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Laboratorio di Architettura Degli Elaboratori

Algebra di Boole e reti combinatorie

Boolean Theorems of Several Var

#	Theorem	Dual	Name
T1	$B \bullet 1 = B$	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	$B + B = B$	Idempotency
T4	$\overline{\overline{B}} = B$	$B = B$	Involution
T5	$B \bullet B = 0$	$B + B = 1$	Complements
T6	$B \bullet C = C \bullet B$	$B+C = C+B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	$(B + C) + D = B + (C + D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	$B + (C \bullet D) = (B+C) (B+D)$	Distributivity
T9	$B \bullet (B+C) = B$	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	$(B+C) \bullet (B+\overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) = (B \bullet C) + (\overline{B} \bullet D)$	$(B+C) \bullet (\overline{B}+D) \bullet (C+D) = (B+C) \bullet (\overline{B}+D)$	Consensus

Teoremi a più variabili: dualità

#	Theorem	Dual	Name
T6	B•C = C•B	B+C = C+B	Commutatività
T7	(B•C) • D = B • (C•D)	(B + C) + D = B + (C + D)	Associatività
T8	B • (C + D) = (B•C) + (B•D)	B + (C•D) = (B+C) (B+D)	Distributività
T9	B • (B+C) = B	B + (B•C) = B	Assorbimento
T10	(B•C) + (B•$\overline{C}$) = B	(B+C) • (B+$\overline{C}$) = B	Combinazione
T11	(B•C) + ($\overline{B}$•D) + (C•D) =(B•C) + ($\overline{B}$•D)	(B+C) • ($\overline{B}$+D) • (C+D) = (B+C) • ($\overline{B}$+D)	Consenso

Principio di dualità: + ←→ · 1←→0

How do we prove these are true?

Demonstration techniques

- There are several methods to demonstrate the equivalence of two expressions.
  - Perfect induction: if the truth tables of two expressions coincide then the two expressions are equivalent
  - use previously proven axioms and theorems to manipulate expressions until equal expressions are obtained

Tecniche di Dimostrazione

- Esistono diversi metodi per dimostrare l'equivalenza di due espressioni.
  - Induzione perfetta: se le tavole di verità di due espressioni coincidono allora le due espressioni sono equivalenti
  - utilizzare assiomi e teoremi precedentemente provati per manipolare le espressioni fino a ottenere espressioni uguali

Proof by Perfect Induction

Also called: **proof by exhaustion**

- Check every possible input value
- If two expressions produce the same value for every possible input combination, the expressions are equal

Dimostrazione attraverso la PERFECT INDUCTION

Chiamato anche: prova per ESAUSTIONE

- Controllare ogni possibile valore di input
- Se due espressioni producono lo stesso valore per ogni possibile combinazione di input, le espressioni sono uguali

Example: Proof by Perfect Induction

Number	Theorem	Name
T6	B•C = C•B	Commutativity

B	***C***	***BC***	***CB***
0	0		
0	1		
1	0		
1	1		

Example: Proof by Perfect Induction

Number	Theorem	Name
T6	B•C = C•B	Commutativity

B	*C*	*BC*	*CB*
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

T7: Associativity

Number	Theorem	Name
T7	(B•C) • D = B • (C • D)	Associativity

T8: Distributivity

Number	Theorem	Name
T8	B • (C + D) = (B•C) + (B•D)	Distributivity

T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Assorbimento

Prove true by:

- **Method 1:** Perfect induction
- **Method 2:** Using other theorems and axioms

T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Covering

Method 1: Perfect Induction

B	*C*	*(B+C)*	*B(B+C)*
0	0		
0	1		
1	0		
1	1		

T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Covering

Method 1: Perfect Induction

B	*C*	*(B+C)*	*B(B+C)*
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Covering

Method 1: Perfect Induction

B	*C*	*(B+C)*	*B(B+C)*
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 2: Prove true using other axioms and theorems.

T9: Covering

Number	Theorem	Name
T9	B• (B+C) = B	Assorbimento

Method 2: Prove true using other axioms and theorems.

B•(B+C)	= B•B + B•C	T8: Distributivity
	= **B** + B•C	T3: Idempotency
	= B•(1 + C)	T8: Distributivity
	= B•(**1**)	T2: Null element
	= B	T1: Identity

T10: Combining

Number	Theorem	Name
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combinazione

Prove true using other axioms and theorems:

T10: Combining

Number	Theorem	Name
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	Combining

Prove true using other axioms and theorems:

$$B \bullet C + B \bullet \overline{C} = B \bullet (C + \overline{C}) \quad \text{T8: Distributivity}$$

$$= B \bullet (\mathbf{1}) \quad \text{T5': Complements}$$

$$= B \quad \text{T1: Identity}$$

T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) =$ $(B \bullet C) + (\overline{B} \bullet D)$	Consensus

Prove true using (1) perfect induction or (2) other axioms and theorems.

Perfect induction: consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) =$ $(B \bullet C) + (\overline{B} \bullet D)$	Consensus

B	C	D	$BC + \overline{B}D + CD$	$BC + \overline{B}D$
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	0	0
1	1	0	1	1
1	1	1	1	1

T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) =$ $(B \bullet C) + (\overline{B} \bullet D)$	Consensus

Prove using other theorems and axioms:

$B \bullet C + \overline{B} \bullet D + C \bullet D$

$= BC + \overline{B}D + (CDB + CD\overline{B})$ **T10: Combining**

$= BC + \overline{B}D + BCD + \overline{B}CD$ **T6: Commutativity**

$= BC + BCD + \overline{B}D + \overline{B}CD$ **T6: Commutativity**

$= (BC + BCD) + (\overline{B}D + \overline{B}CD)$ **T7: Associativity**

$= BC + \overline{B}D$ **T9’: Covering**

Boolean Theorems of Several Vars

#	Theorem	Dual	Name
T6	B•C = C•B	B+C = C+B	Commutativity
T7	(B•C) • D = B • (C•D)	(B + C) + D = B + (C + D)	Associativity
T8	B • (C + D) = (B•C) + (B•D)	B + (C•D) = (B+C) (B+D)	Distributivity
T9	B • (B+C) = B	B + (B•C) = B	Covering
T10	(B•C) + (B•$\overline{C}$) = B	(B+C) • (B+$\overline{C}$) = B	Combining
T11	(B•C) + ($\overline{B}$•D) + (C•D) = (B•C) + ($\overline{B}$•D)	(B+C) • ($\overline{B}$+D) • (C+D) = (B+C) • ($\overline{B}$+D)	Consensus

Warning: T8’ differs from traditional algebra:
OR (+) distributes over AND (•)

Limiti del perfect induction

- La tecnica del perfect induction è semplice ma «priva di intelligenza»
- Al crescere della lunghezza delle espressioni diventa sempre più laboriosa
- Al crescere delle variabili che occorrono nelle espressioni diventa estremamente più laboriosa
  - 4→16 checks
  - 5→32 checks
  - 6→64 checks
  - …

Semplificare un equazione

- Ridurre un'equazione al minor numero di implicanti, dove ogni implicante ha il minor numero di letterali

- Assiomi e teoremi sono utili per semplificare le equazioni.

Minimizzazione di un Equazione

- Ridurre un'equazione al minor numero di implicanti, dove ogni implicante ha il minor numero di letterali

- Ricordatevi che..
  - Implicante: prodotto di letterali

    ABC, AC, BC
  - Letterale: variabile o suo complemento

    A, A, B, B, C, C

Metodi di Semplificazione

- **Distributivity (T8, T8')** — $B(C+D) = BC + BD$
  $B + CD = (B+C)(B+D)$
- **Covering (T9')** — $A + AP = A$
- **Combining (T10)** — $PA + P\overline{A} = P$
- **Expansion** — $P = P\overline{A} + PA$
  $A = A + AP$
- **Duplication** — $A = A + A$
- **"Simplification" theorem** — $P\overline{A} + A = P + A$
  $PA + \overline{A} = P + \overline{A}$

Proving the "Simplification" Theorem

"Simplification" theorem

$PA + \overline{A} = P + \overline{A}$

Method 1:

$$
\begin{aligned}
PA + \overline{A} &= PA + (\overline{A} + \overline{A}P) && \text{T9' Covering (A+AP =A => A'+A'P =A')} \\
&= PA + P\overline{A} + \overline{A} && \text{T6 Commutativity} \\
&= P(A + \overline{A}) + \overline{A} && \text{T8 Distributivity} \\
&= P(1) + \overline{A} && \text{T5' Complements} \\
&= P + \overline{A} && \text{T1 Identity}
\end{aligned}
$$

Proving the "Simplification" Theorem

"Simplification" theorem

$PA + \overline{A} = P + \overline{A}$

Method 2: $PA + \overline{A} = (\overline{A} + A)(\overline{A} + P)$ **T8' Distributivity**

$= 1(\overline{A} + P)$ **T5' Complements**

$= \overline{\mathbf{A}} + \mathbf{P}$ **T1 Identity**

Semplificazioni di funzioni booleane

Esercizi

1. $Y = AB + A\overline{B}$
2. $Y = A(AB + ABC)$
3. $Y = A'BC + A'$ **Recall:** $A' = A$
4. $Y = AB'C + ABC + A'BC$
5. $Y = AB + BC + B'D' + AC'D'$
6. $Y = (A + BC)(A + DE)$ Apply T8′ first when possible: $W+XZ = (W+X)(W+Z)$

Semplificazioni di funzioni booleane

Esercizio 1

$Y = AB + A\overline{B}$

Y = A — T10: Combining

or

$= A(B + \overline{B})$ — T8: Distributivity

$= A(1)$ — T5': Complements

$= A$ — T1: Identity

Semplificazioni di funzioni booleane

Esercizio 2

$$Y = A(AB + ABC)$$

$= A(AB(1 + C))$	T8: Distributivity
$= A(AB(1))$	T2′: Null Element
$= A(AB)$	T1: Identity
$= (AA)B$	T7: Associativity
$= AB$	T3: Idempotency

Semplificazioni di funzioni booleane

Esercizio 3

$Y = A'BC + A'$	**Recall:** $A' = \overline{A}$
$= A'$	T9' Covering: $X + XY = X$
or	
$= A'(BC + 1)$	T8: Distributivity
$= A'(1)$	T2': Null Element
$= A'$	T1: Identity

Semplificazioni di funzioni booleane

Esercizio 4

Y = AB’C + ABC + A’BC	Duplication
= AB’C + **ABC + ABC** + A’BC	T3’: Idempotency
= (AB’C+ABC) + (ABC+A’BC)	T7’: Associativity
= AC + BC	T10: Combining

Semplificazioni di funzioni booleane

Esercizio 5

Y = AB + BC +B'D' + AC'D'

Method 1:

Y = AB + BC + B'D' + AC'D'B + AC'D'B' — T10: Combining (PA' + PA = P → AC'D' =AC'D'B + AC'D'B')

= (AB + ABC'D') + (B'D' + B'D'AC') + BC — T6: Commutativity e T7: Associativity

= (AB + ABC'D') + (B'D' + B'D'C'A) + BC — T9: Covering (A+AP=A → AB + ABC'D'= AB)

T9: Covering (A+AP=A → B'D' + B'D'C'A = B'D')

= AB + B'D' + BC

Method 2:

Y = AB + BC + B'D' + AC'D' + AD' — T6: Commutativity

= BA + B'D' + AD' + AD'C' + BC — T9: Covering (A+AP=A → AD'+ AD'C' =AD')

= BA + B'D' + AD' + BC — T11: Consensus (**BA** + **B'D**' + **AD' = BA** + **B'D**')

= BA + B'D' + BC — Ricorda che: T11: Consensus (**BC + B'D + CD = BC + B'D**)

Semplificazioni di funzioni booleane

Esercizio 6

Apply T8' first when possible: W+XZ = (W+X)(W+Z)

Y = (A + BC)(A + DE)

Make: X = BC, Z = DE and rewrite equation

Y = (A+X)(A+Z)	substitution (X=BC, Z=DE)
= A + XZ	T8': Distributivity
= A + BCDE	substitution

or

Y = (A + BC)(A + DE)	T8: Distributivity
= AA + ADE + BCA + BCDE	T3: Idempotency (AA = A)
= A + ADE + BCA + BCDE	T9': Covering (A+AP=A → A+ADE = A)
= A + ABC + BCDE	T7: Commutativity T9': Covering (A+AP=A → A+ABC = A)
= A + BCDE	