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Laboratorio di Architettura Degli Elaboratori

Algebra di Boole e reti combinatorie

Boolean Theorems of Several Var

#	Theorem	Dual	Name
T1	$B \bullet 1 = B$	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	$B + B = B$	Idempotency
T4	$\overline{\overline{B}} = B$	$B = B$	Involution
T5	$B \bullet \overline{B} = 0$	$B + B = 1$	Complements
T6	$B \bullet C = C \bullet B$	$B + C = C + B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	$(B + C) + D = B + (C + D)$	Associativity
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	$B + (C \bullet D) = (B + C) (B + D)$	Distributivity
T9	$B \bullet (B + C) = B$	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	$(B + C) \bullet (B + \overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) = (B \bullet C) + (\overline{B} \bullet D)$	$(B + C) \bullet (\overline{B} + D) \bullet (C + D) = (B + C) \bullet (\overline{B} + D)$	Consensus

Teoremi a più variabili: dualità

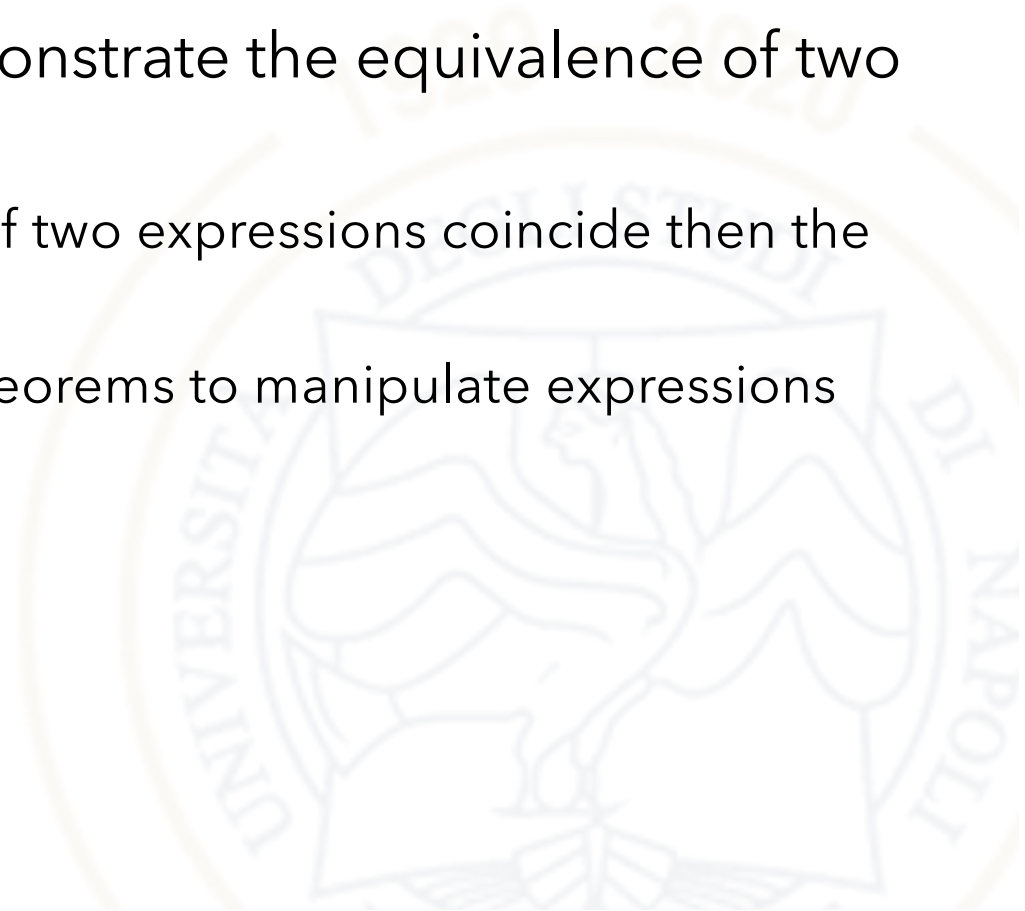
#	Theorem	Dual	Name
T6	$B \bullet C = C \bullet B$	$B + C = C + B$	Commutatività
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	$(B + C) + D = B + (C + D)$	Associatività
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	$B + (C \bullet D) = (B + C)(B + D)$	Distributività
T9	$B \bullet (B + C) = B$	$B + (B \bullet C) = B$	Assorbimento
T10	$(B \bullet C) + (B \bullet \bar{C}) = B$	$(B + C) \bullet (B + \bar{C}) = B$	Combinazione
T11	$(B \bullet C) + (\bar{B} \bullet D) + (C \bullet D) = (B \bullet C) + (\bar{B} \bullet D)$	$(B + C) \bullet (\bar{B} + D) \bullet (C + D) = (B + C) \bullet (\bar{B} + D)$	Consenso

Principio di dualità: $+ \leftrightarrow \cdot$ $1 \leftrightarrow 0$

How do we prove these are true?

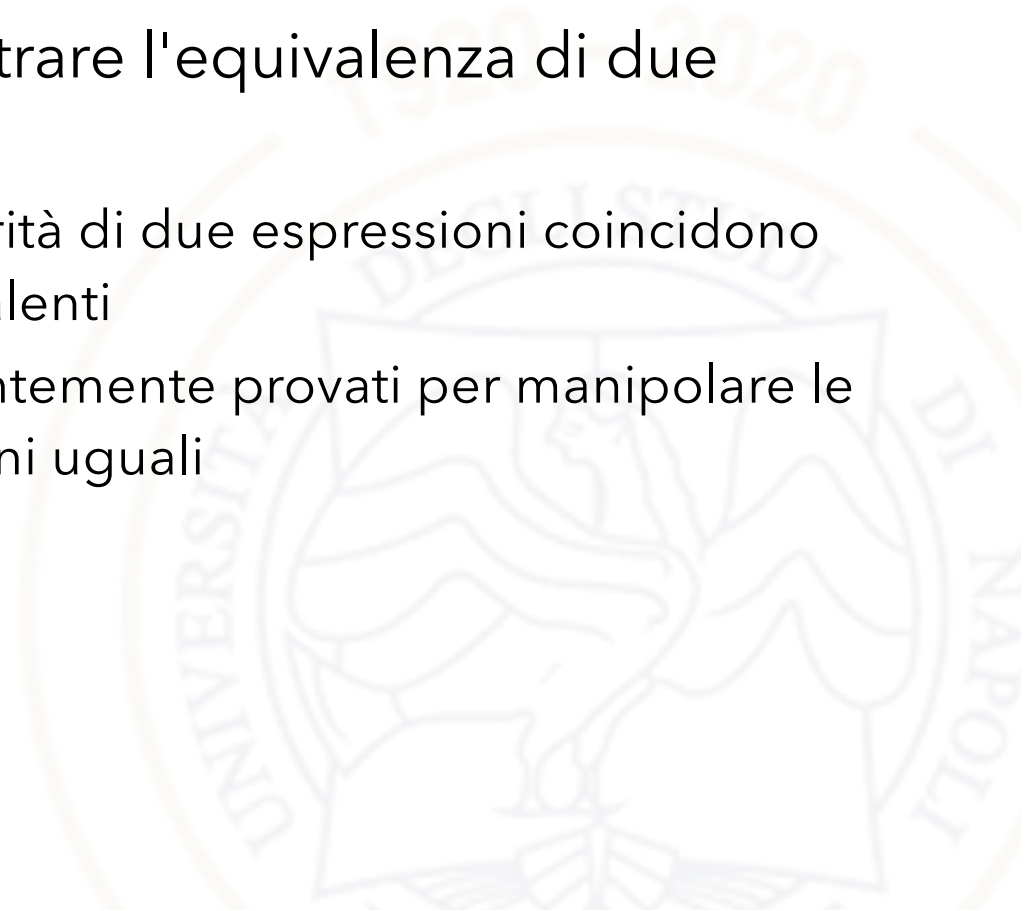
Demonstration techniques

- There are several methods to demonstrate the equivalence of two expressions.
 - Perfect induction: if the truth tables of two expressions coincide then the two expressions are equivalent
 - use previously proven axioms and theorems to manipulate expressions until equal expressions are obtained



Tecniche di Dimostrazione

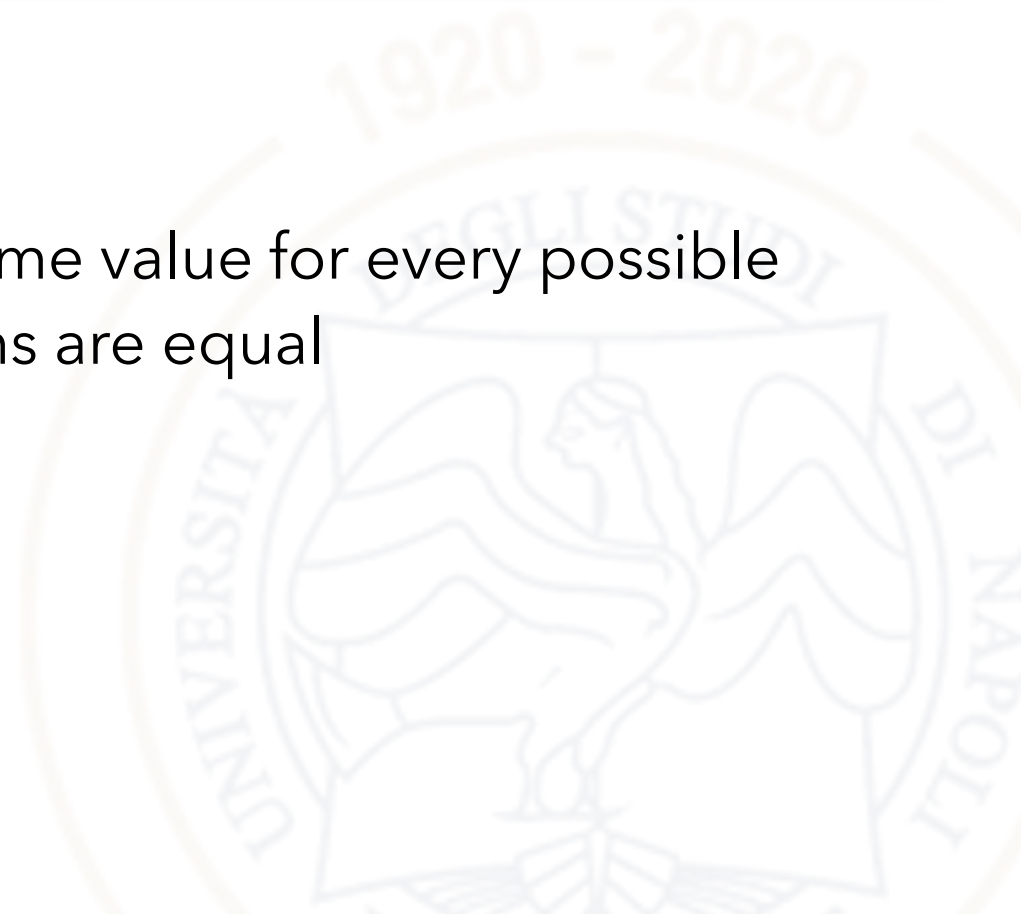
- Esistono diversi metodi per dimostrare l'equivalenza di due espressioni.
 - **Induzione perfetta**: se le tavole di verità di due espressioni coincidono allora le due espressioni sono equivalenti
 - **utilizzare assiomi e teoremi** precedentemente provati per manipolare le espressioni fino a ottenere espressioni uguali



Proof by Perfect Induction

Also called: **proof by exhaustion**

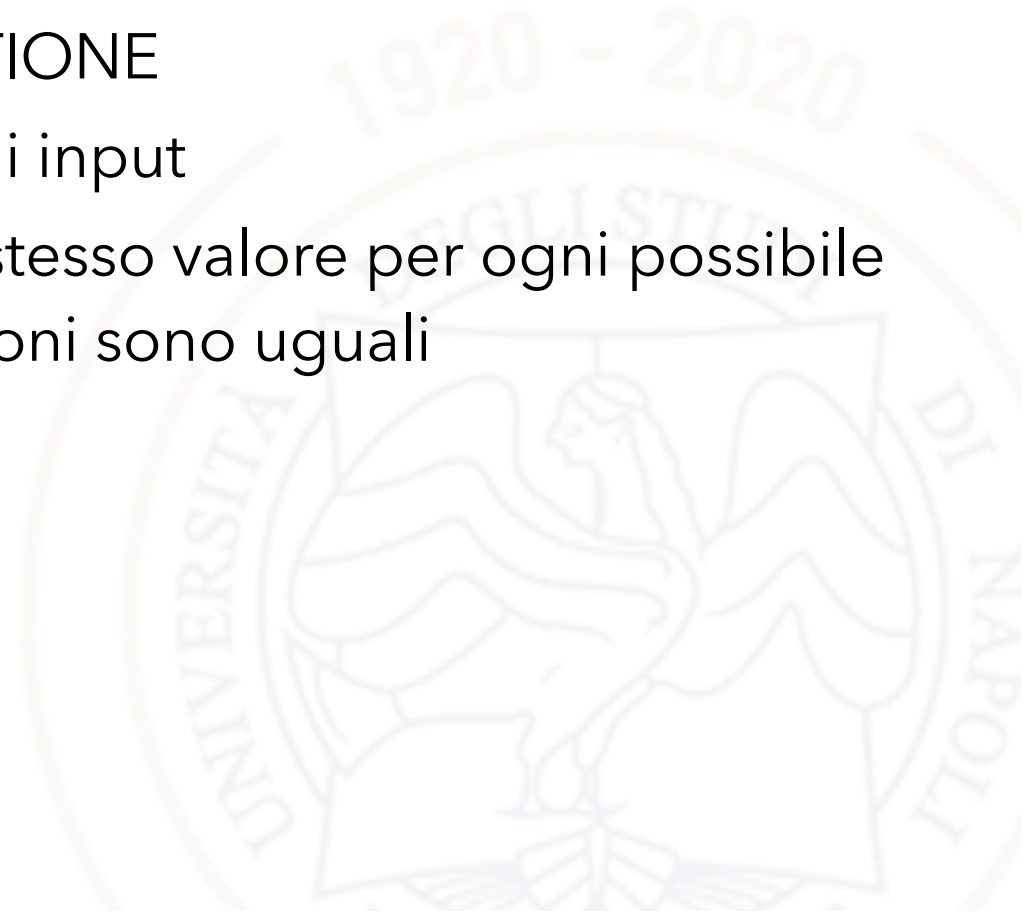
- Check every possible input value
- If two expressions produce the same value for every possible input combination, the expressions are equal



Dimostrazione attraverso la PERFECT INDUCTION

Chiamato anche: prova per ESAUSTIONE

- Controllare ogni possibile valore di input
- Se due espressioni producono lo stesso valore per ogni possibile combinazione di input, le espressioni sono uguali



Example: Proof by Perfect Induction

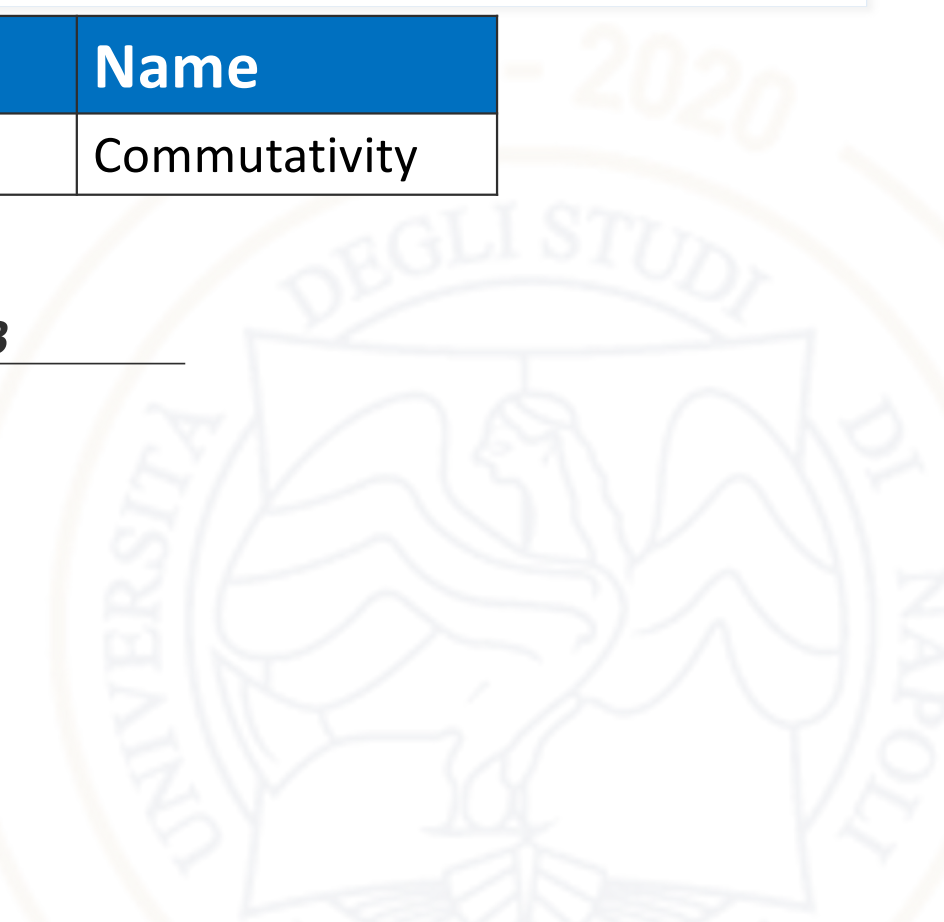
Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity

<i>B</i>	<i>C</i>	<i>BC</i>	<i>CB</i>
0	0		
0	1		
1	0		
1	1		

Example: Proof by Perfect Induction

Number	Theorem	Name
T6	$B \bullet C = C \bullet B$	Commutativity

<i>B</i>	<i>C</i>	<i>BC</i>	<i>CB</i>
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1



T7: Associativity

Number	Theorem	Name
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	Associativity

T8: Distributivity

Number	Theorem	Name
T8	$B \bullet (C + D) = (B \bullet C) + (B \bullet D)$	Distributivity

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Assorbimento

Prove true by:

- **Method 1:** Perfect induction
- **Method 2:** Using other theorems and axioms

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 1: Perfect Induction

<i>B</i>	<i>C</i>	<i>(B+C)</i>	<i>B(B+C)</i>
0	0		
0	1		
1	0		
1	1		

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 1: Perfect Induction

B	C	$(B+C)$	$B(B+C)$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 1: Perfect Induction

B	C	$(B+C)$	$B(B+C)$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Covering

Method 2: Prove true using other axioms and theorems.

T9: Covering

Number	Theorem	Name
T9	$B \bullet (B+C) = B$	Assorbimento

Method 2: Prove true using other axioms and theorems.

$$\begin{aligned} B \bullet (B+C) &= B \bullet B + B \bullet C \\ &= \mathbf{B} + B \bullet C \\ &= B \bullet (1 + C) \\ &= B \bullet (\mathbf{1}) \\ &= B \end{aligned}$$

T8: Distributivity
T3: Idempotency
T8: Distributivity
T2: Null element
T1: Identity

T10: Combining

Number	Theorem	Name
T10	$(B \bullet C) + (B \bullet \bar{C}) = B$	Combinazione

Prove true using other axioms and theorems:

T10: Combining

Number	Theorem	Name
T10	$(B \bullet C) + (B \bullet \bar{C}) = B$	Combining

Prove true using other axioms and theorems:

$$\begin{aligned} B \bullet C + B \bullet \bar{C} &= B \bullet (C + \bar{C}) && \text{T8: Distributivity} \\ &= B \bullet (\mathbf{1}) && \text{T5': Complements} \\ &= B && \text{T1: Identity} \end{aligned}$$

T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\bar{B} \bullet D) + (C \bullet D) = (B \bullet C) + (\bar{B} \bullet D)$	Consensus

Prove true using (1) perfect induction or (2) other axioms and theorems.

Perfect induction: consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\bar{B} \bullet D) + (C \bullet D) = (B \bullet C) + (\bar{B} \bullet D)$	Consensus

<i>B</i>	<i>C</i>	<i>D</i>	$BC + \bar{B}D + CD$	$BC + \bar{B}D$
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	0	0
1	1	0	1	1
1	1	1	1	1

T11: Consensus

Number	Theorem	Name
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D) = (B \bullet C) + (\overline{B} \bullet D)$	Consensus

Prove using other theorems and axioms:

$$\begin{aligned} B \bullet C + \overline{B} \bullet D + C \bullet D &= BC + \overline{BD} + (CDB + C\overline{D}\overline{B}) \\ &= BC + \overline{BD} + BCD + \overline{BCD} \\ &= BC + BCD + \overline{BD} + \overline{BCD} \\ &= (BC + BCD) + (\overline{BD} + \overline{BCD}) \\ &= BC + \overline{BD} \end{aligned}$$

T10: Combining

T6: Commutativity

T6: Commutativity

T7: Associativity

T9': Covering

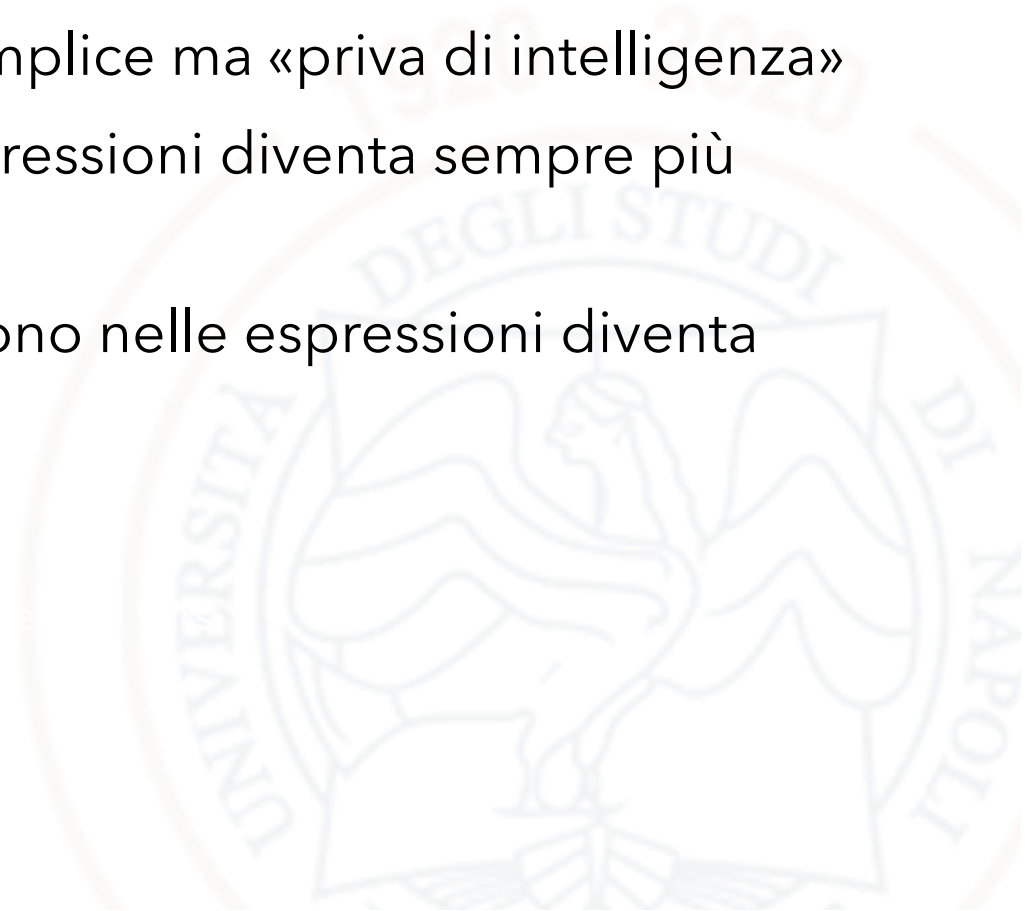
Boolean Theorems of Several Vars

#	Theorem	Dual	Name
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T11	$(B \bullet C) + (\bar{B} \bullet D) + (C \bullet D) = (B \bullet C) + (\bar{B} \bullet D)$	$(B + C) \bullet (\bar{B} + D) \bullet (C + D) = (B + C) \bullet (\bar{B} + D)$	Consensus

Warning: T8' differs from traditional algebra:
OR (+) distributes over AND (•)

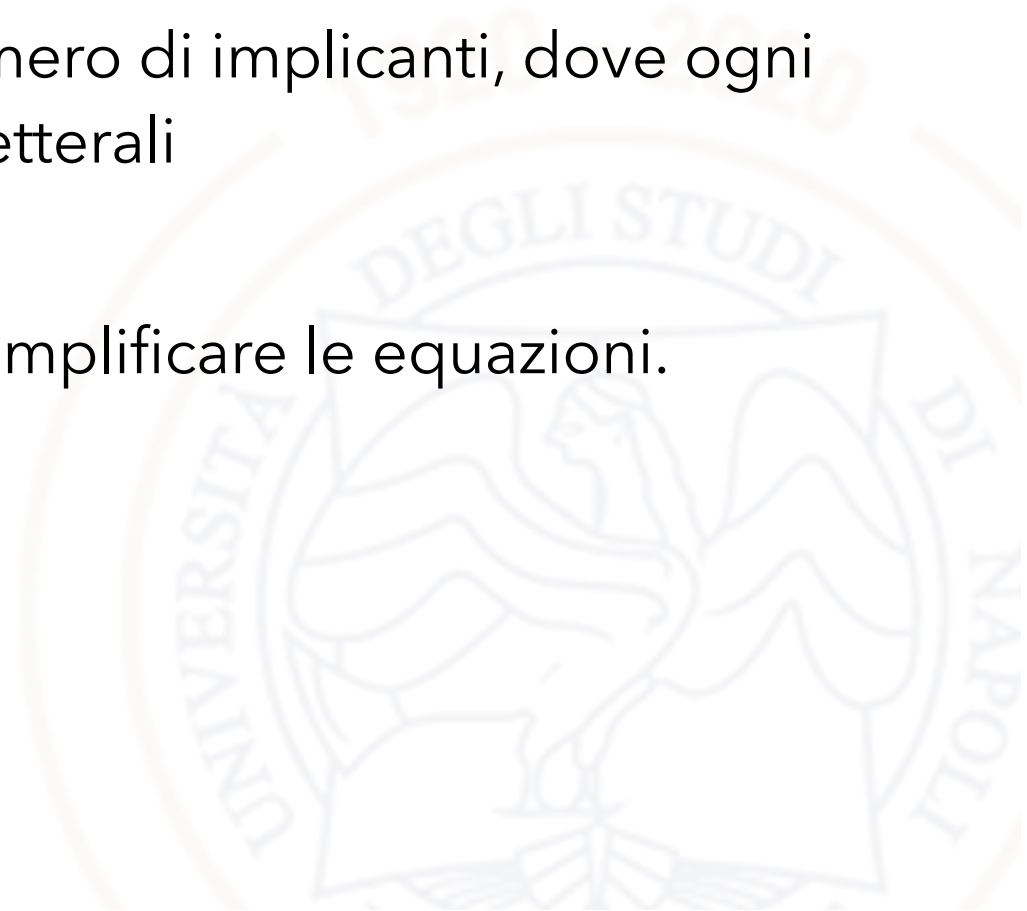
Limiti del perfect induction

- La tecnica del perfect induction è semplice ma «priva di intelligenza»
- Al crescere della lunghezza delle espressioni diventa sempre più laboriosa
- Al crescere delle variabili che occorrono nelle espressioni diventa estremamente più laboriosa
 - 4→16 checks
 - 5→32 checks
 - 6→64 checks
 - ...



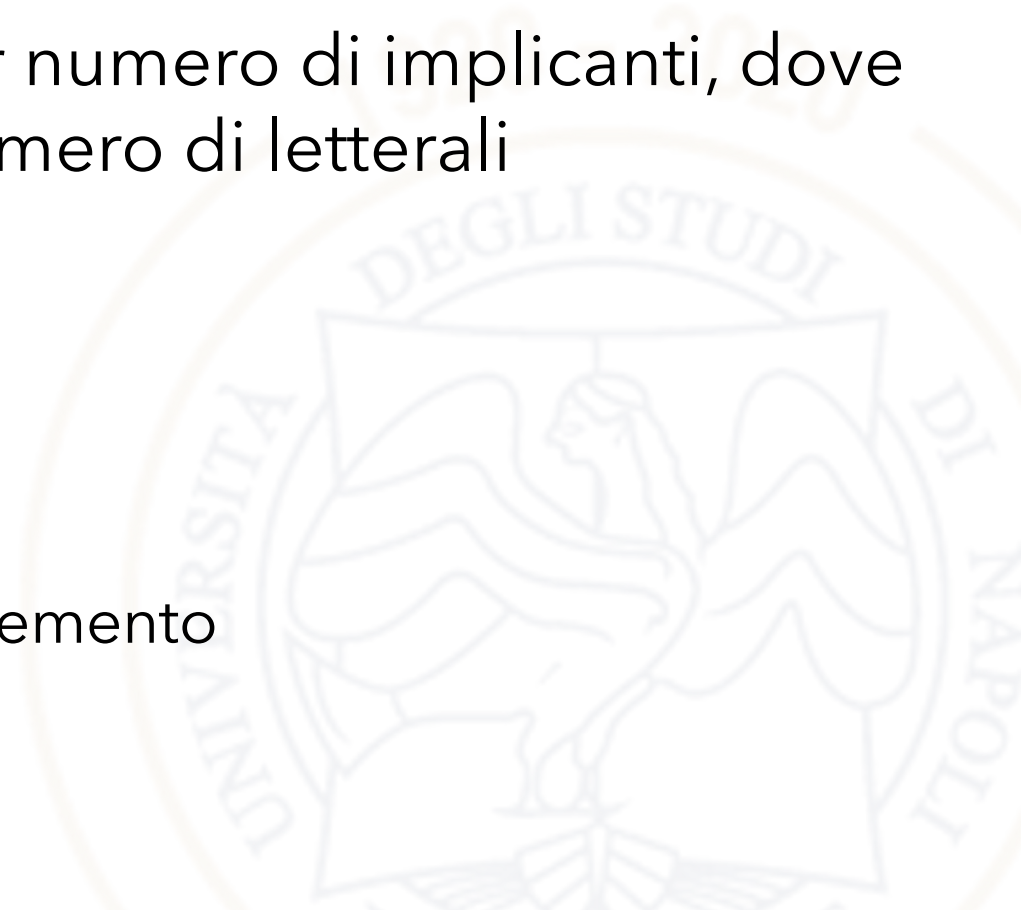
Semplificare un'equazione

- Ridurre un'equazione al minor numero di implicanti, dove ogni implicante ha il minor numero di letterali
- Assiomi e teoremi sono utili per semplificare le equazioni.



Minimizzazione di un Equazione

- Ridurre un'equazione al minor numero di implicanti, dove ogni implicante ha il minor numero di letterali
- Ricordatevi che..
 - **Implicante:** prodotto di letterali
ABC, AC, BC
 - **Letterale:** variabile o suo complemento
A, A, B, B, C, C



Metodi di Semplificazione

- **Distributivity (T8, T8')**

$$B(C+D) = BC + BD$$

$$B + CD = (B+ C)(B+D)$$

- **Covering (T9')**

$$A + AP = A$$

- **Combining (T10)**

$$PA + P\bar{A} = P$$

- **Expansion**

$$P = P\bar{A} + PA$$

$$A = A + AP$$

- **Duplication**

$$A = A + A$$

- **"Simplification" theorem**

$$P\bar{A} + A = P + A$$

$$PA + \bar{A} = P + \bar{A}$$

Proving the “Simplification” Theorem

“Simplification” theorem

$$PA + \bar{A} = P + \bar{A}$$

Method 1: $PA + \bar{A} = PA + (\bar{A} + \bar{A}P)$
 $= PA + P\bar{A} + \bar{A}$
 $= P(A + \bar{A}) + \bar{A}$
 $= P(1) + \bar{A}$
 $= P + \bar{A}$

T9' Covering ($A+AP = A \Rightarrow A'+A'P = A'$)

T6 Commutativity

T8 Distributivity

T5' Complements

T1 Identity

Proving the “Simplification” Theorem

“Simplification” theorem

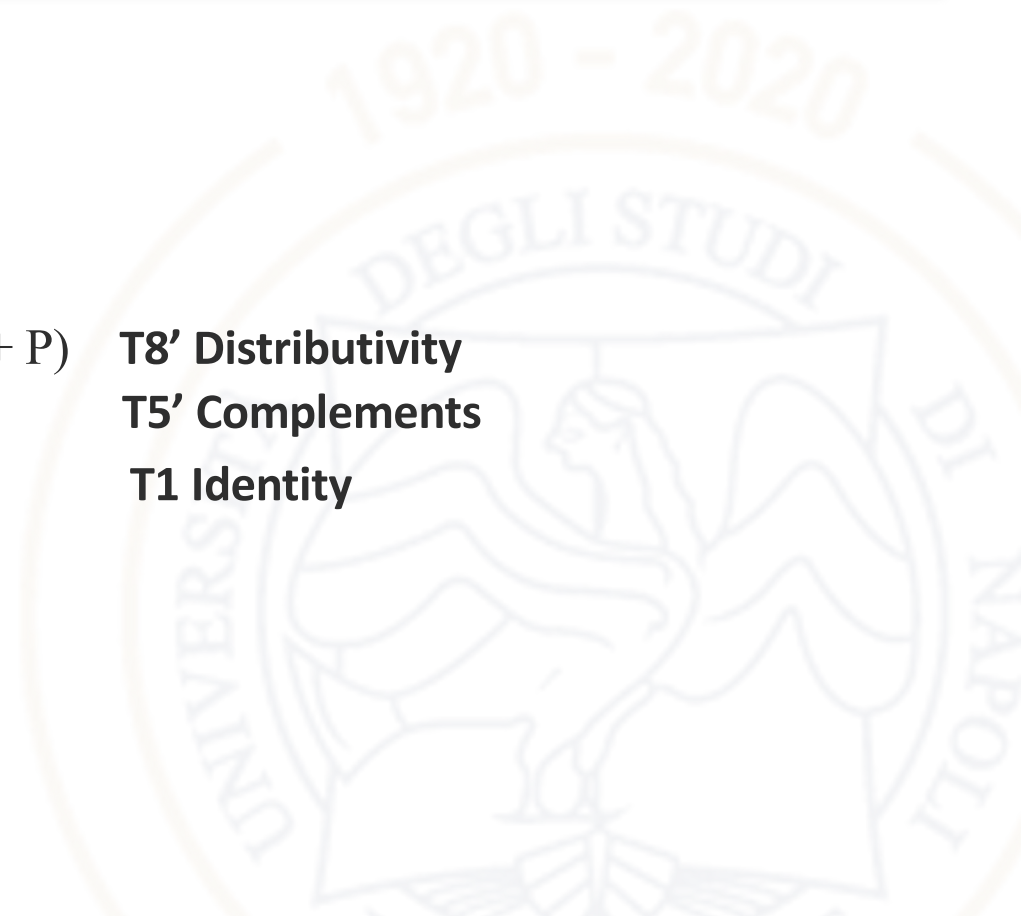
$$PA + \bar{A} = P + \bar{A}$$

Method 2: $PA + \bar{A} = (\bar{A} + A)(\bar{A} + P)$
 $= 1(\bar{A} + P)$
 $= \bar{A} + P$

T8' Distributivity

T5' Complements

T1 Identity



Semplificazioni di funzioni booleane

Esercizi

1. $Y = AB + \overline{AB}$

2. $Y = A(AB + ABC)$

3. $Y = A'BC + A'$

Recall: $A' = A$

4. $Y = AB'C + ABC + A'BC$

5. $Y = AB + BC + B'D' + AC'D'$

6. $Y = (A + BC)(A + DE)$ Apply T8' first when possible: $W + XZ = (W + X)(W + Z)$

Semplificazioni di funzioni booleane

Esercizio 1

$$Y = AB + A\bar{B}$$

$$Y = A$$

T10: Combining

or

$$= A(B + \bar{B})$$

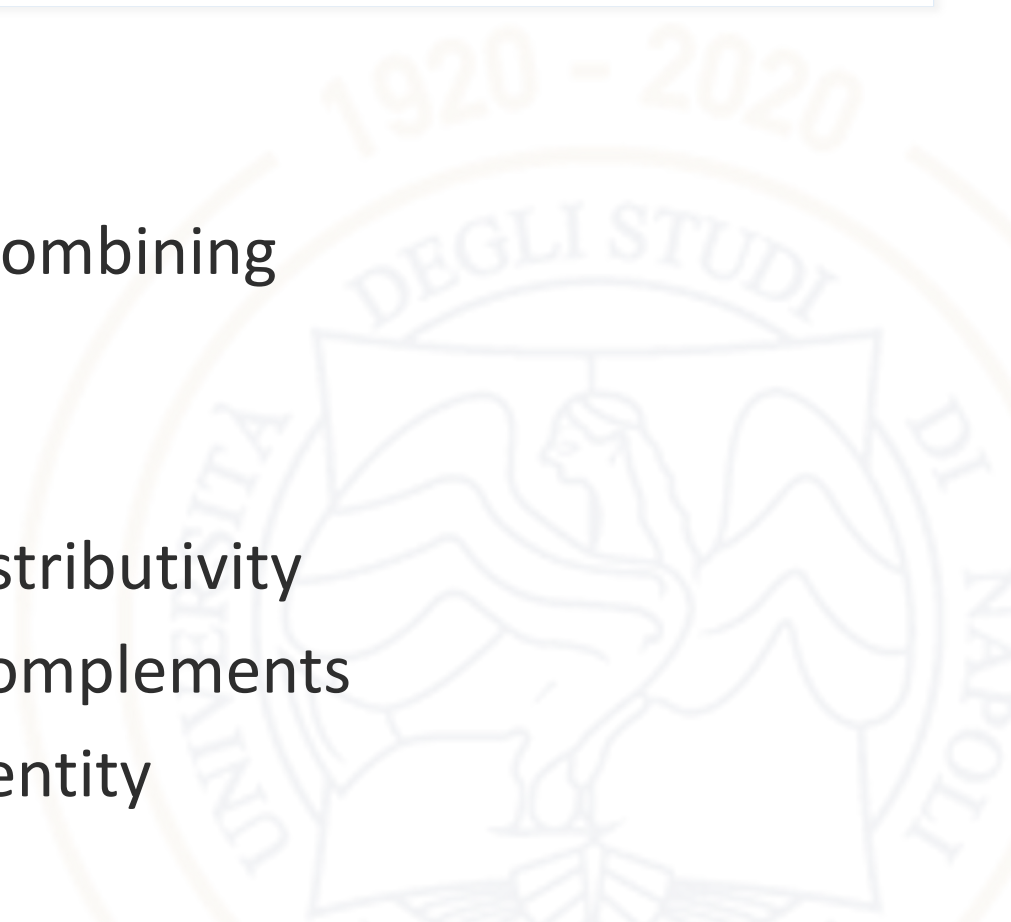
T8: Distributivity

$$= A(1)$$

T5': Complements

$$= A$$

T1: Identity



Semplificazioni di funzioni booleane

Esercizio 2

$$Y = A(AB + ABC)$$

$$= A(AB(1 + C))$$

$$= A(AB(1))$$

$$= A(AB)$$

$$= (AA)B$$

$$= AB$$

T8: Distributivity

T2': Null Element

T1: Identity

T7: Associativity

T3: Idempotency

Semplificazioni di funzioni booleane

Esercizio 3

$$Y = A'BC + A'$$

$$= A'$$

or

$$= A'(BC + 1)$$

$$= A'(1)$$

$$= A'$$

Recall: $A' = \bar{A}$

T9' Covering: $X + XY = X$

T8: Distributivity

T2': Null Element

T1: Identity

Semplificazioni di funzioni booleane

Esercizio 4

$$Y = AB'C + ABC + A'BC$$

$$= AB'C + \mathbf{ABC} + \mathbf{ABC} + A'BC$$

$$= (AB'C+ABC) + (ABC+A'BC)$$

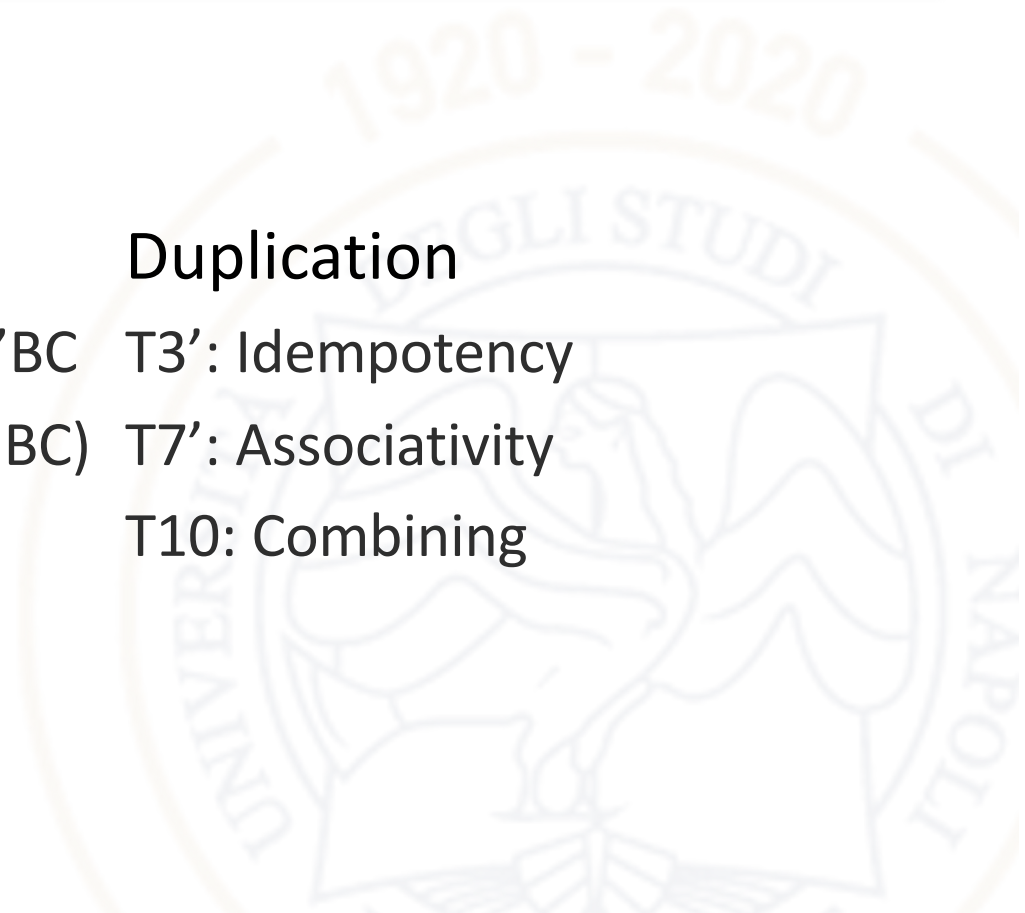
$$= AC + BC$$

Duplication

T3': Idempotency

T7': Associativity

T10: Combining



Semplificazioni di funzioni booleane

Esercizio 5

$$Y = AB + BC + B'D' + AC'D'$$

Method 1:

$$\begin{aligned} Y &= AB + BC + B'D' + AC'D'B + AC'D'B' \\ &= (AB + ABC'D') + (B'D' + B'D'AC') + BC \\ &= (AB + ABC'D') + (B'D' + B'D'C'A) + BC \\ &= AB + B'D' + BC \end{aligned}$$

Method 2:

$$\begin{aligned} Y &= AB + BC + B'D' + AC'D' + AD' \\ &= BA + B'D' + AD' + AD'C' + BC \\ &= BA + B'D' + AD' + BC \\ &= BA + B'D' + BC \end{aligned}$$

T10: Combining ($PA' + PA = P \rightarrow AC'D' = AC'D'B + AC'D'B'$)

T6: Commutativity e T7: Associativity

T9: Covering ($A+AP=A \rightarrow AB + ABC'D' = AB$)

T9: Covering ($A+AP=A \rightarrow B'D' + B'D'C'A = B'D'$)

T6: Commutativity

T9: Covering ($A+AP=A \rightarrow AD' + AD'C' = AD'$)

T11: Consensus ($BA + B'D' + AD' = BA + B'D'$)

Ricorda che: T11: Consensus ($BC + B'D + CD = BC + B'D$)

Semplificazioni di funzioni booleane

Esercizio 6

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$

$$Y = (A + BC)(A + DE)$$

Make: $X = BC$, $Z = DE$ and rewrite equation

$$\begin{aligned} Y &= (A+X)(A+Z) \\ &= A + XZ \\ &= A + BCDE \end{aligned}$$

substitution ($X=BC$, $Z=DE$)

T8': Distributivity

substitution

or

$$\begin{aligned} Y &= (A + BC)(A + DE) \\ &= AA + ADE + BCA + BCDE \\ &= A + ADE + BCA + BCDE \\ &= A + ABC + BCDE \\ &= A + BCDE \end{aligned}$$

T8: Distributivity

T3: Idempotency ($AA = A$)

T9': Covering ($A+AP=A \rightarrow A+ADE = A$)

T7: Commutativity T9': Covering ($A+AP=A \rightarrow A+ABC = A$)