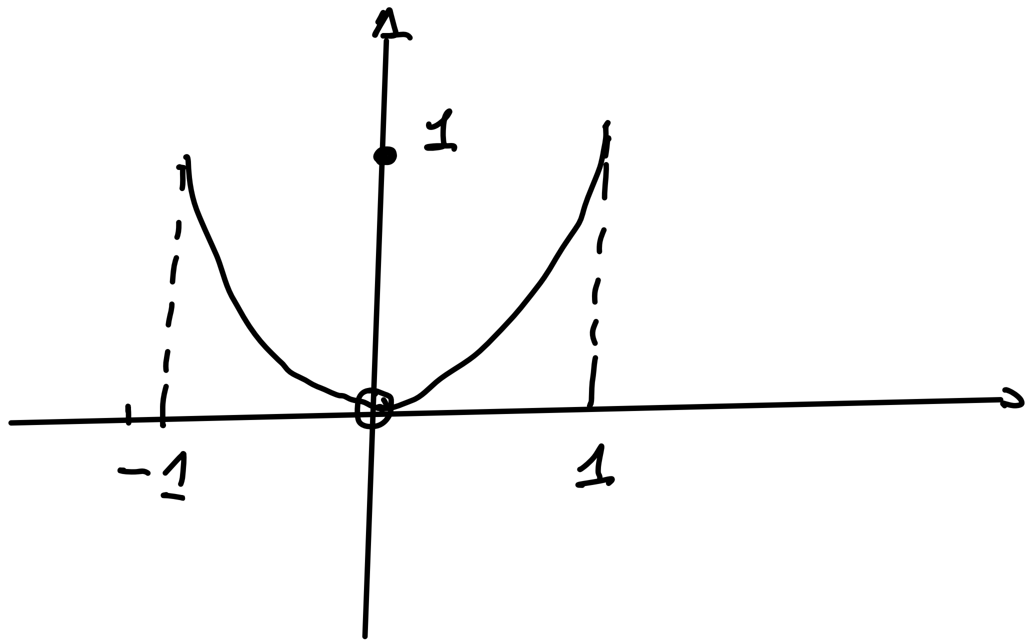


$\min f = 1$

$\max f$ NON ESISTE

$f(x) = 1/x$

$x \in \,]0,1]$



$$f(x) = \begin{cases} x^2 & \text{se } \ x \in [-1,1],\ x \neq 0 \\ 1 & \text{se } \ x = 0 \end{cases}$$

$\max\limits_{[-1,1]} f(x) = 1$, $\quad \min\limits_{[-1,1]} f(x)$ non esiste

$\inf\limits_{[-1,1]} f(x) = 0$

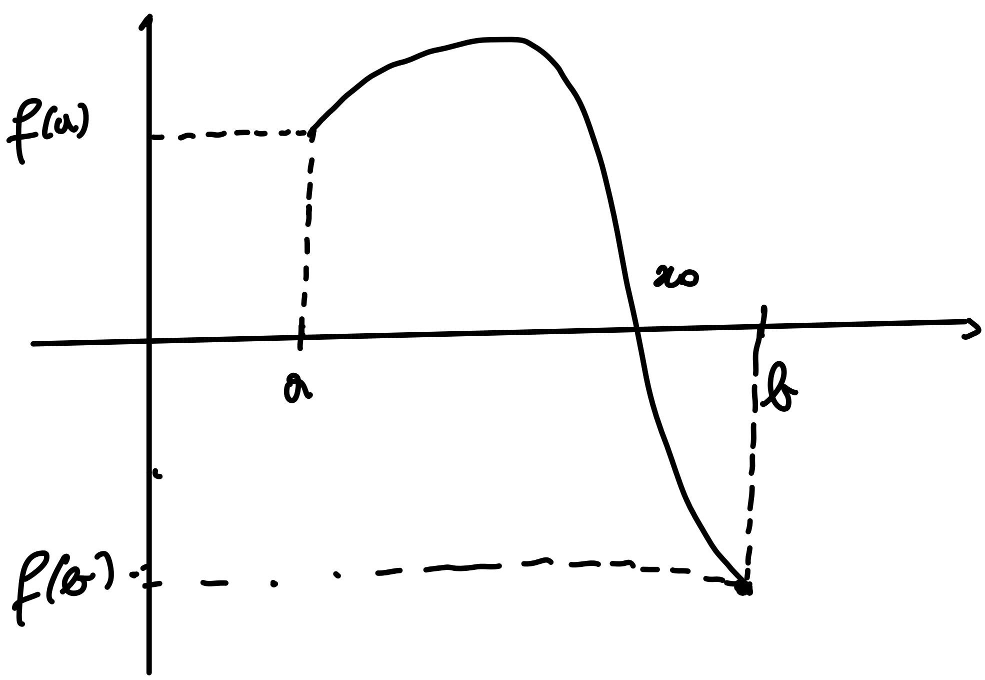
Teorema dell'esistenza degli zeri

Sia $f(x)$ una funzione continua in $[a, b]$.

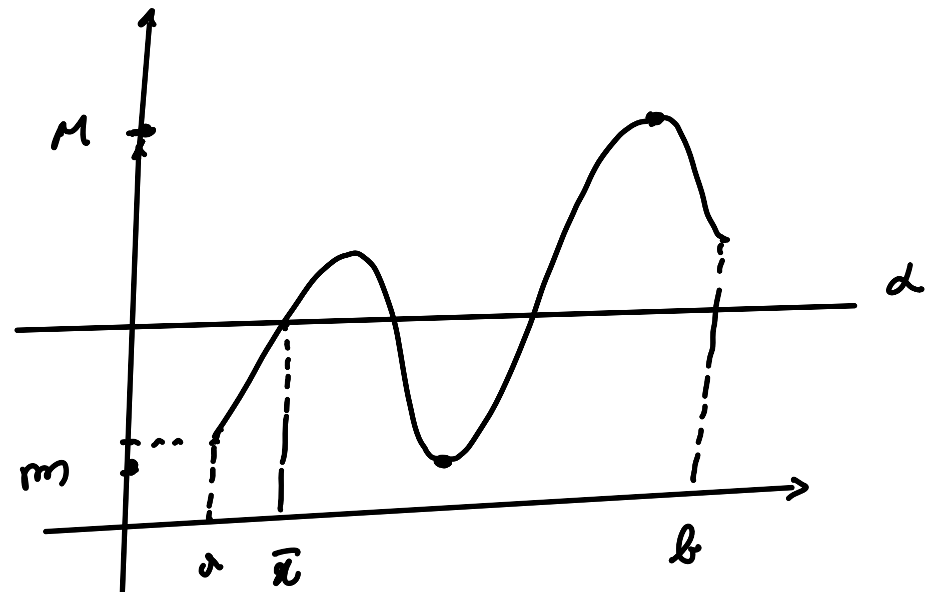
Se $f(a) \cdot f(b) < 0$, allora esiste $x_0 \in]a, b[$

tale che $f(x_0) = 0$

x_0 = zero di $f(x)$



Teorema dei valori intermedi



$$f(\bar{x}) = \alpha$$

Se $f = f(x)$ è una funzione continua in $[a,b]$ essa assume tutti i valori tra il suo minimo ed il suo massimo, cioè $f([a,b]) = [m, M]$

con $m = \min\limits_{x\in[a,b]} f(x)$, $M = \max\limits_{x\in[a,b]} f(x)$