Lezioni del 23/09/2022 - 26/09/2022

Informatica

Serie a termini non negativi

$$\sum_{n=1}^{\infty} a_n \quad , \quad a_n \geq 0 \quad \forall n \in \mathbb{N}$$

Es $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ (s. geometrica) $= 2$

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad , \quad \sum_{n=1}^{\infty} \left(1 - \cos\frac{1}{n}\right)$$

$$-1 \leq \cos\frac{1}{n} \leq 1$$

$$1 - \cos\frac{1}{n} \geq 0$$

Le serie a termini non negativi sono sempre regolari:

Proposizione $\displaystyle\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$: allora o la serie converge oppure diverge positivamente.

Es. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ $\quad \frac{n}{n+1} \xrightarrow[n\to\infty]{} 1 \neq 0$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n+1} = +\infty .$$

Dim. $S_n = a_1 + a_2 + \dots + a_n$

$$S_{n+1} = a_1 + a_2 + \dots + a_n + \underbrace{a_{n+1}}_{\geq 0} \geq$$

$$\geq a_1 + a_2 + \dots + a_n = S_n$$

$\{S_n\}_{n \in \mathbb{N}}$ è monotona crescente

$$\Rightarrow \exists\, S = \lim_{n\to\infty} S_n = \sup_{n \geq 1} S_n \in [0, +\infty]$$

$\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$.

Serie geometrica

$h \in \mathbb{R}$ (campo reali)

$$\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{\pm\infty\}$$

$$1 + h + h^2 + h^3 + \dots + h^n + \dots$$

$$= \sum_{n=0}^{\infty} h^n$$ Serie geometrica di ragione h

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$ S. geometrica di ragione $\frac{1}{2}$

$$\sum_{n=0}^{\infty} 2^n = +\infty$$ " " " 2

$$\lim_{n \to \infty} 2^n = +\infty$$

$$S_n = \underbrace{1 + \underbrace{h + h^2 + h^3 + \dots + h^{n-1}}_{n-1}}_{n}$$

$h \neq 1$

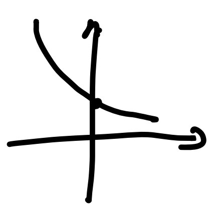
$$= \frac{1 - h^n}{1-h} \qquad \forall n \in \mathbb{N}_0$$

$h = 1$ $\quad \sum_{n=0}^{\infty} h^n = 1 + 1 + 1 + \dots + 1 + \dots$

$$= +\infty$$

$-1 < h < 1$ $\quad$? $\qquad |h| < 1$

$0 < h < 1$ $\qquad \lim_{n\to\infty} h^n = 0$



$$\lim_{n\to\infty} |h|^n = 0$$

$|h|<1$: $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-h^n}{1-h}$

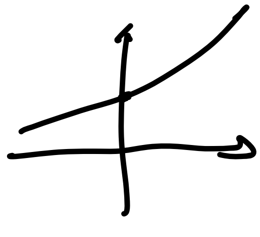
$$= \frac{1}{1-h}$$

$$\sum_{n=0}^{\infty} h^n = \frac{1}{1-h} \qquad -1<h<1$$

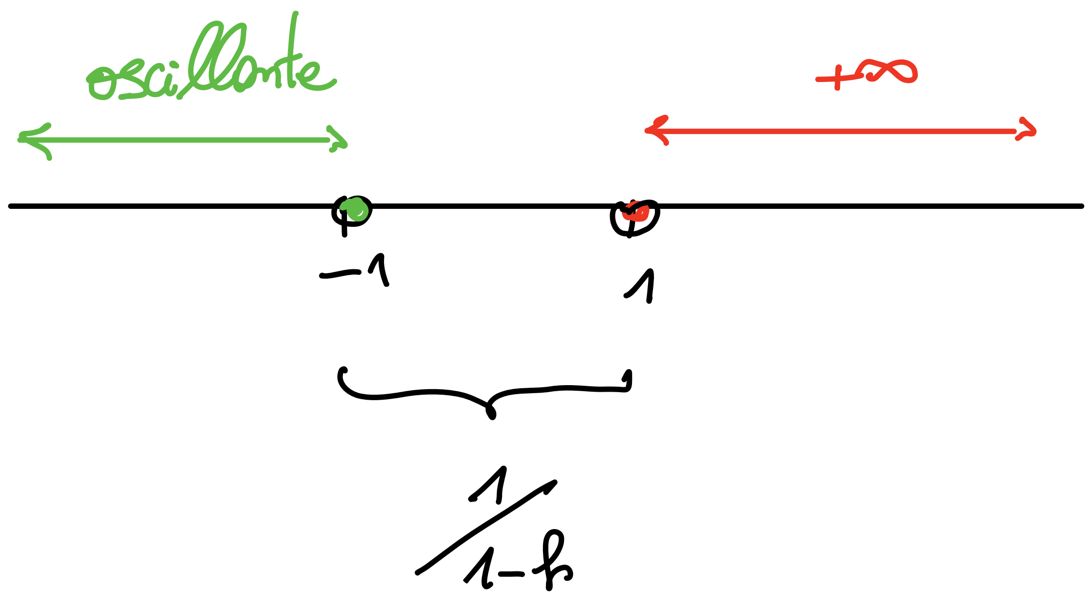
ES. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

$h = \frac{1}{2} < 1$

OSS. $h>1$ $\qquad \lim_{n\to\infty} h^n = +\infty$



$$\Rightarrow \sum_{n=0}^{\infty} h^n = +\infty$$



<u>$h=-1$</u> ? $\sum_{n=0}^{\infty} (-1)^n$ oscillante

$\sum_{n=0}^{\infty} (-2)^n$ "

$$\sum_{n=0}^{\infty} h^n = \begin{cases} +\infty & \text{se } h \geq 1 \\ \frac{1}{1-h} & \text{se } -1 < h < 1 \\ \text{oscillate} & \text{se } h \leq -1 \end{cases}$$

Criterio del confronto

$$\sum_{n=1}^{\infty} a_n \ , \ \sum_{n=1}^{\infty} b_n \ , \qquad a_n, b_n \geq 0$$

$$e \qquad a_n \leq b_n$$

Allora: (i) se $\displaystyle\sum_{n=1}^{\infty} b_n < +\infty$ (convergente)

$$\Downarrow$$

$$\sum_{n=1}^{\infty} a_n < +\infty$$

(ii) se $\displaystyle\sum_{n=1}^{\infty} a_n = +\infty$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} b_n = +\infty$$

ES. $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{2^n}$

$$\frac{|\cos n|}{2^n} \leq \frac{1}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \underbrace{\sum_{n=0}^{\infty} \frac{1}{2^n}} = \frac{1}{2} \cdot 2 = 1$$

$$\frac{1}{2} + \frac{1}{4} + \dots$$

$\Rightarrow$ per il criterio del confronto,

$$\sum_{n=1}^{\infty} \frac{|\cos n|}{2^n} < +\infty$$

Dim. (Criterio del confronto)

$a_n \leq b_n$ e $\displaystyle\sum_{n=1}^{\infty} b_n < +\infty$ (i)

$a_n, b_n \geq 0$

$$S_n = a_1 + a_2 + \cdots + a_n$$
$$\qquad\;\; \wedge \quad\;\; \wedge\!\shortparallel \qquad\quad \wedge\!\shortparallel$$
$$T_n = b_1 + b_2 + \cdots + b_n$$

$$S_n \leq T_n \quad \forall n \in \mathbb{N} \qquad \odot$$

Poiché $\sum_{n=1}^{\infty} b_n$ converge, $T_n \longrightarrow T$
$$= \sup_{n \geq 1} T_n < +\infty$$

da $\odot$ $\quad S_n \leq T_n \leq T$

$$\Rightarrow \qquad \sup_{n\geq 1} S_n < +\infty$$
$$\shortparallel$$
$$\sum_{n=1}^{\infty} a_n = \sup_{n \geq 1} S_n < +\infty$$

(ii) $\sum_{n=1}^{\infty} a_n = +\infty$

$$S_n \xrightarrow{n\to+\infty} +\infty$$

Ma $\quad S_n \leq T_n \Rightarrow T_n \to +\infty$

$S_n \downarrow +\infty$

crit. confronto

$$\Rightarrow \sum_{n=1}^{\infty} b_n = +\infty .$$

Criterio della radice

$a_n \geq 0 \qquad \sum_{n=1}^{\infty} a_n$

$$\exists \lim_{n\to\infty} \sqrt[n]{a_n} = \ell \in [0,+\infty]$$

$$a_1, \sqrt[2]{a_2}, \sqrt[3]{a_3}, \ldots$$

Se $\ell < 1 \Rightarrow \sum_{n=1}^{\infty} a_n < +\infty$

Se $\ell > 1 \Rightarrow \sum_{n=1}^{\infty} a_n = +\infty$

$\ell = 1$?? NULLA SI PUÒ DIRE A PRIORI!

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \qquad a_n = \frac{1}{2^n}$$

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{2^n}} = \frac{1}{\sqrt[n]{2^n}} = \frac{1}{2}$$

$$\lim_{n \to \infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{2^n} < +\infty \quad = 2$$

$$\sum_{n=0}^{\infty} 2^n = +\infty \qquad \sqrt[n]{a_n} = 2$$

$$\lim_{n \to \infty} \sqrt[n]{a_n} = 2 > 1$$

$\Rightarrow$ la serie diverge

Criterio del rapporto

$a_n \underset{\neq}{\geq} 0 \quad \forall n \in \mathbb{N}$

$$\exists \ \ell = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \in [0, +\infty]$$

$$\ell < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n < +\infty$$

$$\ell > 1 \Rightarrow \sum_{n=1}^{\infty} a_n = +\infty$$

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \qquad a_n = \left(\frac{2}{3}\right)^n$$

$$= \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{3} \cdot \frac{1}{1-\frac{2}{3}} = 2$$

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)^n} = \frac{\cancel{\left(\frac{2}{3}\right)^n} \cdot \frac{2}{3}}{\cancel{\left(\frac{2}{3}\right)^n}}$$

$$= \frac{2}{3} < 1$$

$\ell = 1$??

ES. $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ | $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$= \lim_{n\to\infty} \frac{1/_{n+1}}{1/_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \dots$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/_{(n+1)^2}}{1/_{n^2}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$$

Criterio asintotico

$$\sum_{n=1}^{\infty} a_n \ , \ \sum_{n=1}^{\infty} b_n \qquad a_n, b_n \underset{\neq}{>} 0$$

Se $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$

$$a_n \sim b_n$$

allora le serie hanno lo stesso carattere.

(se una serie converge, converge anche l'altra

se una serie diverge, diverge anche l'altra)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\lim_{n \to \infty} \frac{1/n^2}{1/n(n+1)} = 1$$

$$\frac{1}{n(n+1)} \sim \frac{1}{n^2}$$

$\Rightarrow$ per il criterio asintotico,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$$

$$= \pi^2/6$$

$$\sum_{n=1}^{\infty} \underbrace{\left(1 - \cos 1/n\right)}_{\wedge} \sim \sum_{n=1}^{\infty} \frac{1}{2n^2} < +\infty$$

$+\infty$

$$\frac{1-\cos x}{x^2} \xrightarrow[x \to 0]{} \frac{1}{2}$$

$$\lim_{n\to+\infty} \frac{1-\cos 1/n}{\frac{1}{2}\, 1/n^2} = 1$$

<u>Criterio degli infinitesimi</u> $\sum_{n=1}^{\infty} a_n\,, \quad a_n \geq 0$

$$\exists \lim_{n\to+\infty} n^{\alpha} a_n = \ell \in [0, +\infty]$$

$$\frac{a_n}{1/n^{\alpha}}$$

1) <u>Se</u> $\alpha > 1$ e $\ell < +\infty$, la serie converge

2) <u>Se</u> $\alpha \leq 1$ e $\ell \neq 0$, la serie diverge

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty \qquad \lim_{n \to \infty} \frac{1/n}{1/n^{\alpha}} = 1 \quad \alpha = 1$$

$\Rightarrow$ la serie diverge

$$\sum_{n=1}^{\infty} 1/n^2 \qquad \lim_{n \to \infty} \frac{1/n^2}{1/n^{\alpha}} = 1 \quad \alpha = 2$$

$\alpha = 2 > 1$

$\Rightarrow$ la serie converge

$$\sum_{n=1}^{\infty} \left(1 - \cos\frac{1}{n}\right) \qquad \frac{1 - \cos 1/n}{1/n^2} \longrightarrow \frac{1}{2}$$

$\underline{\alpha = 2}$

$\Rightarrow$ la serie converge

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^5+4n+3} \sim \sum_{n=1}^{\infty} \frac{2}{n^4} < +\infty$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{2(n+1)+1}{(n+1)^5+4(n+1)+3}}{\dfrac{2n+1}{n^5+4n+3}}$$

$$= \lim_{n\to\infty} \frac{2(n+1)+1}{(n+1)^5+4(n+1)+3} \cdot \frac{n^5+4n+3}{2n+1} = 1$$

$$\lim_{n\to\infty} n^{\alpha} \frac{2n+1}{n^5+4n+3} \overset{\alpha=4}{=} \lim_{n\to\infty} \frac{2n^5+n^4}{n^5+4n+3} = 2$$

$\alpha = 4 > 1 \Rightarrow$ la serie converge

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$$

$$\lim_{n\to\infty} n^{\alpha} \frac{1}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 1$$

$\alpha = \frac{1}{2} < 1 \Rightarrow$ la série diverge

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

$$+ \dots + \frac{x^n}{n!} + o(x^n)$$

$x > 0$

$$\lim_{x\to 0} \frac{o(x^n)}{x^n} = 0$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$+ o(x - x_0)$$

$x \to x_0$

$$\lim_{x\to x_0} \frac{o(x - x_0)}{x - x_0} = 0$$

$f(x) = e^x$, $x_0 = 0$, $f(x_0) = 1$

$f'(x) = e^x$, $f'(0) = 1$

$$e^x = 1 + 1 \cdot x + o(x) = 1 + x + o(x)$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) +$$

$$+ \frac{f''(x_0)}{2}(x - x_0)^2 + o((x - x_0)^2)$$

$$\lim_{x \to x_0} \frac{o(x - x_0)^2}{(x - x_0)^2} = 0$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

$$+ \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots +$$

$$+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o(x - x_0)^n$$

Sviluppo in formula di Taylor di $f(x)$ di ordine $n \in \mathbb{N}$, di punto iniziale x_0

$$f(x) = e^x \qquad f'(x) = f''(x) = \dots = f^{(n)}(x) = e^x$$

$$x_0 = 0 \qquad f^{(n)}(0) = 1$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$$

$$\sum_{n=0}^{\infty} \underbrace{\frac{x^n}{n!}}_{a_n} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots = e^x$$

$$a_n = \frac{x^n}{n!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} =$$

$$= \lim_{n\to\infty} \frac{x^{n+1}}{\underbrace{(n+1)!}_{(n+1)\cancel{n!}}} \cdot \frac{\cancel{n!}}{x^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{x^n} \cdot x}{n+1} \cdot \frac{1}{\cancel{x^n}} = 0 < 1$$

$\Rightarrow$ la serie converge

$$\lim_{n\to\infty} n^{\alpha} \frac{1}{n^2} = \lim_{n\to\infty} \cancel{n^{\alpha}} \cdot \frac{1}{\cancel{n^2}} = 1$$

$\alpha = ?\ 2$

$$\lim_{n\to\infty} \frac{1-\cos 1/n}{\frac{1}{2}\cdot 1/n^2} = 2 \lim_{n\to\infty} \overbrace{\frac{1-\cos 1/n}{1/n^2}}^{1/2}$$

$$= \cancel{2} \cdot \frac{1}{\cancel{2}} = 1$$