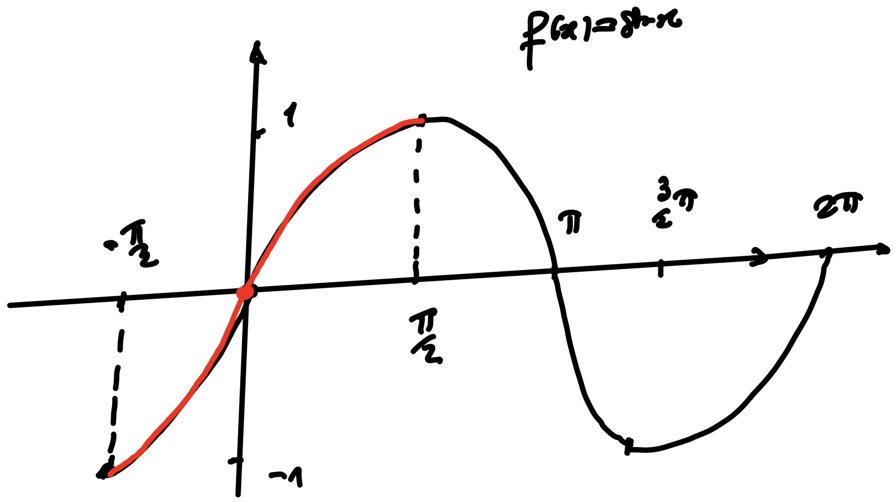
Lezione del 08/11/2022

Inverse delle funzioni trigonometriche



$f(x) = \operatorname{sen} x \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$: f è strettamente crescente

$$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

f è invertibile. Inversa ??

$$f^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$y \in \quad \longrightarrow f^{-1}(y)$$

$y \in [-1,1]$: $f^{-1}(y) = x$ tale che $\overbrace{\operatorname{sen} x = y}$

$$f^{-1}(y) = \arcsin y \quad \forall y \in [-1, 1]$$

$\arcsin 1 = \frac{\pi}{2}$

?

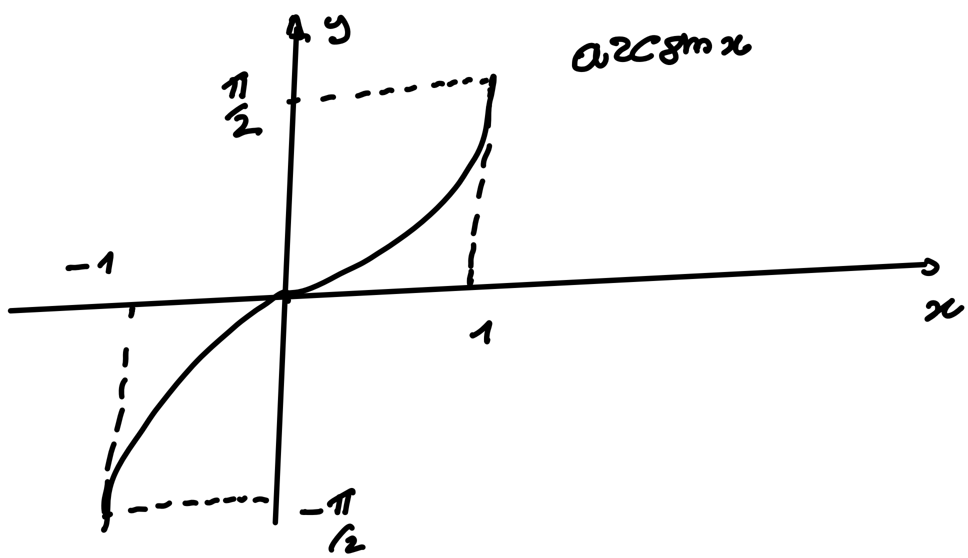
$\arcsin \frac{\sqrt{2}}{2} = \frac{\pi}{4}$

$\arcsin(-1) = -\frac{\pi}{2}$

. . . .

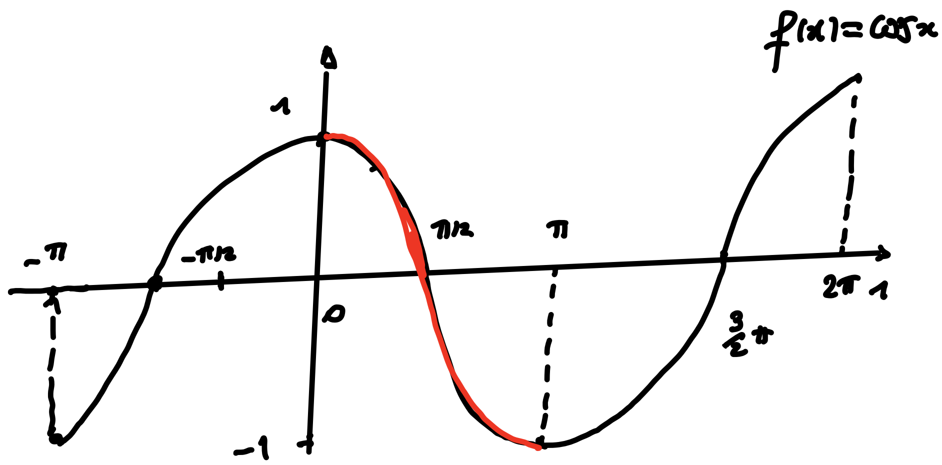
$\forall x \in [-1, 1]$

$g(x) = \arcsin x$



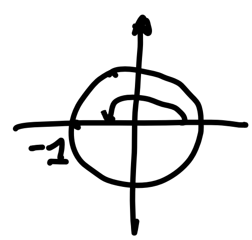
$\sin(\arcsin y) = y \qquad \arcsin(\sin x) = x$

$\forall y \in [-1, 1] \quad , \qquad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



$f(x) = \cos x$, $\forall x \in [0, \pi]$ è strettamente decrescente

$f : [0, \pi] \longrightarrow [-1, 1]$



$f^{-1} : [-1, 1] \longrightarrow [0, \pi]$

$y \in [-1, 1] \longrightarrow f^{-1}(y) \in [0, \pi]$

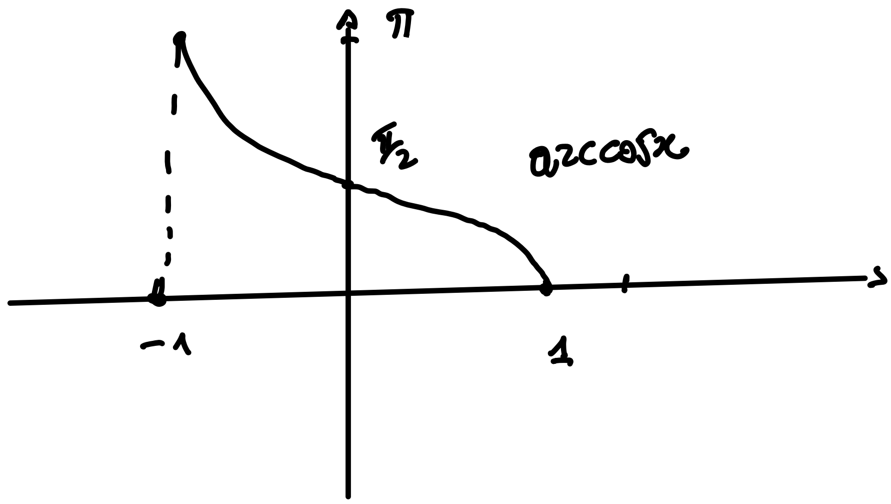
$y \in [-1, 1]$: $f^{-1}(y) = x$ tale che $\cos x = y$

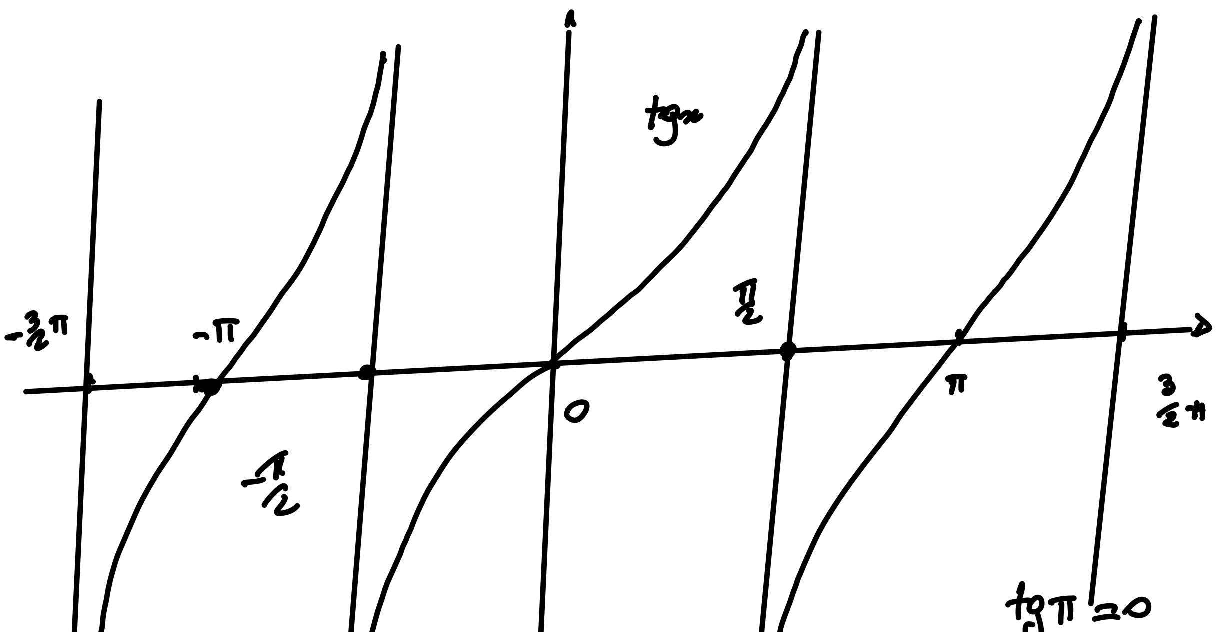
$$f^{-1}(y) = \arccos y$$

$\arccos(0) = \frac{\pi}{2}$

$\arccos(-1) = \pi$

$\arccos 1 = 0$



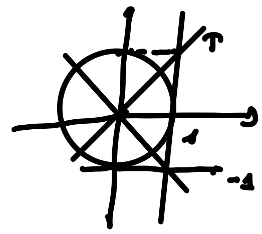


$f(x) = \operatorname{tg} x \quad , \forall x \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$ è invertibile

$f : \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\longrightarrow \mathbb{R}$

$f^{-1} : \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

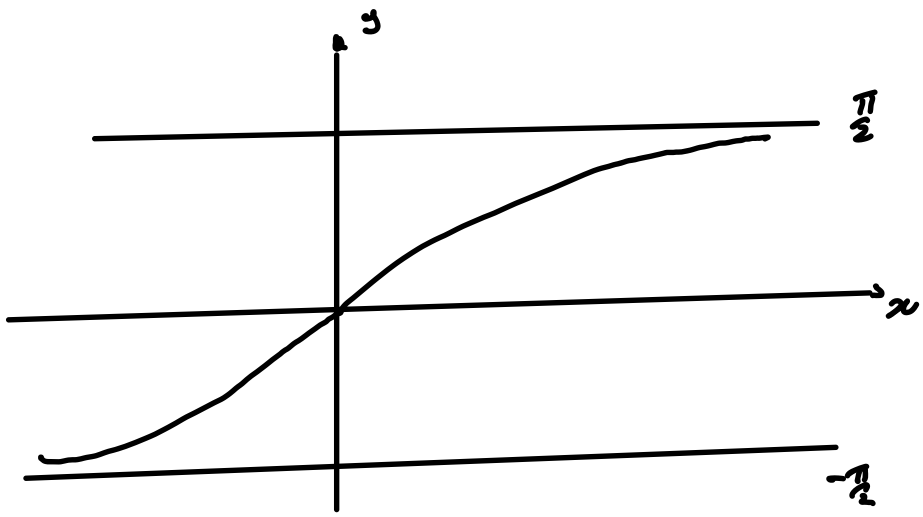
$y \in \mathbb{R}$, $f^{-1}(y) = x$ t.c. $\operatorname{tg} x = y$



$f^{-1}(y) = \operatorname{arctg} y \quad \forall y \in \mathbb{R}$

$\operatorname{arctg} 0 = 0 \qquad \operatorname{arctg} 1 = \frac{\pi}{4}$

$\operatorname{arctg}(-1) = -\frac{\pi}{4}$



Insiemi di definizione

$f(x)$ $\qquad D_f =$ dominio di f o campo di esistenza di f

$$f(x) = \sqrt{x^2-1}$$

$x^2-1 \geq 0 \iff$

$\iff x \leq -1 \cup x \geq 1$

$\iff x \in]-\infty,-1] \cup [1,+\infty[$

$$D_f =]-\infty,-1] \cup [1,+\infty[$$

$$f(x) = \frac{\sqrt{x^2-5x+4}}{|x|} :$$

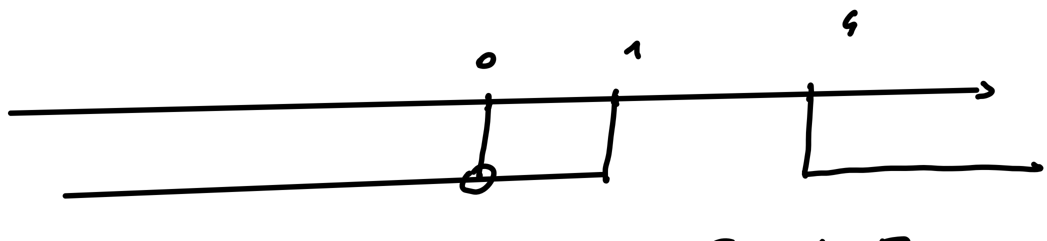
$x^2-5x+4 \geq 0 \quad \odot$

$|x| \neq 0 \iff |x| > 0 \iff x \neq 0$

$\Delta = 25-16 = 9 > 0$

$x_{1,2} = \dfrac{5 \pm 3}{2} \begin{cases} 1 \\ 4 \end{cases}$

$\odot \iff x \leq 1 \cup x \geq 4$



$$D_f =]-\infty, 0[\cup]0,1] \cup [4,+\infty[$$

$D_f = \mathbb{R}$

$f(x) = \log|x|$: $|x| > 0 \iff x \neq 0$

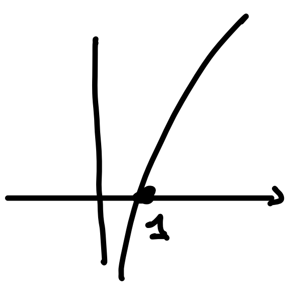
$D_f = \mathbb{R} \setminus \{0\}$

$f(x) = \sqrt{\underbrace{\log x - 1}}$

$(S) \begin{cases} \log x - 1 \geq 0 \\ x > 0 \end{cases}$

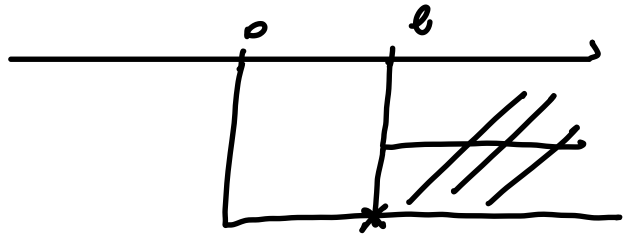
$\log x = \log_e x$

$\log x - 1 \geq 0 \iff \log x \geq 1 = \log e$



$\iff x \geq e$

$(S) \iff \begin{cases} x \geq e \\ x > 0 \end{cases}$



$D_f = [e, +\infty[$

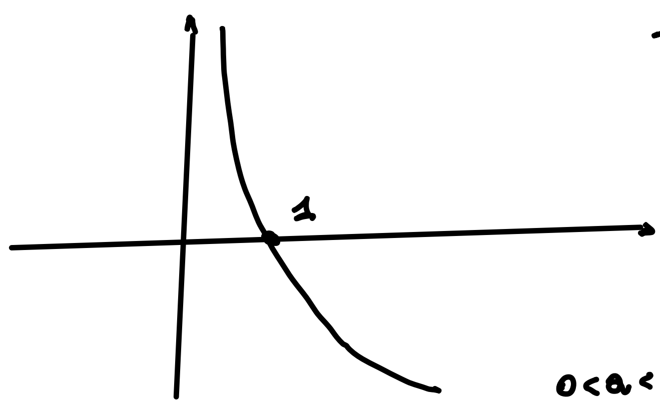
$f(x) = \sqrt{\log_{\frac{1}{2}} x - 8}$

$(S) \begin{cases} x > 0 \\ \log_{\frac{1}{2}} x - 8 \geq 0 \end{cases} \iff \begin{cases} x > 0 \\ x \leq \left(\frac{1}{2}\right)^8 \end{cases}$

$\log_{\frac{1}{2}} x - 8 \geq 0 \iff \log_{\frac{1}{2}} x \geq 8 = \log_{\frac{1}{2}} \frac{1}{2} =$

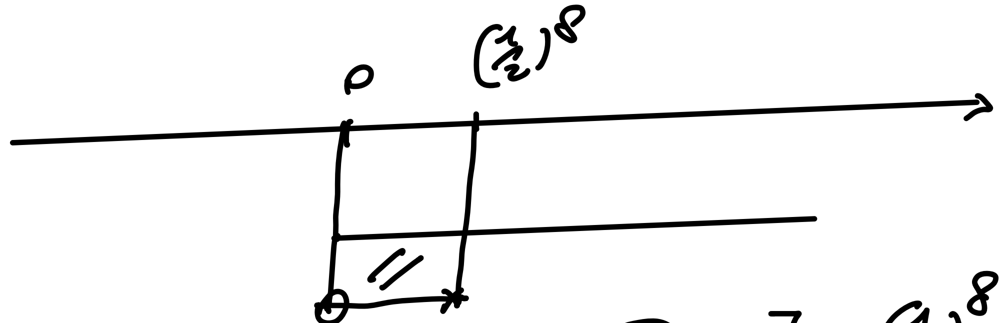
$= \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^8$

$0 < \frac{1}{2} < 1$



$0 < a < 1$

$\iff x \leq \left(\frac{1}{2}\right)^8$



$D_f = \left]0, \left(\frac{1}{2}\right)^8\right]$

$f(x) = \log\left(e^{2x} - e^x\right)$

$e^{2x} - e^x > 0$?

$e^x(e^x - 1) > 0 \iff e^x - 1 > 0$

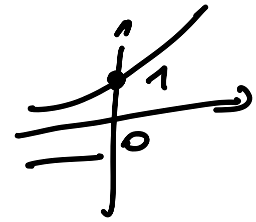
$\iff e^x > 1 \iff x > 0$

$D_f =]0, +\infty[$

$f(x) = \log(e^{2x} - 5e^x + 4)$

$t = e^x$

$e^{2x} - 5e^x + 4 > 0$



$t^2 - 5t + 4 > 0$ $\quad \Delta = 9$

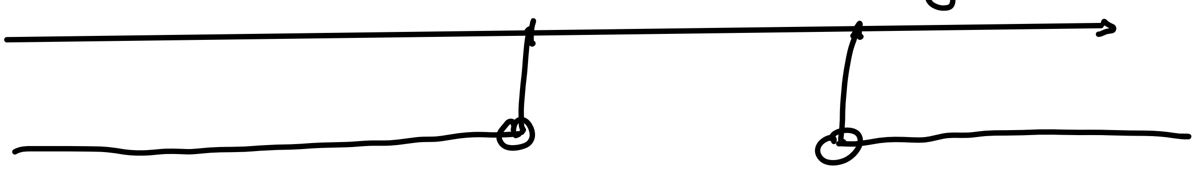
$t_{1,2} = \frac{5 \pm 3}{2}$ $\quad t_1 = 1$, $t_2 = 4$

?

$t < 1 \cup t > 4 \iff e^x < 1 \cup e^x > 4$

$\iff$ $\iff x < 0 \cup e^x > e^{\log 4}$

$\iff \quad x < 0 \quad \cup \quad x > \log 4$



$$D_f =]-\infty, 0[\,\cup\,]\log 4, +\infty[$$

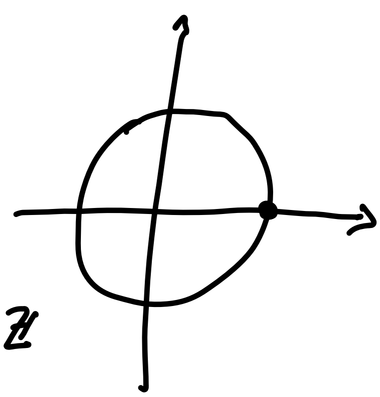
$f(x) = \sin 2^x$? $\quad D_f = \mathbb{R}$

$\sin 2^{\sqrt{x}}$ $\quad x \geq 0 \quad : D_f = [0, +\infty[$

$f(x) = \dfrac{x^2 - 1}{\operatorname{tg} x}$ $\quad \operatorname{tg} x \neq 0$

$\Updownarrow$

$x \neq k\pi, \; k \in \mathbb{Z}$



$D_f = \mathbb{R} \setminus \bigcup_{k \in \mathbb{Z}} \{k\pi\}$

$x \neq 0, \quad x \neq \pm\pi, \pm 2\pi, \pm 3\pi \ldots$

$$f(x) = \arccos(2+x^2)$$

$$-1 \leq 2+x^2 \leq 1$$

$$\iff \quad x^2+1 \leq 0 \quad \underline{\underline{\text{ASSURDO}}}$$

$$x^2+2 \geq 2 > 1$$

$$D_f = \emptyset$$

$$f(x) = (\cos x)^{\log x}$$

$$f(x)^{g(x)} = e^{\log f(x)^{g(x)}} = e^{g(x) \log f(x)}$$

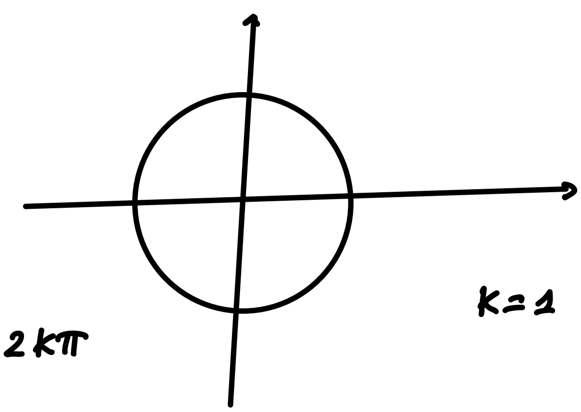
$$\log x^{\alpha} = \alpha \log x$$

$$\begin{cases} f(x) > 0 \\ \text{dominio di } g(x) \end{cases}$$

$$f(x) = e^{\log x \log \cos x}$$

$$\begin{cases} x > 0 \\ \cos x > 0 \ ? \end{cases}$$

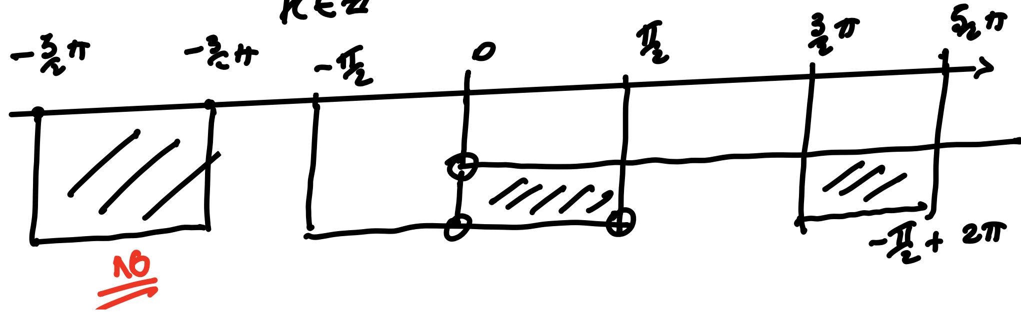
$k=0$



$k=1$

$$\iff -\frac{\pi}{2} + 2k\pi < x < \frac{\pi}{2} + 2k\pi$$

$k \in \mathbb{Z}$



$$-\frac{\pi}{2} + 2\pi = \frac{3}{2}\pi$$

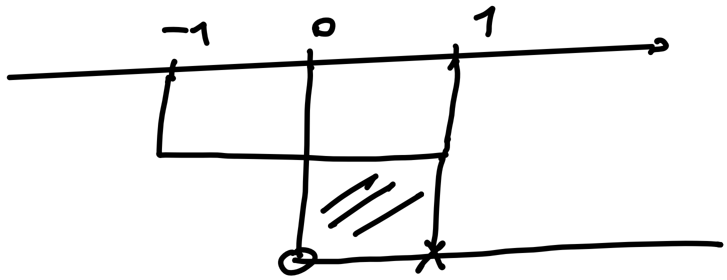
$0 < x < \frac{\pi}{2}$ $\qquad$ $\frac{\pi}{2} + 2\pi = \frac{5}{2}\pi$

$k = -1$ $\qquad$ $-\frac{\pi}{2} - 2\pi = -\frac{5}{2}\pi$

$\qquad\qquad$ $-\frac{\pi}{2} - 2\pi = -\frac{3}{2}\pi$

$$D_f =]0, \tfrac{\pi}{2}[\ \cup \bigcup_{k>0}]-\tfrac{\pi}{2} + 2k\pi, \tfrac{\pi}{2} + 2k\pi[$$

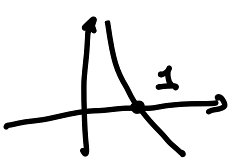
$$f(x) = \underbrace{\log x}_{?} + \arcsin x \qquad \begin{cases} x > 0 \\ -1 \le x \le 1 \end{cases}$$

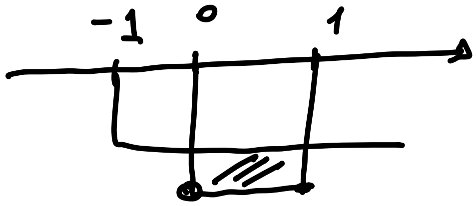


$D_f =]0, 1]$ $\qquad$ $0 < x \le 1$

$$f(x) = \sqrt{\log_{\frac{1}{2}} x} \cdot (x+1)^{\pi} \qquad x^{\alpha} = e^{\alpha \log x} \qquad x > 0$$

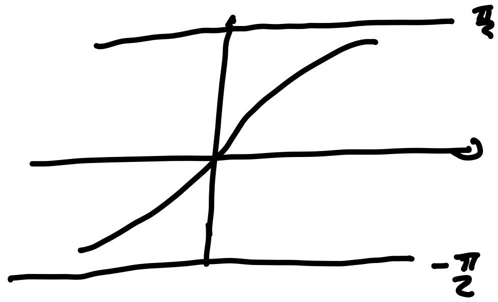
$$\begin{cases} x+1 > 0 \\ \log_{\frac{1}{2}} x \ge 0 \\ x > 0 \end{cases} \iff \begin{cases} x > -1 \\ 0 < x \le 1 \end{cases}$$



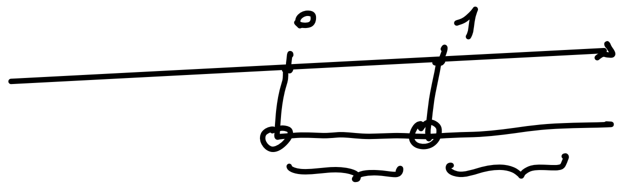


$D_f =]0,1]$

$f(x) = \arctan\left(\frac{x^2+1}{\underbrace{\log x}}\right)$



$\begin{cases} \log x \neq 0 \\ x > 0 \end{cases} \iff \begin{cases} x \neq 1 \\ x > 0 \end{cases}$



$D_f =]0,1[\cup]1,+\infty[.$

$f(x) = \sqrt{|\underbrace{x^2-4x+4}|}$ $\quad \forall x \in \mathbb{R} \quad |x| \geq 0$

$\sqrt{|t|}$

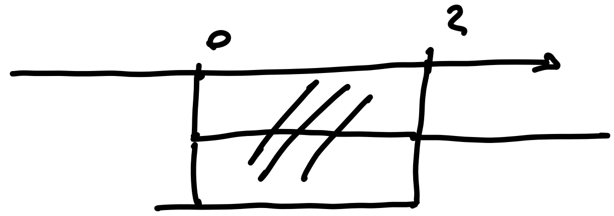
$f(x) = \arcsin(|x-1|)$: $\quad -1 \leq |x-1| \leq 1$

$\iff |x-1| \leq 1$?

$\Downarrow$

$|a| \leq b \iff -b \leq a \leq b \qquad -1 \leq x-1 \leq 1$

$$\begin{cases} x-1 \geq -1 \\ x-1 \leq 1 \end{cases} \iff \begin{cases} x \geq 0 \\ x \leq 2 \end{cases}$$



$D_f = [0, 2]$

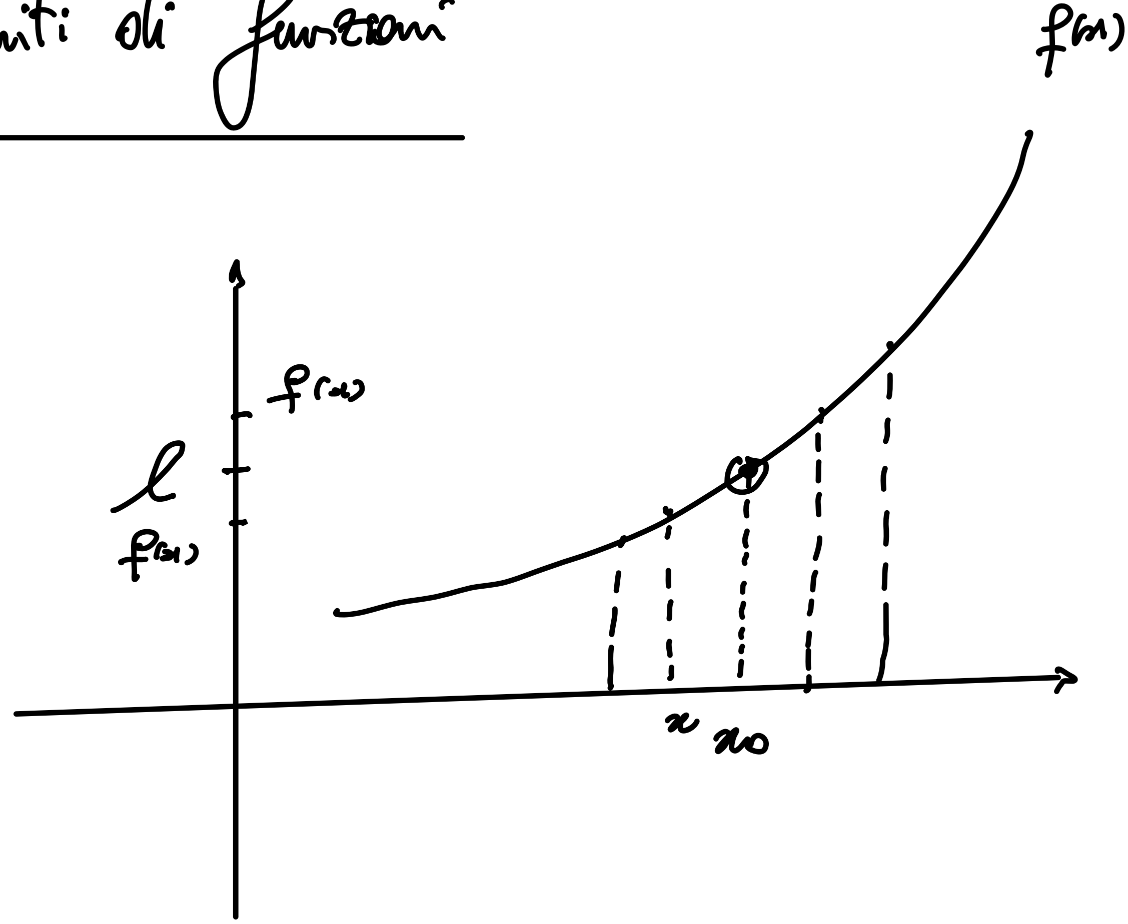
$f(x) = \arccos \log x$: $\begin{cases} -1 \leq \log x \leq 1 \\ x > 0 \end{cases}$ $\qquad e^{-1} = \frac{1}{e}$

$$\begin{cases} \log x \geq -1 \\ \log x \leq 1 \\ x > 0 \end{cases} \iff \begin{cases} \log x \geq -\log e = \log 1/e \\ \log x \leq \log e \\ x > 0 \end{cases}$$

$$\iff \begin{cases} 1/e \leq x \leq e \\ x > 0 \end{cases} \iff \frac{1}{e} \leq x \leq e$$

$D_f = [1/e, e]$.

Limiti di funzioni

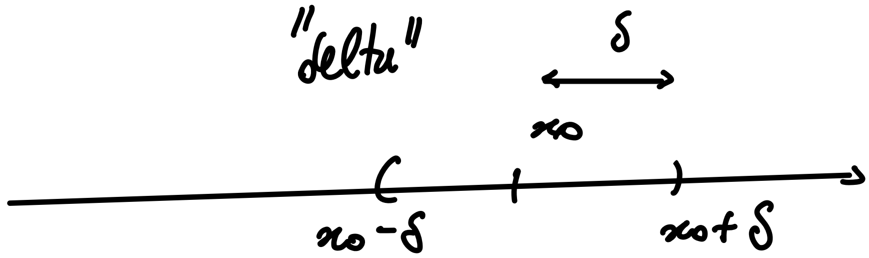


Intorno di un punto

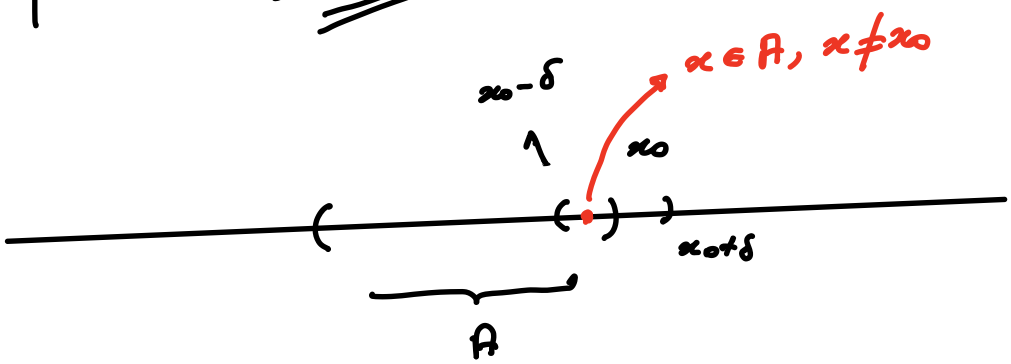
$x_0 \in \mathbb{R}$: un intorno di x_0 è un qualsiasi intervallo del tipo

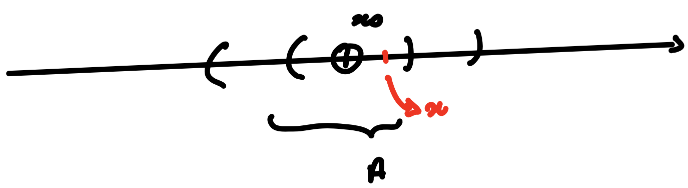
$$]x_0 - \delta, x_0 + \delta[$$

"delta"



Def. $A \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$: si dice che x_0 è di accumulazione per A se "in ogni intorno di x_0 cade almeno un punto di A, diverso da x_0"



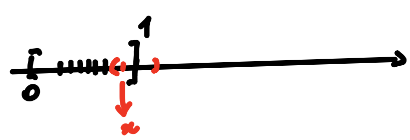


$x_0 \in \mathbb{R}$ è di accumulazione per A

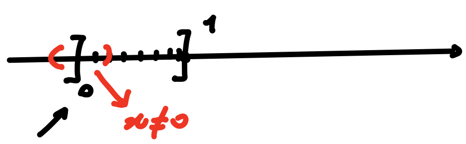
$$\iff \forall \delta > 0 \ \exists x \in A : 0 < |x - x_0| < \delta$$

$$x_0 - \delta < x < x_0 + \delta$$
$$\iff |x - x_0| < \delta$$

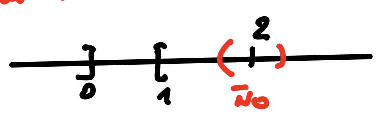
$A = [0,1]$



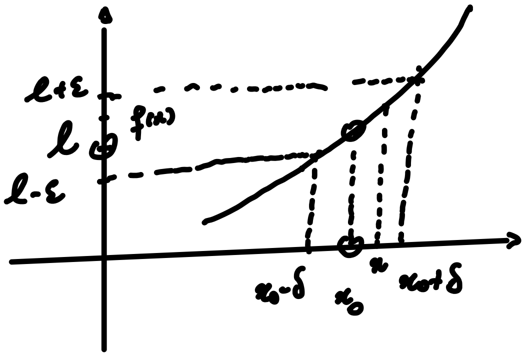
$A =]0,1]$



Punti di accumulazione $= [0,1]$



$[0,1]$



$f: A \subseteq \mathbb{R} \to \mathbb{R}$ $\quad$ x_0 di accumulazione per A

ε "epsilon"

$\varepsilon > 0$ piccolo a piacere

Def Si dice che $f(x)$ tende ad $\ell \in \mathbb{R}$ per x che tende ad x_0 ($x \to x_0$) e scriveremo

$$\lim_{x \to x_0} f(x) = \ell \overset{\text{def.}}{\Longleftrightarrow}$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \forall x \in A \text{ t.c. } 0 < |x - x_0| < \delta \quad ?$$

$$\Downarrow$$

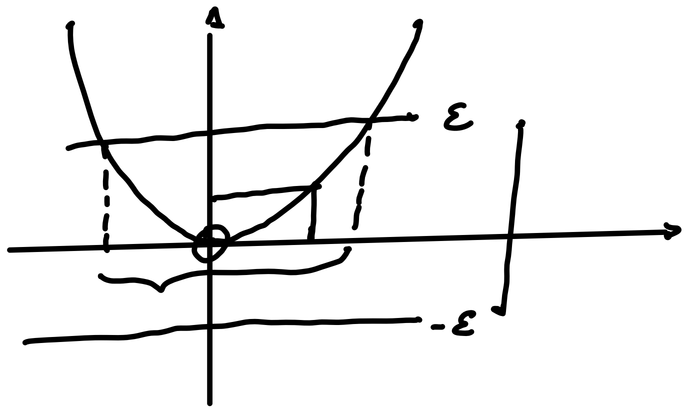
$$|f(x) - \ell| < \varepsilon$$

$$\Leftrightarrow \ell - \varepsilon < f(x) < \ell + \varepsilon$$

Es $\quad f(x) = x^2$ se $x \neq 0$ $\qquad \mathbb{R} \setminus \{0\} = D_f$

$y = x^2$?

parabola



$$\lim_{x \to 0} f(x) = 0$$

Capiamolo dalla definizione (ε, δ)

Vogliamo $\varepsilon > 0$ fissato : allora

$$|f(x) - l| = |x^2| = x^2 < \varepsilon \ ?$$

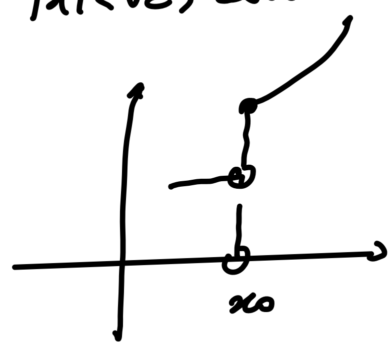
$$\Leftrightarrow \quad x^2 < \varepsilon \Leftrightarrow x^2 - \varepsilon < 0 \Leftrightarrow$$

$$\Leftrightarrow \quad -\sqrt{\varepsilon} < x < \sqrt{\varepsilon}$$

$\delta = \delta(\varepsilon) = \sqrt{\varepsilon}$: perché quando $|x| < \sqrt{\varepsilon}$, abbiamo

$$|f(x) - l| = |x^2| < \varepsilon$$

$$l = 0$$



Teorema di unicità del limite

Se $\exists \lim_{x \to x_0} f(x) = l$, l è unico

Dim. Supponiamo che $\exists\, l_1, l_2$ due limiti di

$f(x)$: $\varepsilon > 0 \quad \exists \delta > 0 : \forall x : 0 < |x - x_0| < \delta$

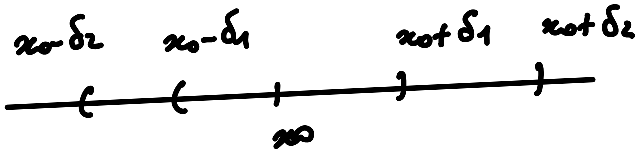
$$\Rightarrow |f(x) - l| < \varepsilon$$

l_1 " $\exists \delta_1 > 0 : \forall x \in A : 0 < |x - x_0| < \delta_1$

$$\Rightarrow |f(x) - l_1| < \varepsilon$$

l_2 " $\exists \delta_2 > 0 : \forall x \in A : 0 < |x - x_0| < \delta_2$

$$\Rightarrow |f(x) - l_2| < \varepsilon$$



$\delta_1 < \delta_2$

$\delta = \min\{\delta_1, \delta_2\}$: $\forall x \in A$ t.c. $0 < |x - x_0| < \delta$ si

ha
$$|f(x) - l_1| < \varepsilon$$
$$|f(x) - l_2| < \varepsilon$$

Allora $|l_1 - l_2| = |(l_1 - f(x)) + (f(x) - l_2)|$

$\leq$ (dis. triangolare) $\leq |\underbrace{f(x) - l_1}_{\hat{\varepsilon}}| + |\underbrace{f(x) - l_2}_{\hat{\varepsilon}}|$

$< 2\varepsilon \qquad \forall \varepsilon > 0$

$$\Rightarrow \quad |l_1 - l_2| = 0 \quad \Rightarrow l_1 = l_2$$