Lezione del 16/12/2022

1ª REGOLA DI SOSTITUZIONE

$$\int f(\varphi(x))\,\overbrace{\varphi'(x)\,dx}^{dt} = \left[\int f(t)\,dt\right]_{t=\varphi(x)}$$

$t = \varphi(x)$, $dt = \varphi'(x)\,dx$

$$\int (\sin x)^3 \cos x\,dx = \int t^3\,dt = \left[\frac{t^4}{4}\right]_{t=\sin x} = \frac{\sin^4 x}{4} + C$$

$\varphi(x) = \sin x$, $f(t) = t^3$, $\varphi'(x) = \cos x$

$t = \sin x$

$$\int e^{tg x} \cdot \overbrace{\frac{1}{\cos^2 x}\,dx}^{dt} = \int e^t\,dt = \left[e^t\right]_{t=tg x} + C$$

$t = tg x$ $\quad dt = \frac{1}{\cos^2 x}\,dx$ $\quad = e^{tg x} + C$

$$\int \frac{\cos x}{\underbrace{1+\sin x}_{}}\,dx = \int \frac{dt}{t} = \left[\log|t|\right]_{t=1+\sin x} + C$$

$t = 1+\sin x$

$$= \log|1+\sin x| + C$$

$$\int \frac{\log|x|}{x}\,dx = \quad = \int t\,dt = \left(\frac{t^2}{2}\right)_{t=\log|x|} + C$$

$t = \log|x|$

$$= \frac{\log^2|x|}{2} + C$$

$$\int \sin 3x \, dx = \frac{1}{3}\int (\sin 3x)\, 3\, dx = \frac{1}{3}\int \sin t \, dt$$

$t = 3x \quad , dt = 3\, dx$

$$= -\frac{1}{3}\left(\cos t\right)_{t=3x} + c = -\frac{1}{3}\cos(3x) + c$$

$$\int \cos 5x \, dx = \frac{1}{5}\int 5\cos 5x \, dx = \frac{1}{5}\sin 5x + c$$

$t = 5x$

$$\int \frac{1}{\sqrt{4-x^2}}\, dx = \int \frac{dx}{\sqrt{4\left(1-\left(\frac{x}{2}\right)^2\right)}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-\underbrace{\left(\frac{x}{2}\right)}_{t}^2}} =$$

$$dt = \frac{dx}{2}$$

$$= \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t \,\Big|_{t=\frac{x}{2}} + c = \arcsin\left(\frac{x}{2}\right) + c$$

$$\int \frac{dx}{\underbrace{(3x-2)}^2} = \qquad \int \frac{1}{t^2} \qquad = \frac{1}{3}\int \frac{3\, dx}{\underbrace{(3x-2)}_{t=3x-2}^2} = \frac{1}{3}\int \frac{dt}{t^2}$$

$$= -\frac{1}{3}\cdot \frac{1}{t}\,\Big|_{t=3x-2} + c = -\frac{1}{3(3x-2)} + c$$

$$\int \frac{3x^2}{1+x^6}\, dx = \int \frac{3x^2}{1+\underbrace{(x^3)}_{t}^2}\, dx = \int \frac{dt}{1+t^2}$$

$dt = 3x^2 dx$

$$= [\operatorname{arctg} t]_{t=x^3} + c = \operatorname{arctg} x^3 + c.$$

$$\sin 2t = 2 \sin t \cos t$$

$$\int \frac{1}{\operatorname{sen} x}\, dx = \int \frac{dx}{\sin 2\left(\frac{x}{2}\right)} = \text{(f.le di duplicazione)}$$

$$= \int \frac{1 \cdot dx}{2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)} = \int \frac{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}\right)}{2 \sin \frac{x}{2} \cos \frac{x}{2}}\, dx$$

$$= \frac{1}{2}\left[\int \frac{\sin \frac{x}{2}}{\underbrace{\cos \frac{x}{2}}}\, dx + \int \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}}\, dx\right]$$

$$D\left(\cos \frac{x}{2}\right) = -\sin\left(\frac{x}{2}\right) \cdot \frac{1}{2}$$

$$= -\frac{1}{2} \underbrace{\int \frac{-\sin \frac{x}{2}}{\cos \frac{x}{2}}\, dx} + \frac{1}{2} \int \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}}\, dx =$$

$$= -\log\left|\cos \frac{x}{2}\right| + \log\left|\sin \frac{x}{2}\right| + c$$

$$= \log\left|\operatorname{tg} \frac{x}{2}\right| + c.$$

$$\int \frac{dx}{x^2-1} = \int \frac{1}{x^2-1}\, dx = \int \frac{(1-x)+x}{x^2-1}\, dx = \int \frac{1-x}{x^2-1}\, dx + \int \frac{x}{x^2-1}\, dx$$

$$= -\int \frac{\cancel{x-1}}{(\cancel{x-1})(x+1)}\, dx + \int \frac{x}{x^2-1}\, dx = -\int \frac{dx}{x+1} + \frac{1}{2} \int \frac{2x\, dx}{x^2-1}$$

$$= -\log|x+1| + \frac{1}{2}\log|x^2-1| + c = \log\sqrt{|x^2-1|} - \log|x+1| + c$$

$$= \log\sqrt{\frac{|x^2-1|}{|x+1|^2}} + c = \log\sqrt{\left|\frac{x-1}{x+1}\right|} + c$$

$$\int \frac{1}{\cos x}\,dx = \int \frac{1}{\cos 2\left(\frac{x}{2}\right)}\,dx = \int \frac{1}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\,dx$$

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$= 2\int \frac{1/2}{\cos^2\frac{x}{2}\left(1 - \tg^2\frac{x}{2}\right)}\,dx = 2\int \frac{dt}{1-t^2} = -2\int \frac{dt}{t^2-1}$$

$$t = \tg\frac{x}{2}, \quad dt = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}\,dx$$

$$= -2\log\sqrt{\left|\frac{t-1}{t+1}\right|} + c =$$

$$= -2\log\sqrt{\left|\frac{\tg\frac{x}{2} - 1}{\tg\frac{x}{2} + 1}\right|} + c$$

$$\int\left(e^{2x} + \frac{7}{3x+5}\right)dx$$

$$\int e^{2x}\,dx = \frac{1}{2}\int 2e^{2x}\,dx = \frac{1}{2}e^{t}\Big|_{t=2x} =$$

$$= \frac{1}{2}e^{2x} + c$$

$$\int \frac{3x+1}{2x+5}\,dx$$

$$\begin{array}{l|l} 3x+1 & 2x+5 \\ \hline -3x-\frac{15}{2} & \frac{3}{2} \\ \hline 1-\frac{15}{2} & \\ = -\frac{13}{2} & \end{array}$$

$$3x+1 = \frac{3}{2}(2x+5) - \frac{13}{2}$$

$$\int \frac{3x+1}{2x+5}\,dx = \int \frac{\frac{3}{2}\cancel{(2x+5)}\,dx}{\cancel{2x+5}} - \frac{13}{2}\cdot\frac{1}{2}\int \frac{2\,dx}{2x+5}$$

$$= \frac{3}{2}x - \frac{13}{4}\log|2x+5| + c.$$

$$\sin^2\frac{t}{2} = \frac{1-\cos t}{2}$$

$$\int \sin^2 x\,dx = \int \sin^2\frac{(2x)}{2}\,dx$$

$$= \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + c = \frac{1}{2}\left(x - \sin x \cos x\right) + c$$

$$\int \cos^2 x\,dx$$

$$\int \cos^3 x = \int \cos^2 x \cdot \cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$$

$$= \int \cos x\,dx - \int \underbrace{\sin^2 x}\cos x\,dx =$$

$$= \sin x - \frac{1}{3}\left(\sin^3 x\right) + c$$

$$\shortparallel$$

$$t = \sin x : \quad \int t^2\,dt = \frac{t^3}{3} + c \Big|_{t=\sin x}$$

$$\int \frac{1}{x^2+3x-4}\,dx \overset{(*)}{=}$$

$$(x+\tfrac{3}{2})^2$$

$$\shortparallel$$

$$\Delta = 9 + 16 = 25 > 0$$

$$x^2+3x-4 = x^2+2\left(\tfrac{3}{2}\right)x-4 = x^2+\overbrace{2\left(\tfrac{3}{2}\right)x+\tfrac{9}{4}}$$

$$-\frac{9}{4}-4 = \left(x+\frac{3}{2}\right)^2-\frac{25}{4} = \frac{1}{4}\left(\left(2x+3\right)^2-25\right)$$

$$\overset{*}{=} \int \frac{dx}{\frac{1}{4}\left[(2x+3)^2-25\right]} = 4\int\frac{dx}{(2x+3)^2-\underbrace{25}}$$

$$= 4\int\frac{dx}{25\left[\left(\frac{2x+3}{5}\right)^2-1\right]} = \frac{4}{25}\int\frac{dx}{\left(\frac{2x+3}{5}\right)^2-1}$$

$$= \frac{2}{5}\cdot\int\frac{\frac{2}{5}\,dx}{\left(\frac{2x+3}{5}\right)^2-1} = \frac{2}{5}\int\frac{dt}{t^2-1}$$ già fatto!

$$= \frac{2}{5}\log\sqrt{\left|\frac{\frac{2x+3}{5}-1}{\frac{2x+3}{5}+1}\right|}+c =$$

$$= \frac{2}{5}\log\sqrt{\left|\frac{2x-2}{2x+8}\right|}+c$$

$$\int \frac{dx}{x^2+x+1} \qquad \Delta = 1-4 = -3 < 0$$

$$x^2+x+1 = x^2 + 2\cdot\left(\frac{1}{2}\right)x + 1 =$$

$$= \overbrace{x^2 + 2\left(\frac{1}{2}\right)x + \frac{1}{4}} - \frac{1}{4} + 1 =$$

$$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{(2x+1)^2}{4} + \frac{3}{4} =$$

$$= \frac{1}{4}\left[(2x+1)^2 + 3\right]$$

$$\downarrow$$

$$\int - = 4\int \frac{dx}{(2x+1)^2+3} = \frac{4}{3}\int \frac{dx}{\underbrace{\left(\frac{2x+1}{\sqrt{3}}\right)}_{t}^2+1}$$

$$dt = \frac{2}{\sqrt{3}}\,dx \qquad = \frac{2}{\sqrt{3}}\int \frac{\frac{2}{\sqrt{3}}\,dx}{''}$$

$$= \frac{2}{\sqrt{3}}\,\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + c.$$

$$\int \frac{dx}{5x^2+4} \qquad\qquad \int \frac{dt}{1+t^2}$$

<u>FRATTI SEMPLICI</u>

$$\int \frac{P(x)}{Q(x)}\,dx = \qquad \frac{P(x)}{Q(x)} \quad \text{f.ne razionale}$$

$$gr(P(x)) > gr(Q(x))$$

$$P(x) = Q(x)\underbrace{S(x)}_{\text{quoziente}} + \underbrace{R(x)}_{\text{resto}}, \quad gr\,R(x) < gr\,P(x)$$

$$= \underbrace{\int S(x)\,dx}_{\text{facile}} + \int \frac{R(x)}{Q(x)}\,dx, \qquad gr(R(x)) < gr(Q(x))$$

$$\int \frac{4x^3+x^2+x-2}{4x^2-3x-1}\,dx = \int (x+1)\,dx + \int \frac{5x+1}{4x^2-3x-1}\,dx$$

$$\begin{array}{r|l} 4x^3+x^2+x-2 & 4x^2-3x-1 \\ \cline{2-2} -4x^3+3x^2+x & x+1 \\ \cline{1-1} // \quad 4x^2+2x & \\ -4x^2+3x+1 & \\ \cline{1-1} // \quad 5x+1 & \end{array}$$

$$4x^3+x^2+x-2 = (4x^2-3x-1)(x+1) + 5x+1$$

$$4x^2-3x-1 = 0 \qquad \Delta = 9+16 = 25$$

$$x_{1/2} = \frac{3 \pm 5}{8} \quad , x_1 = -\frac{1}{4} \; , \; x_2 = 1$$

$$4x^2-3x-1 = 4\left(x+\frac{1}{4}\right)(x-1)$$

$$\frac{5x+1}{4x^2-3x-1} = \frac{A}{x+\frac{1}{4}} + \frac{B}{x-1} \qquad (\ast)$$

$$\int \frac{1}{x^2+3x-4}\,dx$$

$$x^2+3x-4=0 \qquad \Delta = 25$$

$$x_{1,2} = \frac{-3 \pm 5}{2} \begin{cases} -4 \\ 1 \end{cases}$$

$$\frac{1}{x^2+3x-4} = \frac{A}{x+4} + \frac{B}{x-1}$$

$$x^2+3x-4 = (x+4)(x-1)$$

$$1 = A(x-1) + B(x+4) = Ax - A + Bx + 4B$$

$$= (A+B)x + 4B - A$$

$$\Leftrightarrow \begin{cases} A+B=0 \\ 4B-A=1 \end{cases} \Leftrightarrow \begin{cases} A = -B = -\frac{1}{5} \\ B = \frac{1}{5} \end{cases}$$

$$\int \frac{1}{x^2+3x-4}\,dx = -\frac{1}{5}\int \frac{dx}{x+4} + \frac{1}{5}\int \frac{dx}{x-1}$$

$$= \frac{1}{5}\left[\log|x-1| - \log|x+4|\right]$$

$$\frac{5x+1}{4x^2-3x-1} = \frac{A}{x+\frac{1}{4}} + \frac{B}{x-1}$$

$$5x+1 = 4A(x-1) + 4B\left(x+\frac{1}{4}\right)$$

$$5x+1 = 4Ax + 4Bx - 4A + B$$

$$=$$

$$= 4\underbrace{(A+B)}_{}x + B - 4A$$

$$\begin{cases} 4(A+B) = 5 \\ B - 4A = 1 \end{cases} \iff \begin{cases} A+B = \frac{5}{4} \\ B - 4A = 1 \end{cases}$$

$$+\begin{cases} 4A + 4B = 5 \\ B - 4A = 1 \end{cases} \iff \begin{cases} 5B = 6 \\ B - 4A = 1 \end{cases}$$

$$\iff \begin{cases} B = \frac{6}{5} \\ 4A = B - 1 = \frac{6}{5} - 1 = \frac{1}{5} \end{cases}$$

$$A = \frac{1}{20}$$

$$\int \frac{5x+1}{4x^2-3x-1} = \frac{1}{20} \int \frac{dx}{x+\frac{1}{4}}$$

$$+ \frac{1}{5} \int \frac{dx}{x-1}$$

$$= \frac{1}{20}\log\left|x+\frac{1}{4}\right| + \frac{1}{3}\log|x-1|$$

$$\int \frac{x^3+x^2}{x^2+1}\,dx$$

$$\int \frac{4x+3}{\underbrace{(x+1)^2}(x^2+1)}\,dx$$

radici reali : $x = -1$ radice doppia

$x^2+1 \neq 0 \quad \forall x \in \mathbb{R}$ radici complesse coniugate

$x = \pm i$

i = unità immaginaria di $\mathbb{C}$

$$\frac{4x+3}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$$

$$\Leftrightarrow 4x+3 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$$

Formula di integrazione per parti

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$$\int x e^x \, dx$$

$f(x) = e^x \quad g'(x) = x$

$f'(x) = e^x \quad g(x) = \frac{x^2}{2}$

$$= \frac{e^x \cdot x^2}{2} - \frac{1}{2} \int e^x \cdot x^2 \, dx$$

$\underline{\underline{\text{no}}}$!

$f(x) = x, \quad g'(x) = e^x$

$f'(x) = 1 \quad g(x) = e^x$

$$= x e^x - \int e^x \, dx$$

$$= x e^x - e^x + C$$

$$\int \log|x|\,dx = \int 1 \cdot \log|x|\,dx$$

$$f(x) = \log|x| \;, \quad g'(x) = 1$$

$$f'(x) = \frac{1}{x} \qquad g(x) = x$$

$$= x\log|x| - \int \frac{1}{\cancel{x}} \cdot \cancel{x}\,dx$$

$$= x\log|x| - x + C$$

$$\int \operatorname{arctg} x\,dx \qquad \text{(DA FARE)}$$

$$\int \sqrt{1-x^2}\,dx = \int 1 \cdot \sqrt{1-x^2}\,dx$$

$$f(x) = \sqrt{1-x^2} \qquad g'(x) = 1$$

$$f'(x) = \frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x) \quad , \; g(x) = x$$

$$= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx$$

$$= x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$$

$$= x\sqrt{1-x^2} - \int \frac{\overbrace{1-x^2}^{\sqrt{(1-x^2)^2}}}{\sqrt{1-x^2}}\,dx + \overbrace{\int \frac{dx}{\sqrt{1-x^2}}}^{\arcsin x}$$

$$= x\sqrt{1-x^2} - \underbrace{\int \sqrt{1-x^2}}_{\text{PRIMO MEMBRO}} + \arcsin x$$

$$2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \arcsin x + C$$

$$\int \sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2} + \arcsin x}{2} + C$$

$$\int \sin^2 x \qquad \int \cos^2 x$$

$$\int \sin x \cdot \sin x \qquad f = \sin x \quad g' = \sin x$$

$$f' = \cos x \quad g = -\cos x$$

$$= -\sin x \cos x + \int \cos^2 x$$

$$= -\sin x \cos x + \int (1 - \sin^2 x)\,dx$$

$$= -\sin x \cos x + x - \int \sin^2 x\,dx$$

$$\int \sin^2 x \, dx = \frac{x - \sin x \cos x}{2} + C$$