Lezione del 28/09/2022

Informatica

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$ serie armonica generalizzata

$\alpha \in \mathbb{R}$

$\alpha = 1$ $\quad \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$

$1 < \alpha < 2$

$\alpha = 2$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$

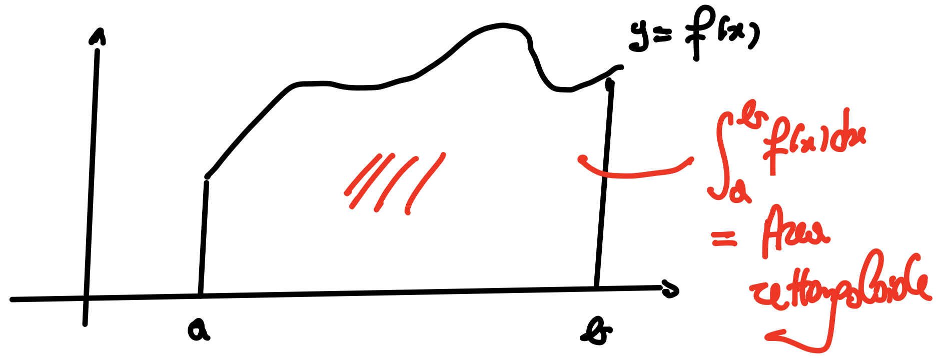
$\alpha < 1$?

$\alpha > 2$

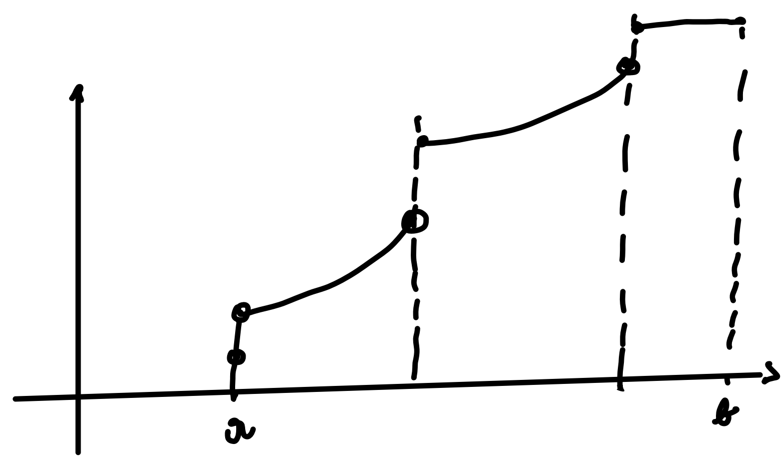
$f(x)$, $f: [a,b] \longrightarrow \mathbb{R}$, $f \geq 0$

continua

$\int_a^b f(x)\,dx$? Integrale di Riemann di $f(x)$

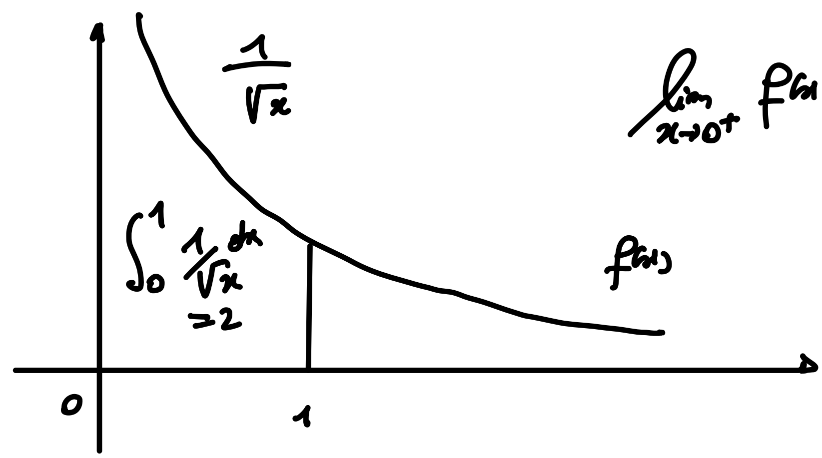


$f(x)$ monotona (cresente o decresente)



$f(x) \geq 0$

$$\int_a^b f(x)\,dx =$$



$$\lim_{x \to 0^+} f(x) = +\infty$$

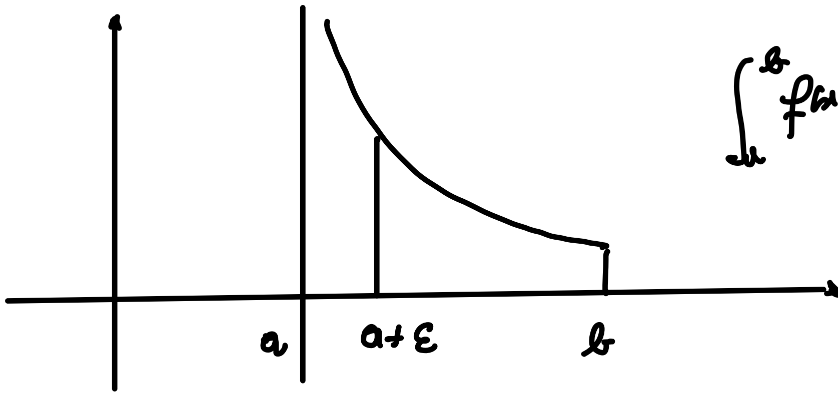
$$\int_0^1 f(x)\,dx \; ? \qquad \int_0^1 \frac{1}{\sqrt{x}}\,dx =$$

$$= \int_0^1 x^{-\frac{1}{2}}\,dx = \left(\frac{x^{1-\frac{1}{2}}}{\frac{1}{2}}\right)_0^1 = 2\left[\sqrt{x}\right]_0^1$$

$$= 2$$

Integrali impropri o generalizzati

$f = f(x)$, $\quad f :]a,b] \longrightarrow \mathbb{R}$, f continua



$$\int_a^b f(x)\,dx \ ?$$

$\varepsilon > 0 \qquad \lim_{x \to a^+} f(x) = +\infty$

$$\lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} f(x)\,dx$$: se questo limite esiste ed è finito, $f(x)$ si dice integrabile in senso improprio in $]a,b]$ e scriveremo

$$\int_a^b f(x)\,dx := \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} f(x)\,dx$$

Es. $\quad f(x) = \dfrac{1}{\sqrt{x}}$, $x \in]0,1]$

$a = 0$, $b = 1$

$$\int_{\varepsilon}^{1} \frac{1}{\sqrt{x}}\,dx = 2\left[\sqrt{x}\right]_{\varepsilon}^{1} =$$
$$= 2\left[1-\sqrt{\varepsilon}\right]$$

$$\lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{1} \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0^+} 2(1-\sqrt{\varepsilon}) = 2$$
$$= \int_{0}^{1} \frac{1}{\sqrt{x}}\,dx$$

$f(x) = \frac{1}{x}, \quad x \in]0,1]$

$$\int_{\varepsilon}^{1} \frac{1}{x}\,dx = \left[\log x\right]_{\varepsilon}^{1} = -\log\varepsilon$$

$$\lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{1} \frac{1}{x}\,dx = -\lim_{\varepsilon\to 0^+} \log\varepsilon = +\infty$$

$\frac{1}{x}$ <u>non</u> è integrabile impropriamente

$f(x) = \frac{1}{x^{\alpha}}$ ($\alpha = 1$ non è integrabile)

$]0,1]$

$\alpha \neq 1$

$$\int_{\varepsilon}^{1} \frac{1}{x^{\alpha}}\,dx = \int_{\varepsilon}^{1} x^{-\alpha}\,dx$$

$$= \left(\frac{x^{1-\alpha}}{1-\alpha} \right]_{\varepsilon}^{1}$$

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{1}{x^{\alpha}}\, dx = \frac{1}{1-\alpha} \lim_{\varepsilon \to 0^+} \left(1 - \varepsilon^{1-\alpha}\right)$$

se $\underline{\alpha < 1}$ $\quad \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{1}{x^{\alpha}}\, dx = \frac{1}{1-\alpha}$

$$\left(\lim_{\varepsilon \to 0^+} \varepsilon^{1-\alpha} = 0 \, ! \right)$$

se $\alpha > 1$ $\quad \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{1}{x^{\alpha}}\, dx = +\infty$

$\boxed{1-\alpha > 0}$ $\qquad \varepsilon^{1-\alpha} \xrightarrow[\varepsilon \to 0^+]{} +\infty$

$$\int_0^1 \frac{1}{x^{\alpha}}\, dx = \begin{cases} \frac{1}{1-\alpha} & \text{se } 0 < \alpha < 1 \\ +\infty & \text{se } \alpha \geq 1 \end{cases} \quad \Big\|$$

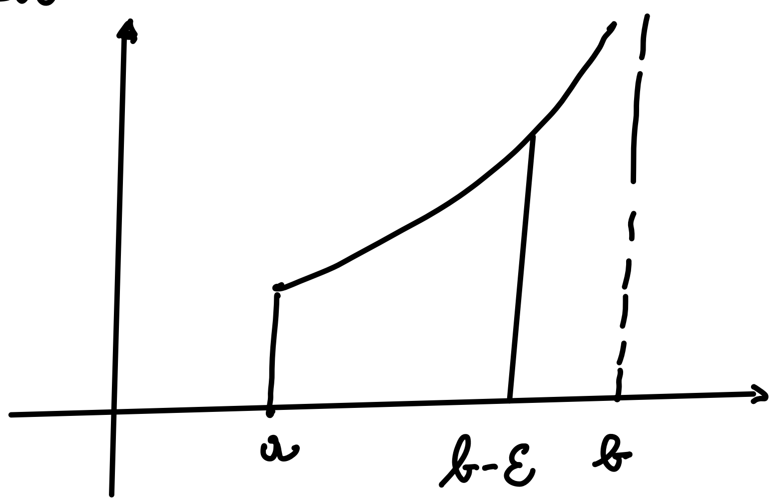
$\alpha = \frac{1}{2}$ $\quad \frac{1}{\sqrt{x}}$ è integrabile

$\alpha = 1 \qquad \frac{1}{x}$ <u>non</u> è integrabile (impropriamente)

$f: [a, b[\to \mathbb{R}$

$f(x)$ continua in $[a, b[$

$\lim_{x \to b^-} f(x) = +\infty$



Se $\lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)\,dx$ esiste finito,

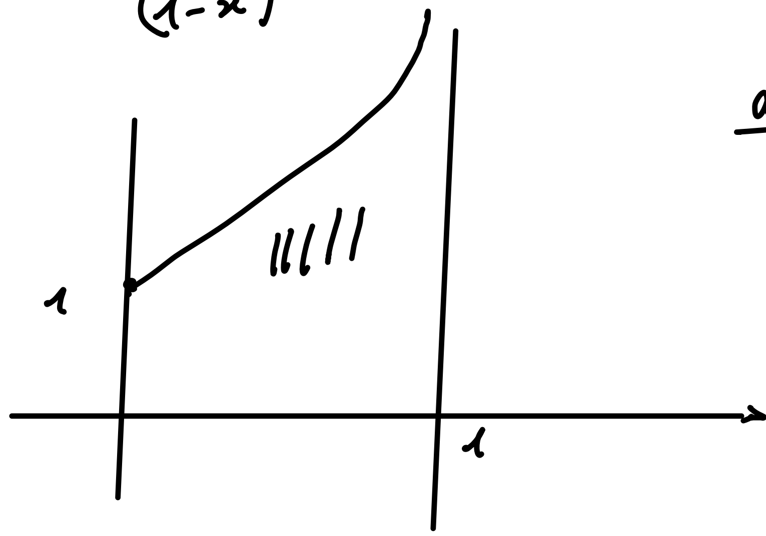
$f(x)$ si dice integrabile in senso improprio in $[a, b[$

e $$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)\,dx.$$

Es.

$$f(x) = \frac{1}{(1-x)^\alpha} \qquad x \in [0,1[$$

$\alpha > 0$



$$f(x) = \frac{1}{(1-x)^\alpha} \quad \text{è integrabile se } \alpha < 1$$

$$\frac{1}{\sqrt{1-x}}$$

non è integrabile se $\alpha \geq 1$

$a=0,\ b=1$

$$\frac{1}{1-x}$$

$$\int_0^{1-\varepsilon} \frac{1}{(1-x)^\alpha}\,dx = \int_0^{1-\varepsilon} (1-x)^{-\alpha}\,dx$$

$$= \dots$$

$\alpha \neq 1$

$$\frac{1}{1-\alpha} < 0$$

$$\int (1-x)^{-\alpha} dx = -\frac{(1-x)^{1-\alpha}}{1-\alpha}$$

$$\int_0^{1-\varepsilon} - = -\frac{1}{1-\alpha}\left[(1-x)^{1-\cancel{\alpha}}\right]_{x=0}^{x=1-\varepsilon}$$

$$= -\frac{1}{1-\alpha}\left[(\cancel{1}-\cancel{1}+\varepsilon)^{1-\alpha} - 1\right]$$

$$= -\frac{1}{1-\alpha}\left[\underset{\substack{\downarrow \\ +\infty}}{\varepsilon^{1-\alpha}} - 1\right] = \left(\frac{1}{1-\alpha}\right)$$

$\alpha < 1$

$\alpha > 1$ $\quad \underset{\substack{\downarrow \\ 0}}{\frac{1}{\alpha-1}} \quad \varepsilon^{1-\alpha} \to +\infty$

$\searrow +\infty$

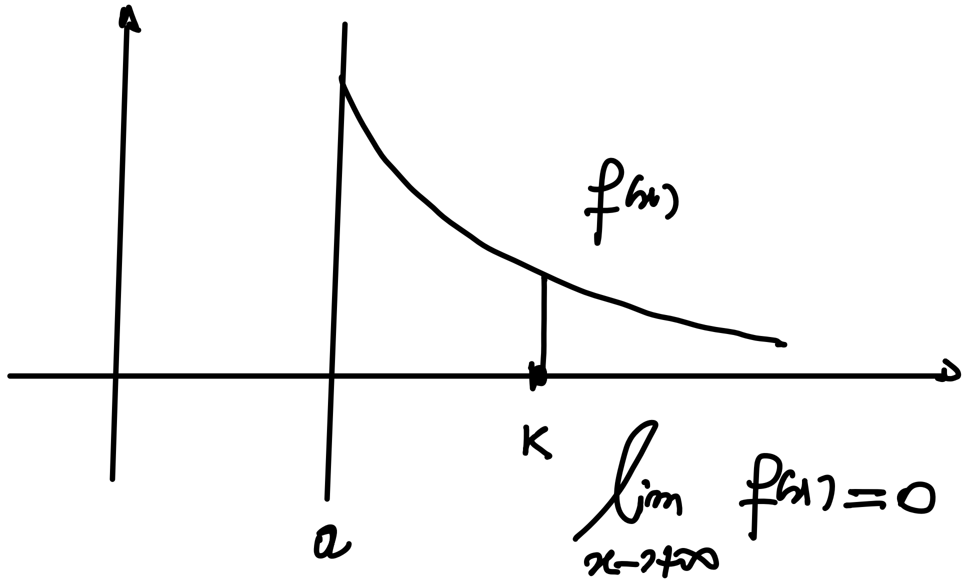
ε

$$\frac{1}{(1-x)^{\alpha}} = (1-x)^{\overbrace{-\alpha}^{>0}} = \text{continue in } [0,1]$$

$\alpha < 0$ $\qquad \int \quad < +\infty$

$$\lim_{x \to 0} (1-x)^{-\alpha} = 1$$

$$\int_{1}^{+\infty} f(x)\,dx \quad ?$$



$$\lim_{x\to+\infty} f(x)=0$$

$$\int_a^K f(x)\,dx$$

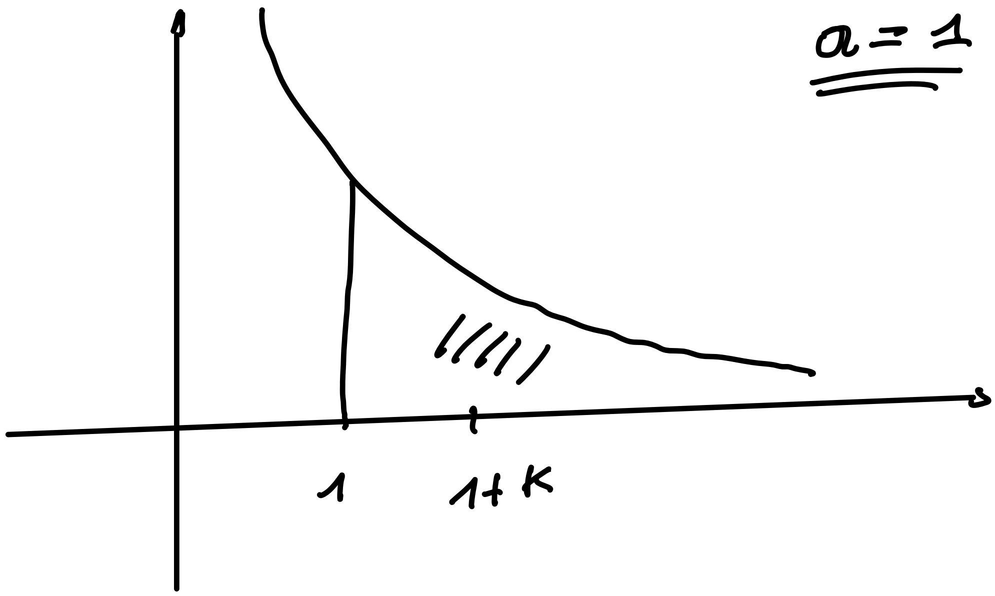
$$\lim_{K\to+\infty}\int_a^K f(x)\,dx$$ se esiste

finito, $f(x)$ si dice integrabile in senso improprio in $[a,+\infty[$ e scriveremo

$$\int_a^{+\infty} f(x)\,dx = \lim_{K\to+\infty}\int_a^K f(x)\,dx$$

$$f(x) = \frac{1}{x^\alpha} \quad , \quad \alpha > 0$$

$$x \in [1, +\infty[$$



$\underline{\underline{a = 1}}$

$$\alpha = 1$$

$$\int_1^K \frac{1}{x^\alpha}\, dx =$$

$$= \int_1^K \frac{1}{x}\, dx = \left[\log x\right]_{x=1}^{x=K}$$

$$\lim_{K \to +\infty} \int_1^K \frac{1}{x}\, dx = \lim_{K \to +\infty} \log K = +\infty$$

$\underline{\alpha \neq 1}$ $$\int_1^k \frac{1}{x^\alpha}\,dx = \int_1^k x^{-\alpha}\,dx =$$

$$= \left(\frac{x^{1-\alpha}}{1-\alpha}\right)_1^k = \frac{1}{1-\alpha}\left[k^{1-\alpha} - 1\right]$$

$\underline{\text{Se } \alpha > 1}$ $$\lim_{k\to+\infty} \int_1^k \frac{1}{x^\alpha}\,dx =$$

$$= \frac{1}{1-\alpha} \lim_{k\to+\infty} \left[k^{1-\alpha} - 1\right] =$$

$$= \frac{1}{1-\alpha} \lim_{k\to+\infty} \left[\underbrace{\frac{1}{k^{\alpha-1}}}_{\downarrow\, 0} - 1\right]$$

$$= \frac{1}{\alpha - 1}$$

$f(x) = \dfrac{1}{x^\alpha}$ integrabile in $[1, +\infty[$ $\underline{\text{se}}$ $\alpha > 1$

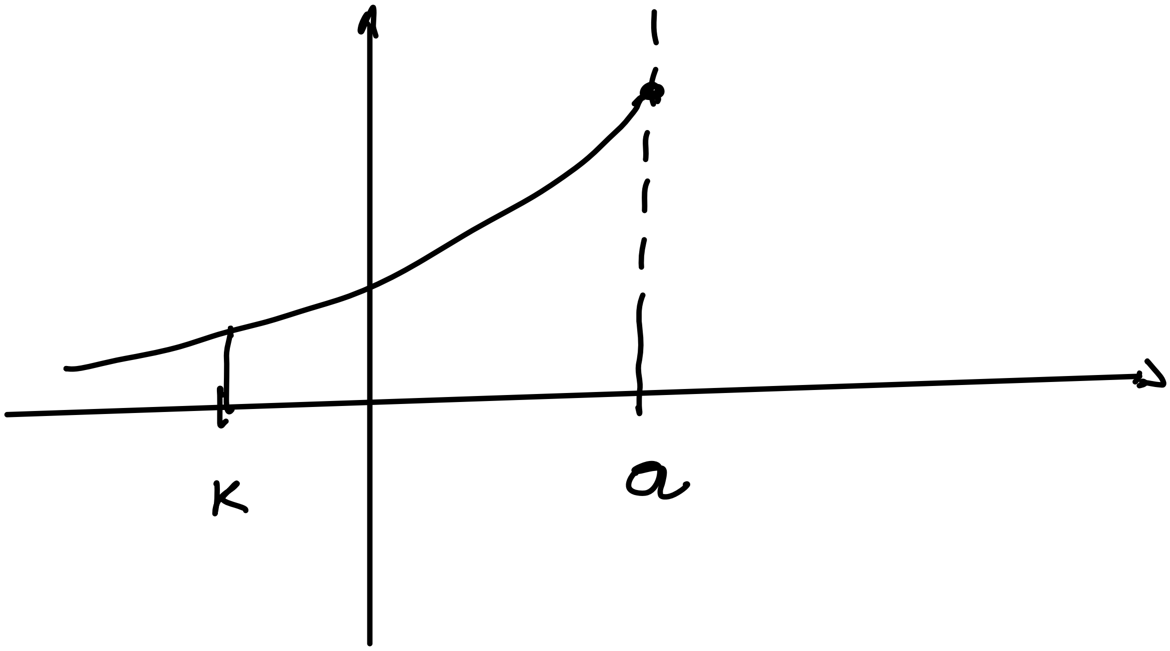
$\alpha < 1$

$$\lim_{x \to +\infty} \int_1^x \frac{1}{x^\alpha}\,dx = +\infty$$

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \dfrac{1}{\alpha-1} & \text{se } \alpha > 1 \\ +\infty & \text{se } \alpha \leq 1 \end{cases}$$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \frac{1}{2-1} = 1$$

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = +\infty$$



$$\lim_{x \to -\infty} f(x) = 0$$

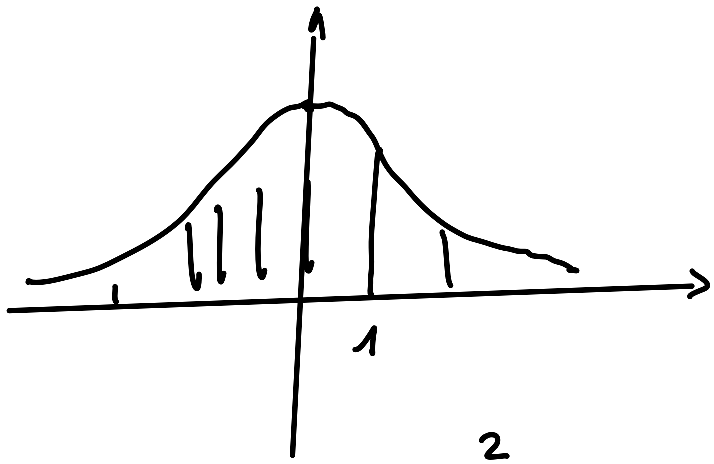
$$\int_{-\infty}^{a} f(x)\,dx \quad ?$$

$$\int_{K}^{a} f(x)\,dx$$

$$\lim_{K \to -\infty} \int_K^a f(x)\,dx$$

$$= \int_{-\infty}^{a} f(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx$$



e^{-x^2}

$\lim_{x \to \pm\infty} f(x) = 0$ $\qquad \int_{-\infty}^{\infty} e^{-x^2}\,dx$

$f(x)$ integrabile in $]-\infty, +\infty[$ se $f(x)$

è " in $]-\infty, 1]$ e $[1, +\infty[$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{+\infty} f(x)\,dx$$

<u>Criteri di integrabilità al finito</u>

$$0 \leq f(x) \leq g(x) \qquad]a,b]$$

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = +\infty$$

<u>se</u> $g(x)$ è integrabile, anche $f(x)$ è integrabile e $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

<u>se</u> $f(x)$ non è integrabile, anche $g(x)$ non è integrabile $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx = +\infty$

se $\quad 0 \leq f(x) \leq g(x) \quad [a, +\infty[$

se $g(x)$ è integrabile, anche $f(x)$
è integrabile e

$$\int_a^{+\infty} f(x)dx \leq \int_a^{+\infty} g(x)dx$$

se $f(x)$ non è integrabile,
$g(x)$ " "

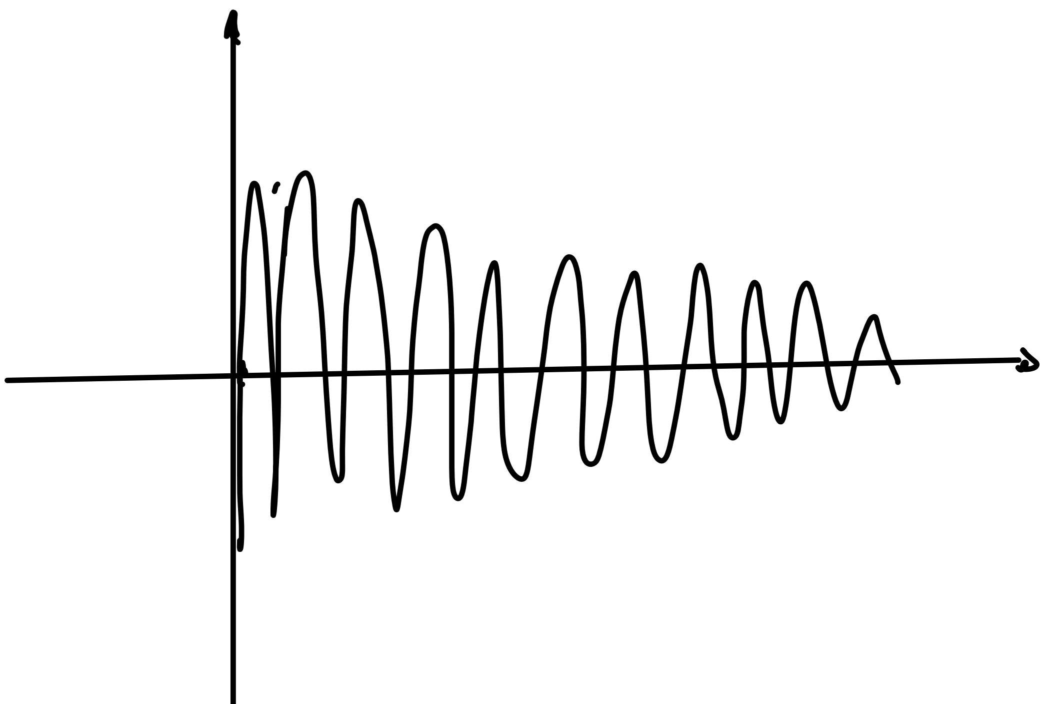
ES. $\quad f(x) = \frac{1}{\sqrt{x}} \left| \underbrace{\sin \frac{1}{x}}_{\leq 1} \right|$

$x \in]0, 1]$

$f(x) \leq \frac{1}{\sqrt{x}}$ integrabile $\Rightarrow f(x)$

integrabile

$$\frac{1}{\sqrt{x}} \sin \frac{1}{x}$$



Criterio asintotico

$f(x), g(x)$ $\quad]a,b]$ $\quad f(x), g(x) \geq 0$

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = +\infty$$

$$f(x) \sim g(x) \quad x \to a^+$$

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = 1 \text{ . Allora}$$

$f(x)$ integrabile $\Leftrightarrow$ $g(x)$ integrabile

$$\int_0^1 \frac{dx}{\sqrt[3]{(1-x^2)}} \qquad [0,1[$$

$$\frac{1}{\sqrt[3]{(1-x^2)}} = \frac{1}{\sqrt[3]{(1+x)(1-x)}} = \frac{1}{\sqrt[3]{1+x}} \cdot \frac{1}{\sqrt[3]{1-x}}$$

$$\frac{1}{\sqrt[3]{1+x}} \to \cancel{\frac{1}{\sqrt[3]{2}}}$$

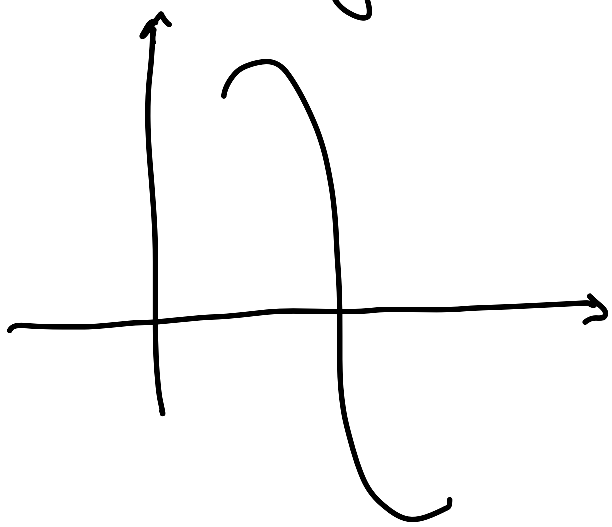
$$f(x) = \frac{1}{\sqrt[3]{1-x}} = \frac{1}{(1-x)^{1/3}} \qquad \frac{1}{(1-x)^\alpha}$$

$\alpha = 1/3 \longrightarrow f(x)$ è integrabile $\qquad \underline{\underline{\alpha < 1}}$

$$\cancel{\frac{1}{\sqrt[3]{1-x^2}}}$$

$$\lim_{x\to 1^-} \frac{\frac{1}{\sqrt[3]{1-x^2}}}{\frac{1}{\sqrt[3]{2}}\,\frac{1}{\sqrt[3]{1-x}}} = 1 \Rightarrow \frac{1}{\sqrt[3]{1-x^2}} \text{ è integrabile in } [0,1[$$

Consideriamo il caso di funzioni che cambiano segno:



Se $\int |f(x)|\,dx < \infty$ si dice

che $f(x)$ è sommabile oppure assolutamente integrabile

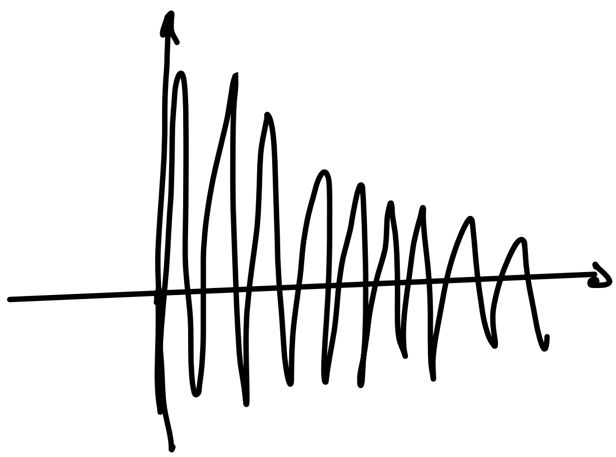
Se f è assolutamente integrabile

$$\Downarrow$$

f è integrabile in senso improprio

$$f(x) = \frac{1}{\sqrt{x}} \sin \frac{1}{x}$$

$x > 0$



$$|f(x)| = \left| \frac{1}{\sqrt{x}} \sin \frac{1}{x} \right| \leq \frac{1}{\sqrt{x}} ?$$

$x \to 0^+$

in $]0,1]$

è integrabile

$(\alpha = 1/2)$

$\Rightarrow$ $|f(x)|$ è integrabile $\Rightarrow$ $f(x)$ è integrabile

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad : \quad \alpha = 1 \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

$$\alpha = 2 \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$$

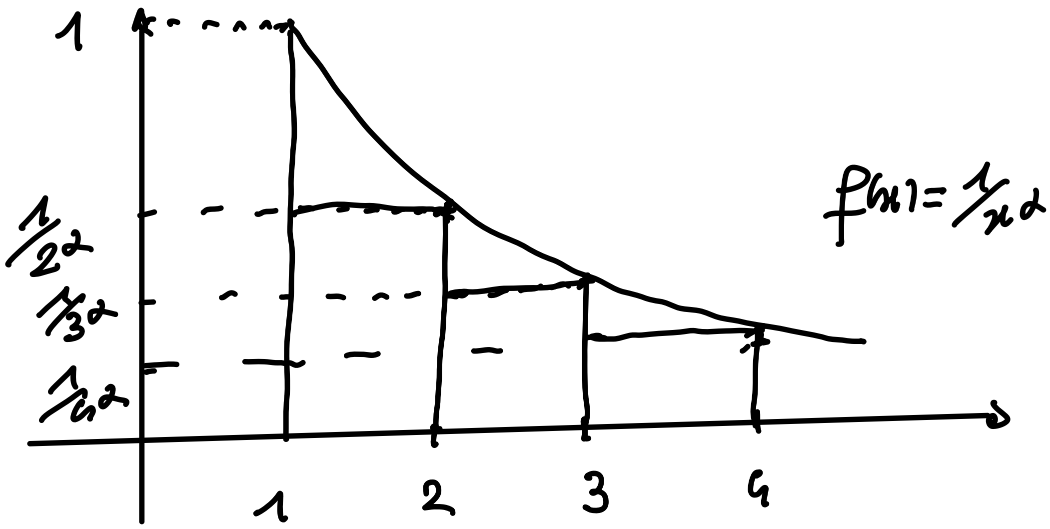
$$\boxed{1 < \alpha < 2}\ ? \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$\alpha < 1$? $\qquad\qquad$ $\underline{1 < \alpha < 2}$

$$f(x) = \frac{1}{x^\alpha} \quad \text{in} \quad [1, +\infty)$$

$$\int_1^{+\infty} \frac{1}{x^\alpha}\, dx = \begin{cases} \dfrac{1}{\alpha - 1} & \text{se } \alpha > 1 \\ +\infty & \text{se } \alpha \leq 1 \end{cases}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = \begin{cases} < +\infty & \text{se } \alpha > 1 \\ +\infty & \text{se } \alpha \leq 1 \end{cases}$$



[1,2] , [2,3] , [3,4]

$$\int_1^4 \frac{1}{x^\alpha}\,dx = \int_1^2 \frac{1}{x^\alpha}\,dx + \int_2^3 \frac{1}{x^\alpha}\,dx$$

$$+ \int_3^4 \frac{1}{x^\alpha}\,dx \geq \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \frac{1}{4^\alpha}$$

[1, N]

$$\int_1^N \frac{1}{x^\alpha}\,dx \geq \frac{1}{2^\alpha} + \overbrace{\frac{1}{3^\alpha} + \cdots +}^{\frac{1}{m^\alpha}} \frac{1}{N^\alpha}$$

$$= \sum_{m=2}^{N} \frac{1}{m^\alpha}$$

$$\sum_{m=2}^{N} \frac{1}{m^\alpha} \leq \int_1^N \frac{1}{x^\alpha}\,dx$$

$$S_N = \sum_{m=1}^{N} \frac{1}{m^\alpha} = 1 + \sum_{m=2}^{N} \frac{1}{m^\alpha}$$

$$\leq 1 + \int_1^N \frac{1}{x^\alpha}\,dx$$

$\underline{N \to +\infty}$

$$\int_1^N \frac{1}{x^\alpha}\,dx \longrightarrow \int_1^{+\infty} \frac{1}{x^\alpha}\,dx < +\infty$$

$$S_N \leq 1 + \int_1^N \frac{1}{x^\alpha}\,dx$$

$$\xrightarrow{N \to \infty} 1 + \int_1^\infty \frac{1}{x^\alpha}\,dx < +\infty$$

$$\Rightarrow \lim_{N \to \infty} S_N < +\infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad \underline{\underline{\text{converge}}}$$

per $\underline{\alpha > 1}$

$\underline{\alpha < 1}$?

$$n^\alpha \leq n$$

$$\frac{1}{n^\alpha} \geq \frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

$$\Downarrow$$

per il criterio del confronto, $\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = +\infty$

Serie a segni alterni

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots + (-1)^{n-1} \frac{1}{n} + \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$ = serie armonica a segni alterni $\underline{\underline{\text{converge}}}$

$$a_n = \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

$n=1$

$$a_n \geq 0$$

$$a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{n-1} a_n + \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad \text{serie a segni alterni}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \big(\underbrace{1 - \cos 1/n}_{a_n \geq 0} \big)$$

<u>Criterio di Leibniz</u> $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad , a_n \geq 0$

Se $\{a_n\}$ è definitivamente decrescente ($a_n \geq a_{n+1}$ $\forall n \geq n_1$, $n_1 \in \mathbb{N}$)

e $\displaystyle\lim_{n \to \infty} a_n = 0$

allora la serie converge a $S \in \mathbb{R}$. Inoltre,

$$| \underbrace{S_n} - S | \leq a_{n+1}$$

<u>ES.</u> $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

$$a_n = \frac{1}{n}$$

$$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$$

$$\lim_{n \to \infty} a_n = 0$$

$\Rightarrow$ criterio Leibniz $\Rightarrow$ la serie converge

Quanti termini dobbiamo sommare in modo che la somma parziale S_n differisca dalla somma S per meno di $1/100$

$$|S_n - S| \leq 1/100$$

Dal criterio di Leibniz, $|S_n - S| \leq a_{n+1} = \frac{1}{n+1}$

$$\frac{1}{n+1} \leq \frac{1}{100} \iff n+1 \geq 100$$
$$\iff \underline{\underline{n \geq 99}}$$

2) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$ $\qquad a_n = \frac{n-1}{n^2+n} \geq 0$

convergente

$$\lim_{n\to\infty} \frac{n-1}{n^2+n} = 0 \quad \to a_n \text{ è infinitesima}$$

$$a_n > a_{n+1} \iff \frac{n-1}{n^2+n} > \frac{(n+1)-1}{(n+1)^2+(n+1)} = \frac{n}{(n+1)(n+2)}$$

$$\frac{n-1}{n\cancel{(n+1)}} > \frac{n}{\cancel{(n+1)}(n+2)}$$

$$\Leftrightarrow (n-1)(n+2) > n \cdot n = n^2$$

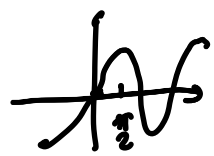
$$\Leftrightarrow \cancel{n^2} + 2n - n - 2 > \cancel{n^2} \Leftrightarrow \underline{\underline{n > 2}}$$

$n=1 \qquad \frac{n-1}{n^2+n} = 0$

$0 < \frac{1}{6}$

$n=2 \qquad \frac{n-1}{n^2+n} = \frac{1}{6}$

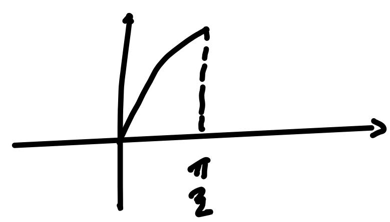
3) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \sin \frac{1}{\sqrt{n}} \qquad a_n = \sin \frac{1}{\sqrt{n}}$



- $a_n > 0$

$n \geq 1 \qquad \frac{1}{\sqrt{n}} \leq 1 < \frac{\pi}{2}$

$\sin$



$\Rightarrow \sin \frac{1}{\sqrt{n}} > 0$

$$\lim_{n\to\infty} \sin \frac{1}{\sqrt{n}} = \sin 0 = 0$$

a_n <u>decrescente</u> ? $\qquad \sin \frac{1}{\sqrt{n}} > \sin \frac{1}{\sqrt{n+1}}$

$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} \Rightarrow \underbrace{\sin \frac{1}{\sqrt{n}}}_{a_n} > \underbrace{\sin \frac{1}{\sqrt{n+1}}}_{a_{n+1}}$$

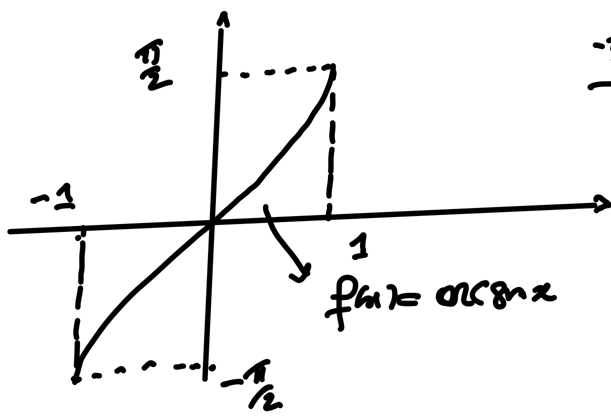
$\Rightarrow$ crit. di Leibniz $\Rightarrow$ la serie converge

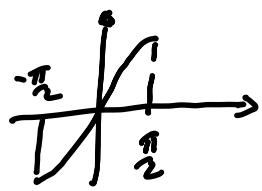
4) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ $\qquad a_n = \frac{1}{n^3} \xrightarrow[n\to\infty]{} 0$

decrescente

5) $\sum_{n=1}^{\infty} (-1)^{n-1} \arcsin \frac{1}{n}$ $\qquad a_n = \arcsin \frac{1}{n}$

$$\lim_{n\to\infty} \arcsin \frac{1}{n} = \arcsin 0 = 0$$





$$a_n = \arcsin \frac{1}{n} > \arcsin \frac{1}{n+1} = a_{n+1}$$

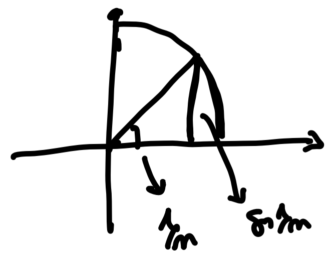
$$\frac{1}{n} > \frac{1}{n+1}$$

$\Rightarrow$ la serie converge.

6) $\sum_{n=1}^{\infty} (-1)^{n-1} \underbrace{\left(1 - n \sin \frac{1}{n}\right)}_{a_n}$

$a_n = 1 - n \sin\frac{1}{n} \geq 0 \iff n \sin\frac{1}{n} \leq 1$

$\iff \sin\frac{1}{n} \leq \frac{1}{n} < \frac{\pi}{2}$



$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 - n \sin\frac{1}{n}\right)$$

$$= 1 - \lim_{n\to\infty} n \sin\frac{1}{n} = 1 - \lim_{n\to\infty}\left(\frac{\sin 1/n}{1/n}\right) \left[\frac{0}{0}\right]$$

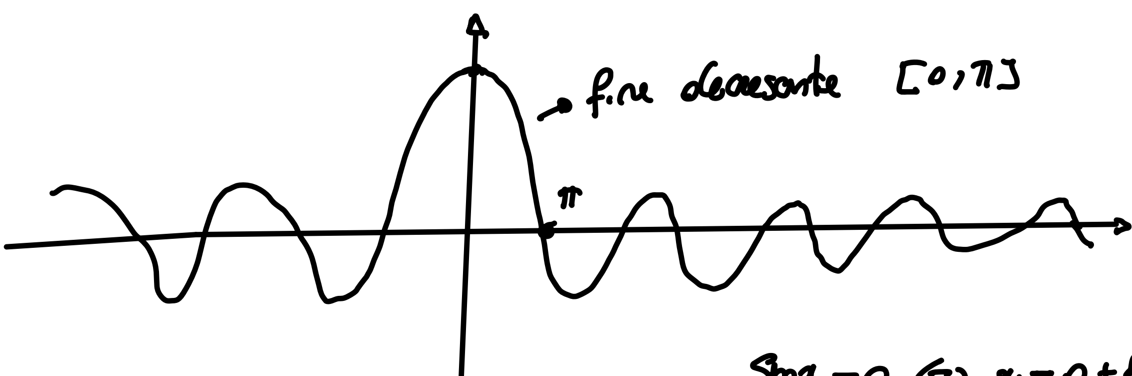
f.i. $\infty \cdot 0$

$= 0$

$x = 1/n$: $n \to \infty$, $x \to 0$ $\qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$

$a_n > a_{n+1}$? $\qquad f(x) = \frac{\sin x}{x}$

$x \to \pm\infty$, $f(x) \to 0$



$\sin x = 0 \iff x = 0 + k\pi$

$= k\pi$

$x \neq 0$

$k \in \mathbb{Z}$

$x = 0$, $x = \pi, 2\pi, -\pi, -2\pi, \ldots$

$\frac{1}{n} > \frac{1}{n+1}$, $\frac{1}{n} < \pi$

$$\Rightarrow \frac{\sin 1/n}{1/n} < \frac{\sin 1/n+1}{1/n+1} \Rightarrow n \sin 1/n < (n+1) \sin \frac{1}{n+1}$$

$$a_n = 1 - n \sin \frac{1}{n} > 1 - (n+1) \sin \frac{1}{n+1} = a_{n+1}$$

7) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \log\left(1 + \sin \frac{1}{n}\right)$

$$= - \sum_{n=1}^{\infty} (-1)^{n+1} \text{———}$$