Lezioni del 14/10/2022 – 17/10/2022

Funzioni di più variabili

$f = f(x) \quad , x \in [a,b]$
$\in I \subseteq \mathbb{R}$

$$f : I \to \mathbb{R}$$

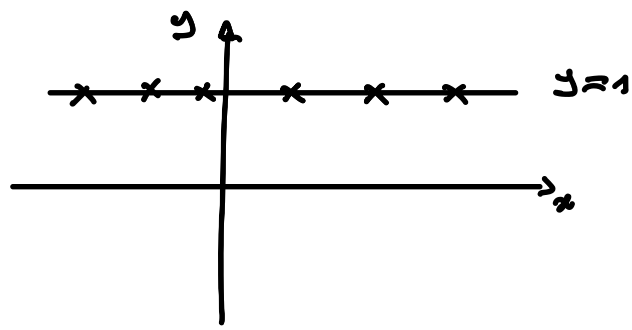
$f = f(\underbrace{x, y})$: **Es.** $\underline{f(x,y) =}\ x^2 + y^2$

$(x,y) \in X \subseteq \mathbb{R}^2$ $\qquad \forall x \in \mathbb{R},\ \forall y \in \mathbb{R}$

$\forall (x,y) \in \mathbb{R}^2$

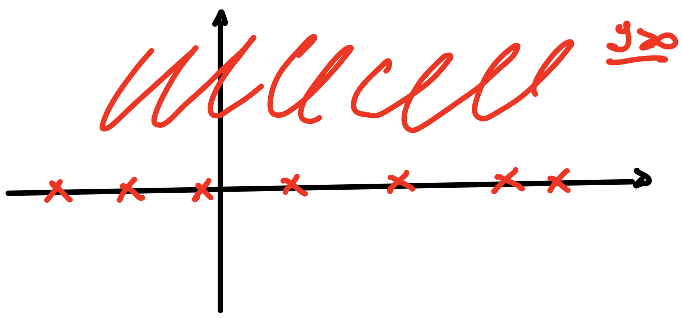
$\underbrace{f(x,y)}= \dfrac{x}{y-1}$ $\qquad f(\underset{x}{\overset{\shortparallel}{1}}, \underset{y}{\overset{\shortparallel}{2}}) = \dfrac{1}{2-1} = 1$

$y - 1 \neq 0 \iff y \neq 1$ $\qquad \mathbb{R}^2 \setminus \{ y = 1 \}$



$f(x,y) = x \log y$ $\qquad f(2,1) = 0$

$\underline{y > 0}$ $\qquad f(2,2) = 2 \log 2 \ \ldots$



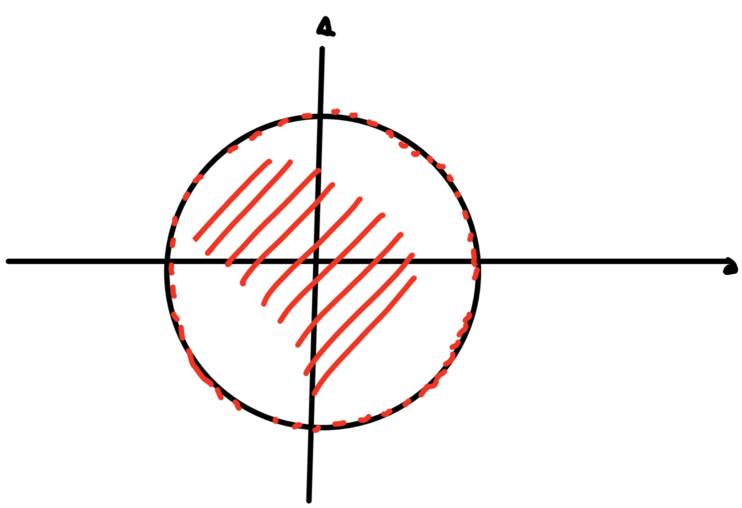
$$f(x,y) = x \log(1-y) \qquad 1-y>0 \iff \underline{\underline{y<1}}$$

$$f(x,y) = x\sqrt{1-y} \quad : \quad 1-y \geq 0 \iff y \leq 1$$

$$f(x,y) = \sqrt{\underbrace{1-x^2-y^2}} \quad : \quad 1-x^2-y^2 \geq 0$$

$$\iff \qquad x^2+y^2 \leq 1$$

$$x^2+y^2 = 1$$

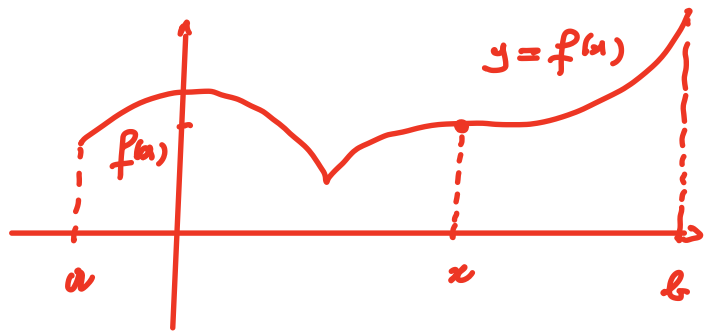


$f = f(x,y)$, $f : X \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

$X \subseteq \mathbb{R}^2$

<u>Grafico di f</u> : per le funzioni di una sola variabile

$y = f(x)$



$$G_f = \{ (x, f(x)) : x \in [a,b] \}$$

$f(x,y)$ il grafico sarà l'insieme

$$G_f = \{ (\underset{(x,y)}{x, y}, f(x,y)) : (x,y) \in X \}$$

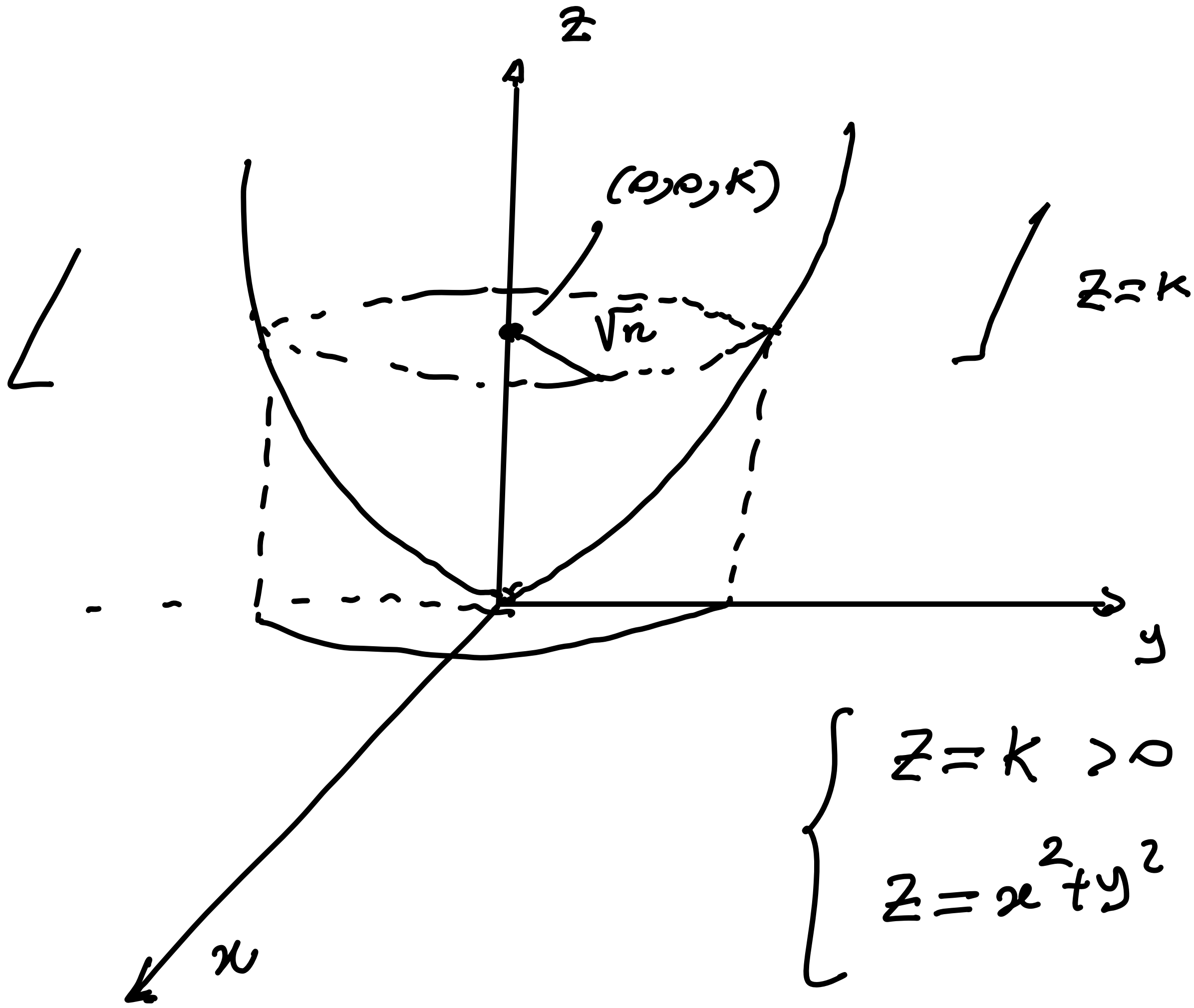
L'equazione del grafico sarà del tipo

$$\underbrace{z = f(x,y)}_{\text{superficie}}$$

Equazione cartesiana del grafico di f

$$z = f(x,y) = x^2+y^2 \geq 0$$

$z = x^2+y^2$ paraboloide ellittico



$$\begin{cases} z = k > 0 \\ z = x^2+y^2 \end{cases}$$

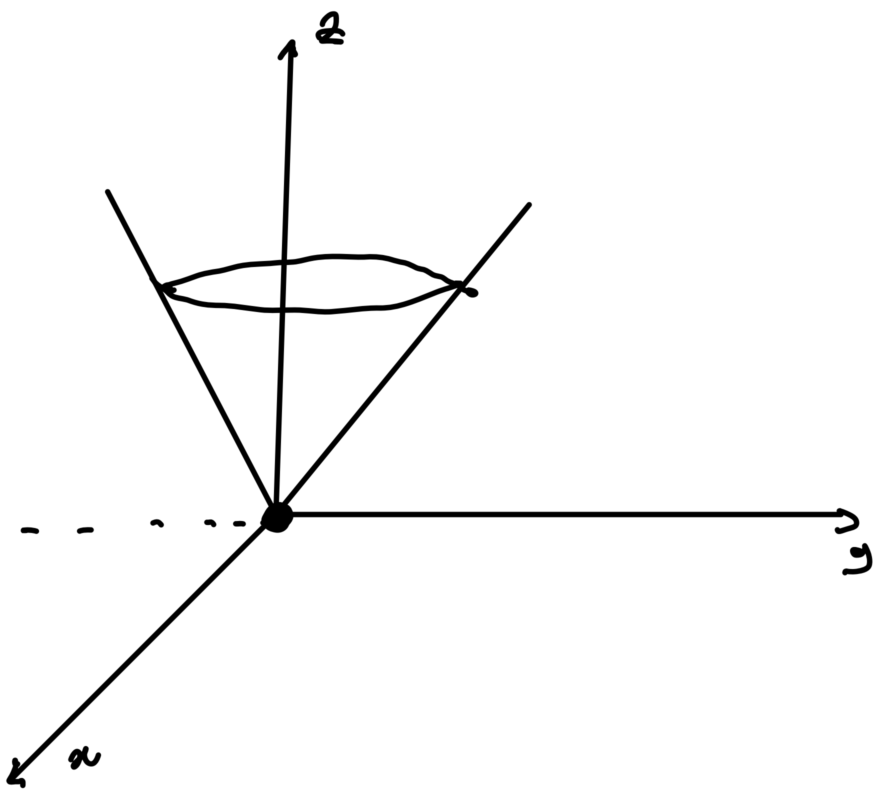
$$\begin{cases} z = k \\ \underbrace{x^2+y^2} = k \end{cases}$$

eq. circonferenza di raggio $\sqrt{K}$

$$z = \sqrt{x^2+y^2} = f(x,y) \,/\, x^2+y^2 \geq 0 \quad \text{sempre}$$

$$D_f = \text{dominio di } f = \mathbb{R}^2$$

Cono con vertice nell'origine



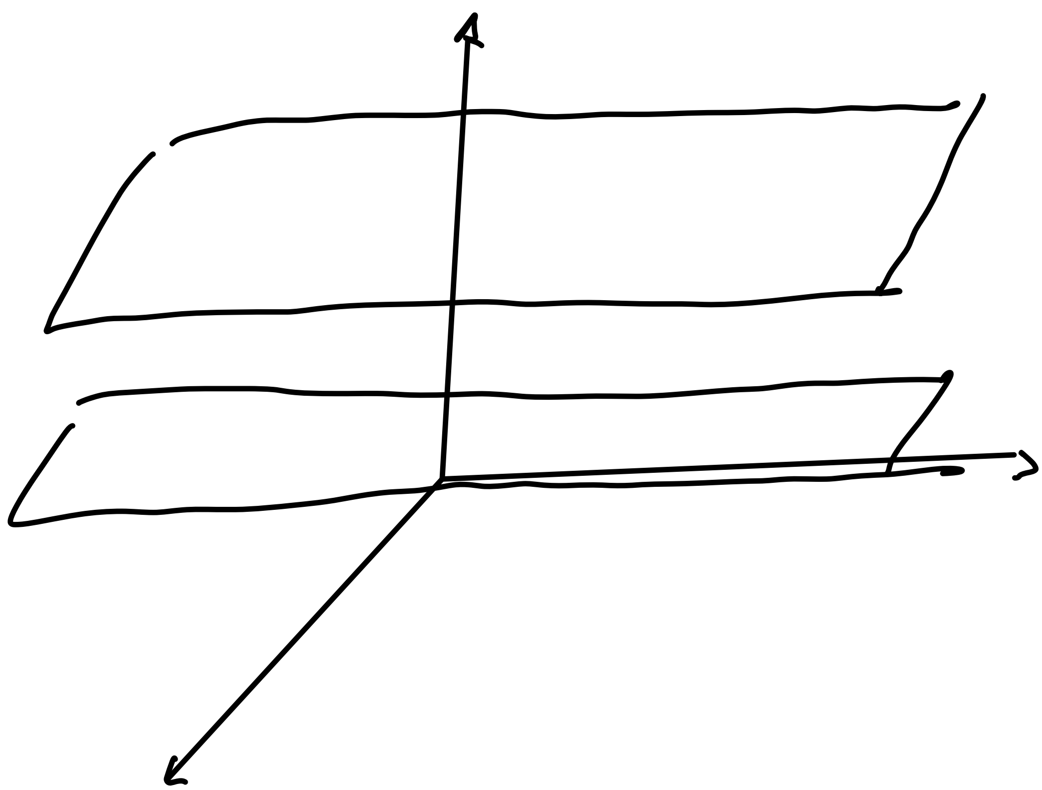
$$z = k \in \mathbb{R}$$

$$= f(x,y)$$

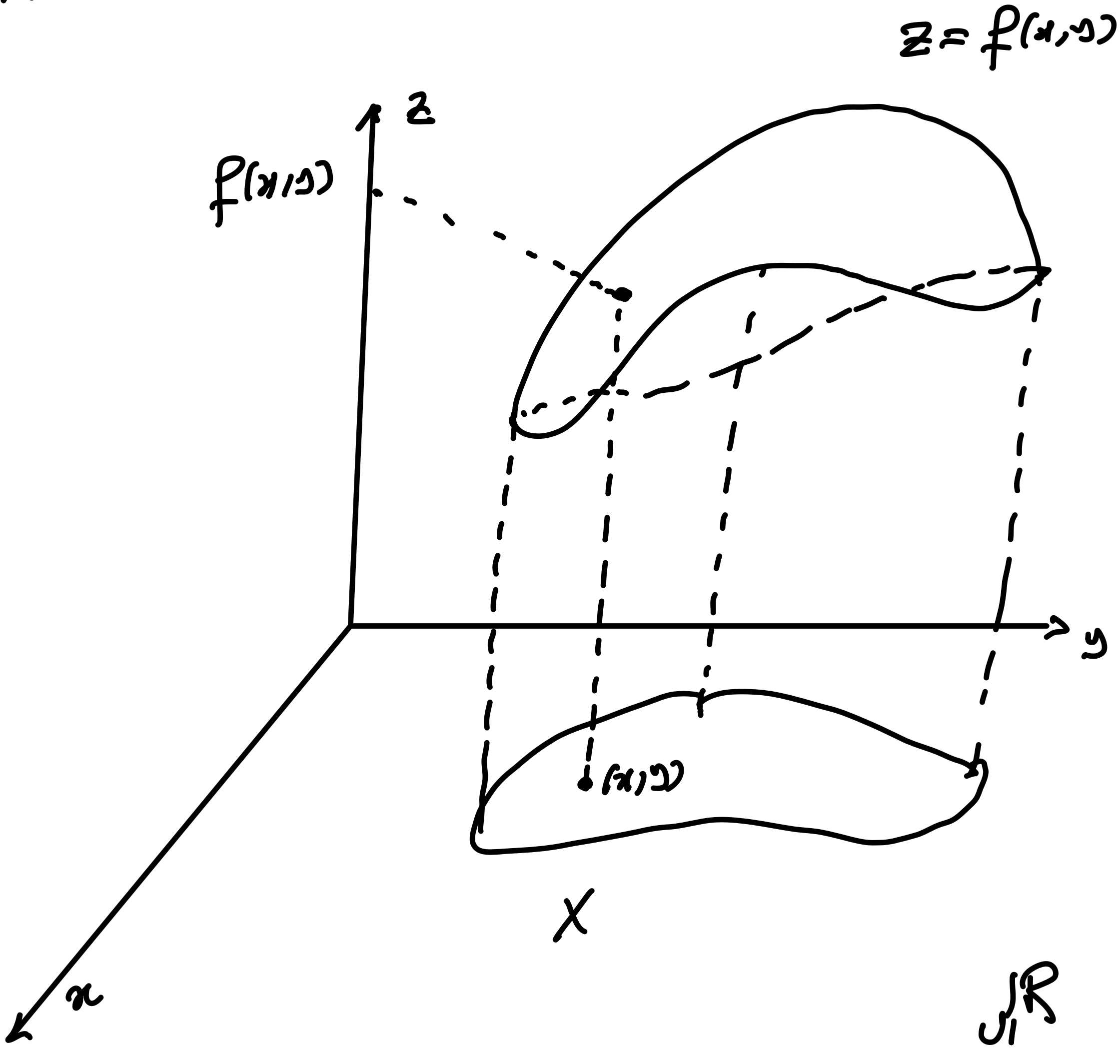
$$f(x,y) = k$$

$z = k$

piani paralleli al piano xy



$f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ $\qquad X \subseteq \mathbb{R}^2$

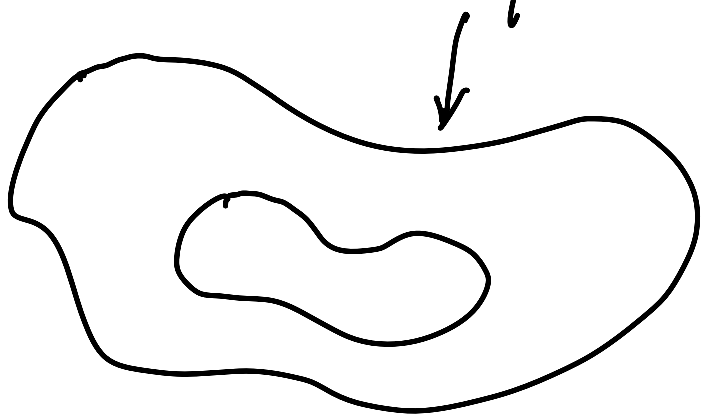


$$\text{Codominio di } f = \underbrace{f(X) = \{ f(x,y) : (x,y) \in X \}}_{\subseteq \mathbb{R}}$$

sottoinsieme dell'asse z.

$f(x,y)$ = quota rispetto al livello del mare in (x,y)

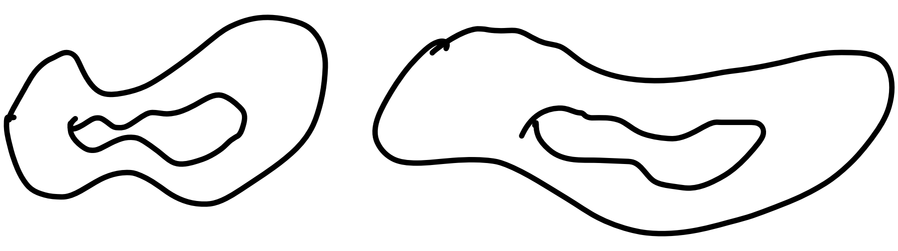
quota è costante



$\{(x,y) : f(x,y) = c > 0\}$

curve di livello

$f(x,y)$ = pressione atmosferica



isobare

$f(x,y,z) \qquad f(x,y,z) = x^2+y^2+z^2$

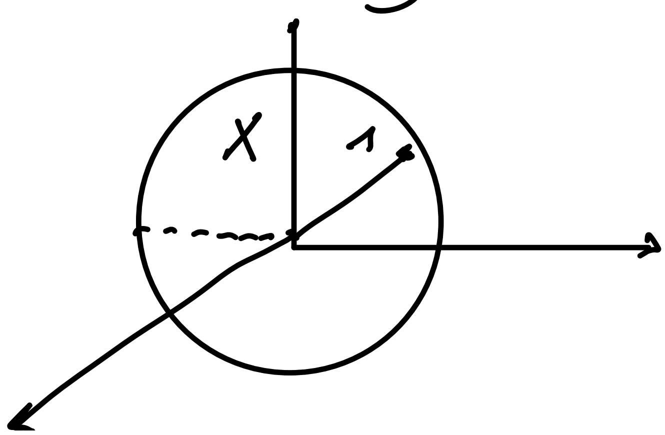
$(x,y,z) \in X \subseteq \mathbb{R}^3$

$$f(x,y,z) = \sqrt{1 \underbrace{-x^2-y^2}_{} -z^2}$$

$$1-x^2-y^2-z^2 \geq 0$$

$$\Longleftrightarrow \quad x^2+y^2+z^2 \leq 1$$

$$\underbrace{x^2+y^2+z^2 = 1}_{\text{sup. sferica unitaria}}$$



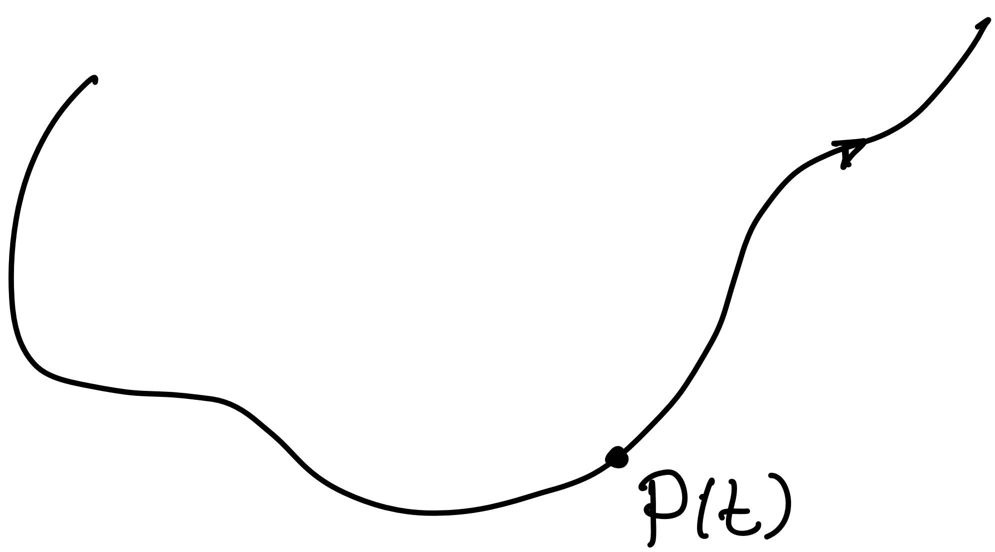
$$f = f(\overbrace{x_1, x_2, \ldots, x_n}^{\in \mathbb{R}^n}) = f(x)$$

$$x \in X \subseteq \mathbb{R}^n$$

$n, m \geq 1$

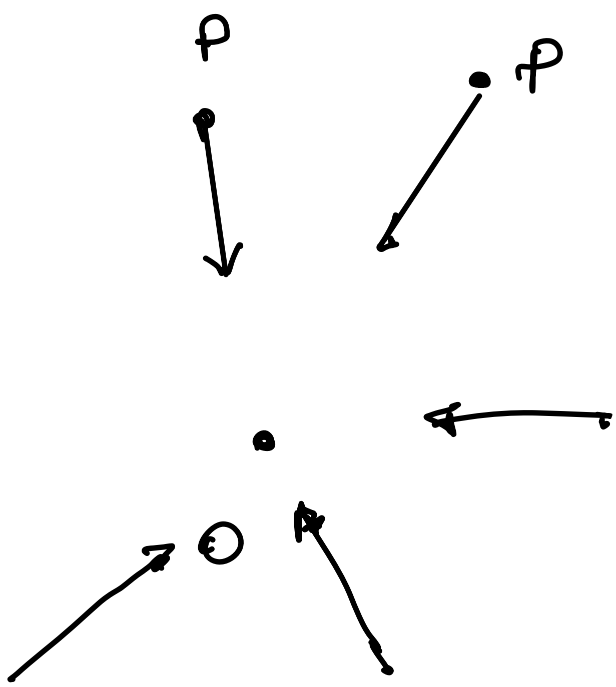
$$f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Es



$$f : t \in I \subseteq \mathbb{R} \longrightarrow P(t) \in \mathbb{R}^3$$

$= (x(t), y(t), z(t))$



$$P \in \mathbb{R}^3 \setminus \{(0,0,0)\} \longrightarrow \underline{F}(P) \in \mathbb{R}^3$$

$$(x,y,z) \longrightarrow \underline{F}(x,y,z) \in \mathbb{R}^3$$

$$= (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)) \in \mathbb{R}^3$$

$$\mathbb{R}^n = \{(x_1,\dots,x_n) : x_i \in \mathbb{R}, \forall i=1,\dots,n\}$$

$+$ spazio vettoriale

$\cdot$

$$\underline{x} = (x_1,\dots,x_n), \quad \underline{y} = (y_1,\dots,y_n)$$

$\underline{x}\cdot\underline{y}$ = prodotto scalare $= x_1\underbrace{y_1+x_2y_2+\dots+x_n}y_n$

(x,y)

$$\underline{x}\cdot\underline{y} = \underline{y}\cdot\underline{x} \qquad \forall \underline{x},\underline{y}\in\mathbb{R}^n$$

$$(\underline{x}+\underline{y})\cdot\underline{z} = (\underline{x}\cdot\underline{z})+(\underline{y}\cdot\underline{z}) \qquad \forall \underline{x},\underline{y},\underline{z}\in\mathbb{R}^n$$

$$\underline{x}\cdot\underline{x} = x_1^2+x_2^2+\dots+x_n^2 \geq 0$$

$= 0 \iff \underline{x}=\underline{0}$

<u>vett nullo</u>

Norma o modulo di $\underline{x}$

$$\|\underline{x}\| = \text{norma di } \underline{x} = \sqrt{\underbrace{\underline{x}\cdot\underline{x}}_{\geq 0}} \geq 0$$

$$\|\underline{x}\| = 0 \iff \underline{x} = \underline{0}$$

$$\|\lambda \underline{x}\| = |\lambda| \, \|\underline{x}\|$$

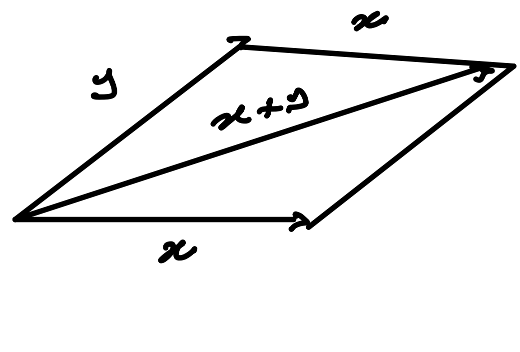
$\lambda \in \mathbb{R}$, $\underline{x} \in \mathbb{R}^n$

$n=1$ $\quad |\lambda x| = |\lambda| \, |x|$

$$|x+y| \leq |x| + |y| \quad \text{dis. triangolare}$$

$\underline{n>1}$

$$\underbrace{\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|}_{\text{dis. triangolare}} \quad \forall \underline{x},\underline{y} \in \mathbb{R}^n$$



Disuguaglianza di Cauchy Schwartz:

$$|\underline{x} \cdot \underline{y}| \leq \|\underline{x}\| \, \|\underline{y}\|$$

$$\forall \underline{x}, \underline{y} \in \mathbb{R}^n$$

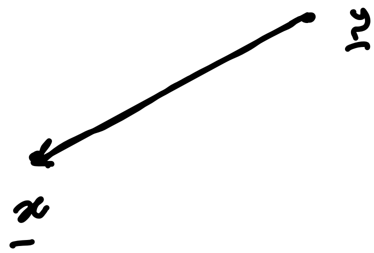
$$\left| \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \, \|\underline{y}\|} \right| \leq 1$$

$\exists \, \theta \in [0, \pi]$ tale che

$$\cos\theta = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \cdot \|\underline{y}\|}$$

$$\underline{x} \cdot \underline{y} = \|\underline{x}\| \, \|\underline{y}\| \cos\theta$$

Distanza $\mathbb{R}^2$ $\quad d(\underline{x}, \underline{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$



$\mathbb{R}^n$ $\quad d(\underline{x},\underline{y})$ = distanza euclidea

$$= \|\underline{x}-\underline{y}\| =$$

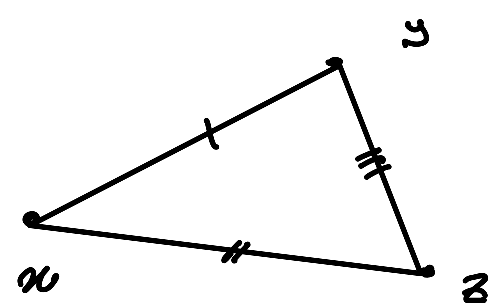
$$= \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\dots+(x_n-y_n)^2} \geq 0$$

$$d(\underline{x},\underline{x}) = 0$$

$$d(\underline{x},\underline{y}) = d(\underline{y},\underline{x})$$

$$d(\underline{x},\underline{y}) = 0 \iff \underline{x} = \underline{y}$$

$$d(x,y) \leq d(x,z) + d(z,y)$$



dis. triangolare

$\mathbb{R}^n$ prod. scalare $\cdot$

$\|\cdot\|$ spazio normato

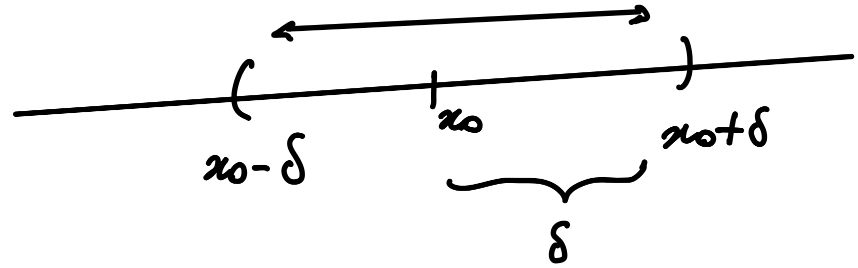
$d(\cdot,\cdot)$ spazio metrico

$\underline{e}_1=(1,0,\dots,0)$, $\underline{e}_2=(0,1,0,\dots,0)$

$\dots$, $\underline{e}_n=(0,0,\dots,0,1)$

base canonica di $\mathbb{R}^n$.

<u>Intorno</u> ($\mathbb{R}$) $\quad x_0\in\mathbb{R}$



$$]x_0-\delta, x_0+\delta[\; = I_\delta(x_0)$$

$$I_\delta(x_0)=\{x\in\mathbb{R} : |x-x_0|<\delta\}$$

$\mathbb{R}^n \quad x_0 \in \mathbb{R}^n, \quad \delta > 0$

$I_\delta(x_0)$ = intorno sferico centrato in x_0 e di raggio δ

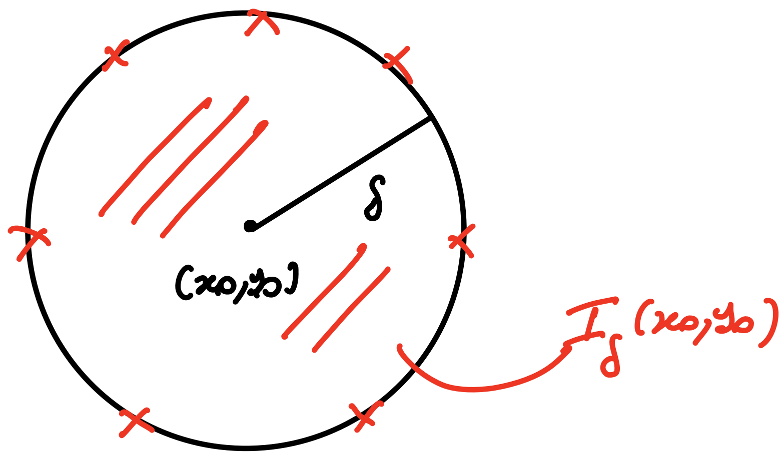
$= \{ x \in \mathbb{R}^n : \|x - x_0\| < \delta \}$

$\underline{n=2}$ $\qquad \overbrace{(x_0, y_0)}^{x_0} \qquad \overbrace{(x,y)}^{x}$

$$\|(x,y) - (x_0,y_0)\| = \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

$$\Leftrightarrow \quad (x-x_0)^2 + (y-y_0)^2 < \delta^2$$

$$\underbrace{(x-x_0)^2 + (y-y_0)^2 = \delta^2}_{\text{circonferenza in } (x_0,y_0) \text{ raggio } \delta}$$



$\underline{m=3}$

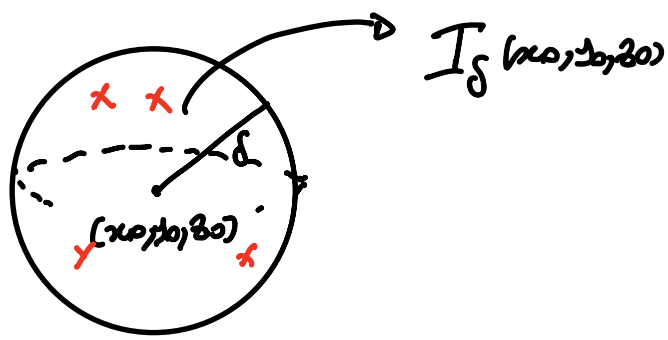
(x_0,y_0,z_0) $\qquad$ (x,y,z)

$$d\Big((x,y,z),\ (x_0,y_0,z_0)\Big) = \big\|(x,y,z) - (x_0,y_0,z_0)\big\|$$

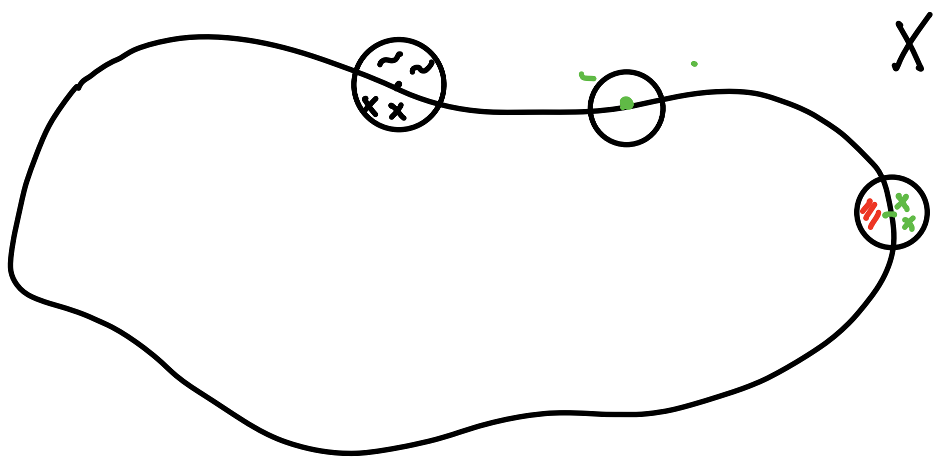
$$= \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} < \delta$$

$$\Leftrightarrow \quad (x-x_0)^2+(y-y_0)^2+(z-z_0)^2 < \delta^2$$

$=$

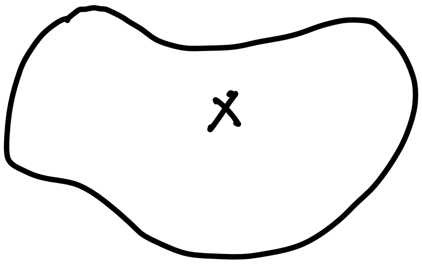


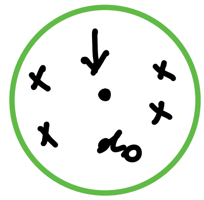
Def $X \subseteq \mathbb{R}^n$: $x_0 \in \mathbb{R}^n$ si dice <u>interno</u> ad X se $\exists\, I_\delta(x_0) \subseteq X$



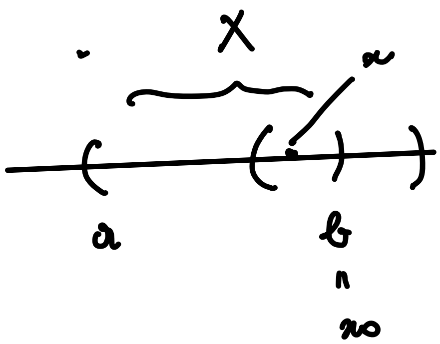
$x_0 \in \mathbb{R}^n$ si dice <u>esterno</u> ad X se è interno al complementare di X : $\mathbb{R}^n \setminus X = X^c$

$x_0 \in \mathbb{R}^n$ si dice di <u>frontiera</u> se <u>non è</u> né interno né esterno ad X $\Longleftrightarrow$ in ogni intorno di x_0 cadono sia punti di X che punti di X^c





$X_1 = X \cup \{x_0\}$

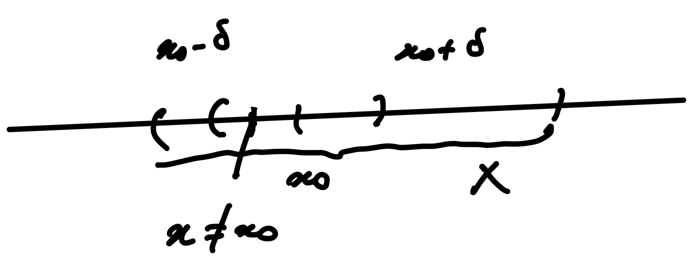


$X \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$ si dice di accumulazione per X

se $\forall \delta > 0 \ \exists x \in X$, $x \neq x_0$ tale che

$$x_0 - \delta < x < x_0 + \delta$$

$$\iff |x - x_0| < \delta \iff x \in I_\delta(x_0)$$

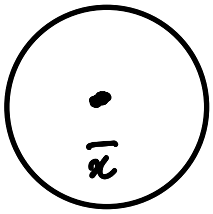


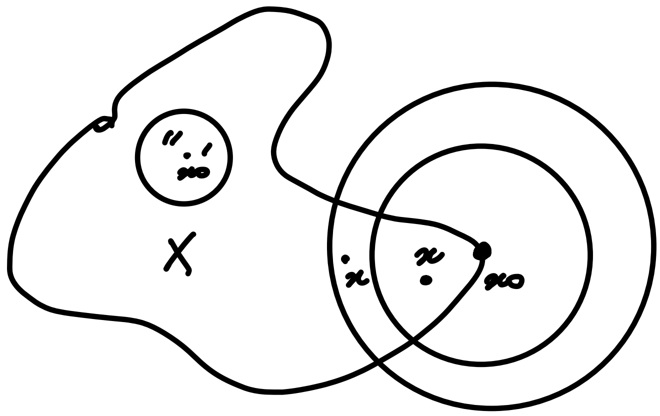
Def.

$X \subseteq \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ si dice di accumulazione per X

se $\forall \delta > 0 \ \exists x \in X$, $x \neq x_0$ t.c. $\|x - x_0\| < \delta$

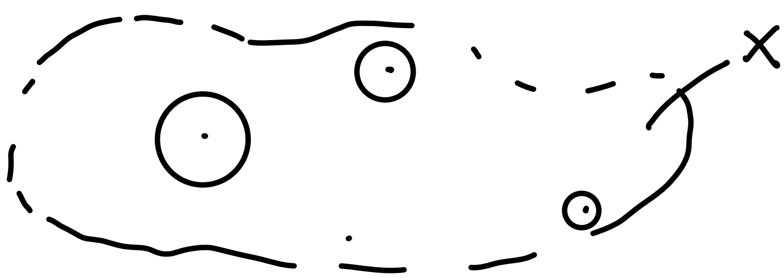
$\iff \forall \delta>0, \; I_\delta(x_0)\setminus\{x_0\}\cap X \neq \emptyset$



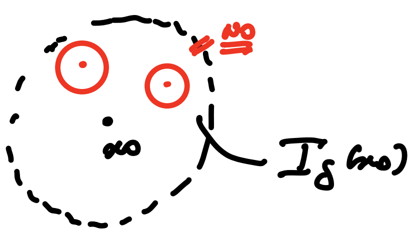


Se $\bar{x}$ è di frontiera per X, allora $\bar{x}$ può essere o può non essere di accumulazione per X: **se** non è di accumulazione, $\bar{x}$ si dice punto **isolato** di X

Def. Un insieme $X \subseteq \mathbb{R}^n$ si dice **aperto** **se** ogni punto di X è **interno** ad X



ES.

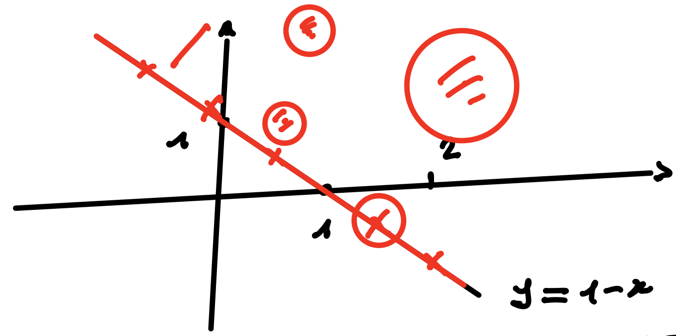


aperto

$X = \{(x,y) \in \mathbb{R}^2 : x+y \underbrace{\geq}_{} 1\}$ rappresentare

$x+y \geq 1$?

$x+y = 1 \iff y = 1-x$



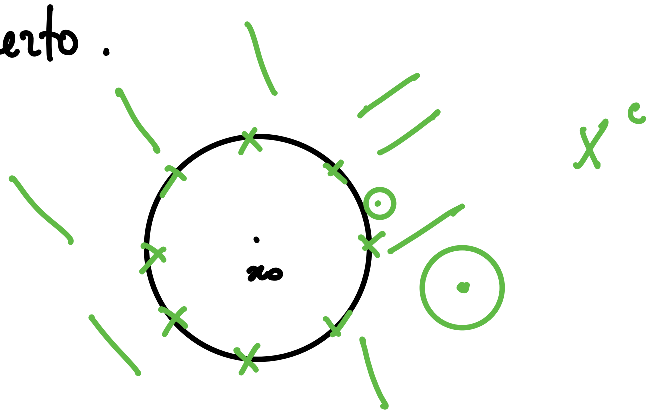
$y = 1-x$

$\iff x+y=1$

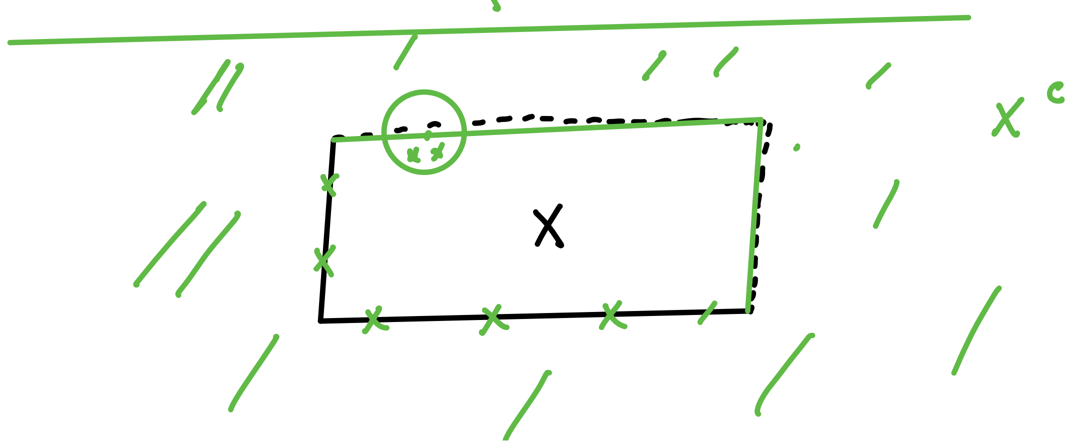
$2+0 = 2 > 1 \qquad (2,0)$

$x+y \quad > 1$

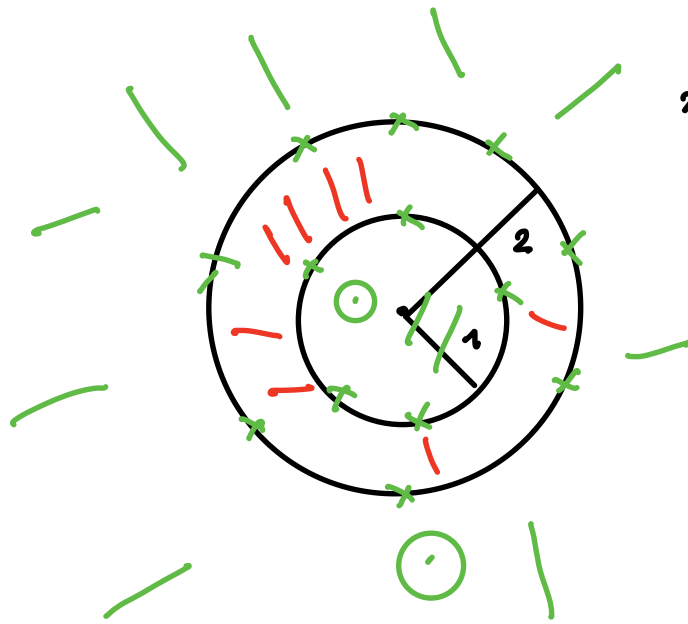
Def $X \subseteq \mathbb{R}^n$ è chiuso se $X^c = \mathbb{R}^n \setminus X$ è aperto.



ES.



$X = \{ (x,y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 9 \}$



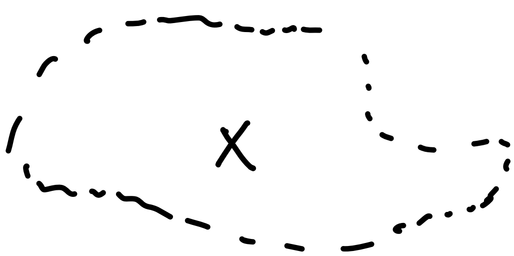
$x^2 + y^2 \ge 1$

$x^2 + y^2 \le 9$

Def (Chiusura di un insieme)

$\overline{X}$ = chiusura di X = $X \cup \{$punti di frontiera$\}$
$= X \cup \partial X$

∂X = frontiera di X



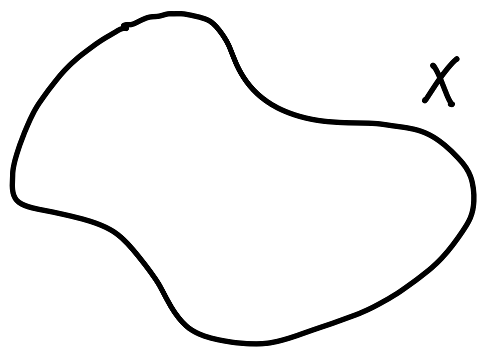
$\overline{X} = X \cup \partial X$

$X \subseteq \overline{X}$, $\overline{X}$ chiuso

OSS. Se X è chiuso, $\overline{X} = X$

$\Longleftarrow$

Def $\mathring{X}$ = interno di X = {insieme dei punti interni ad X}

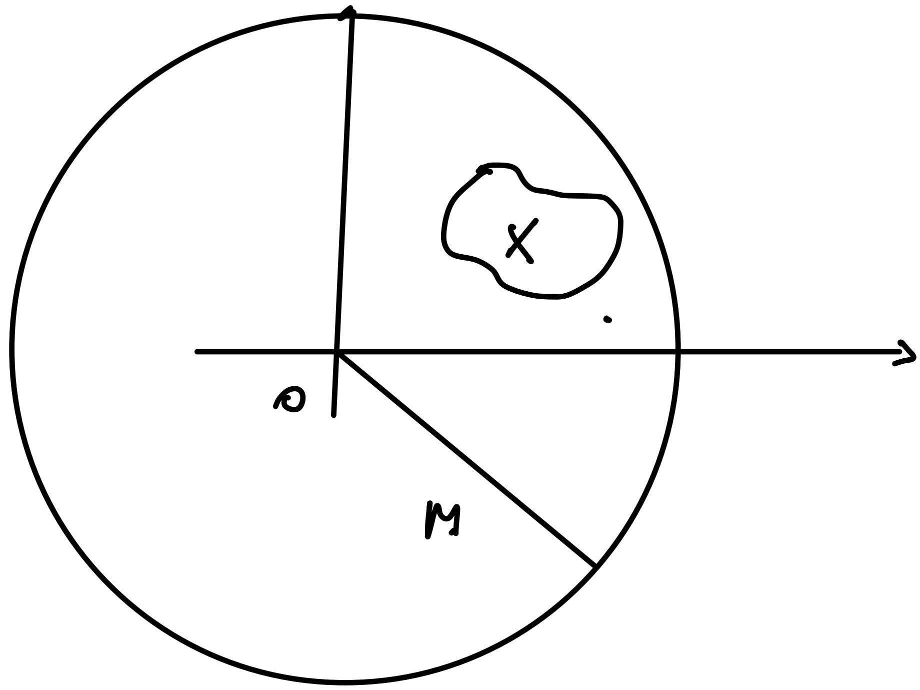


X aperto $\Longleftrightarrow$ $X = \mathring{X}$

Def (Insieme limitato)

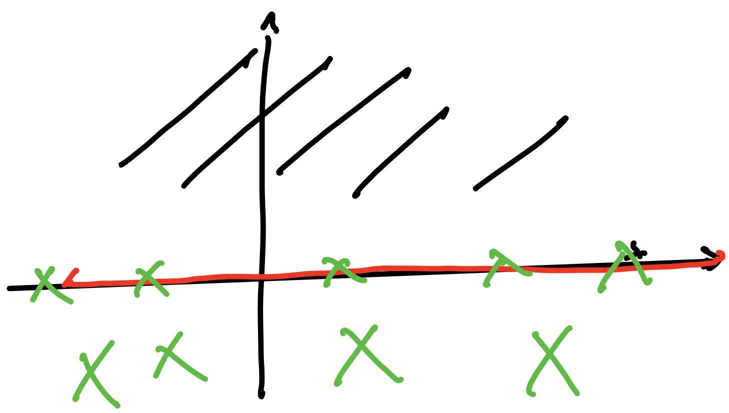
$X \subseteq \mathbb{R}^n$ limitato se $\exists M > 0 : \|x\| \leq M, \forall x \in X$

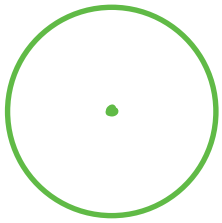
$n=1$ $\exists M > 0$ t.c. $|x| \leq M$, $\forall x \in X$



Def. $X \subseteq \mathbb{R}^n$ chiuso e limitato si dice compatto di $\mathbb{R}^n$

ES. $\{y \geq 0\} = X$





X compatto

ES. $f(x,y) = \log(x^4 - y^2)$

$x^4 - y^2 > 0$ $\qquad y^2 = x^4 \; \cancel{\iff y = \pm x^2}$

$(x^2)^2 - y^2 > 0 \iff (x^2+y)(x^2-y) > 0$

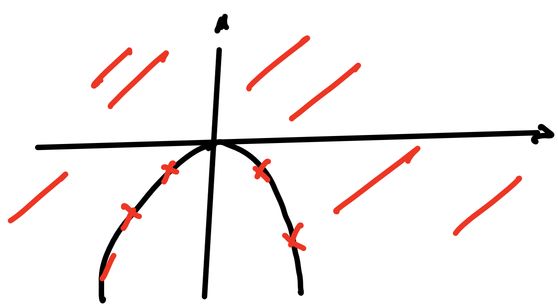
$$\iff \begin{cases} x^2 + y > 0 \\ x^2 - y > 0 \end{cases} \cup \begin{cases} x^2 + y < 0 \\ x^2 - y < 0 \end{cases} \Downarrow \; \emptyset$$

$y < -x^2$

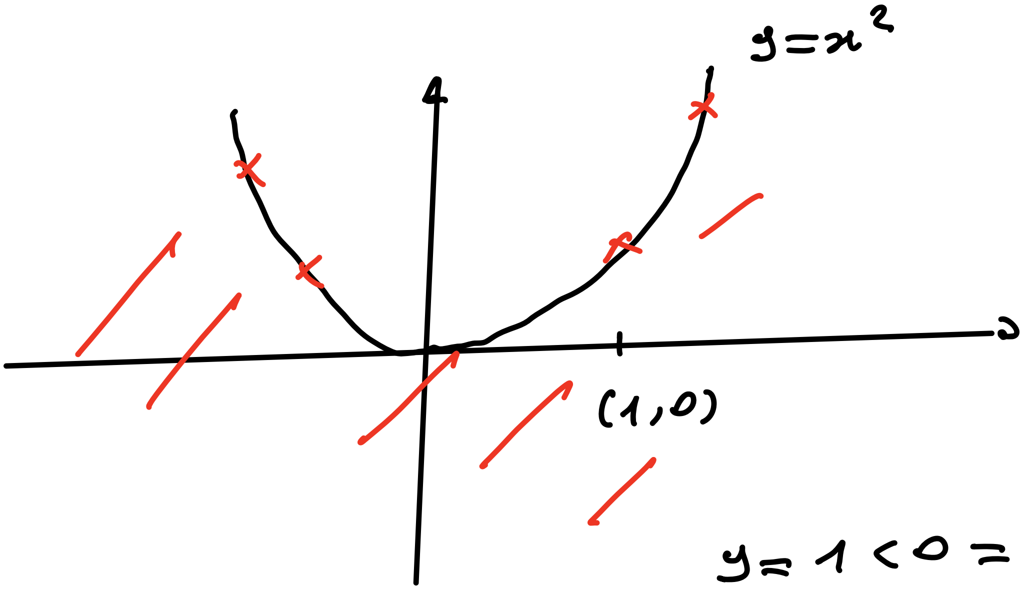
$y > x^2$

$y + x^2 > 0 \iff y > -x^2$

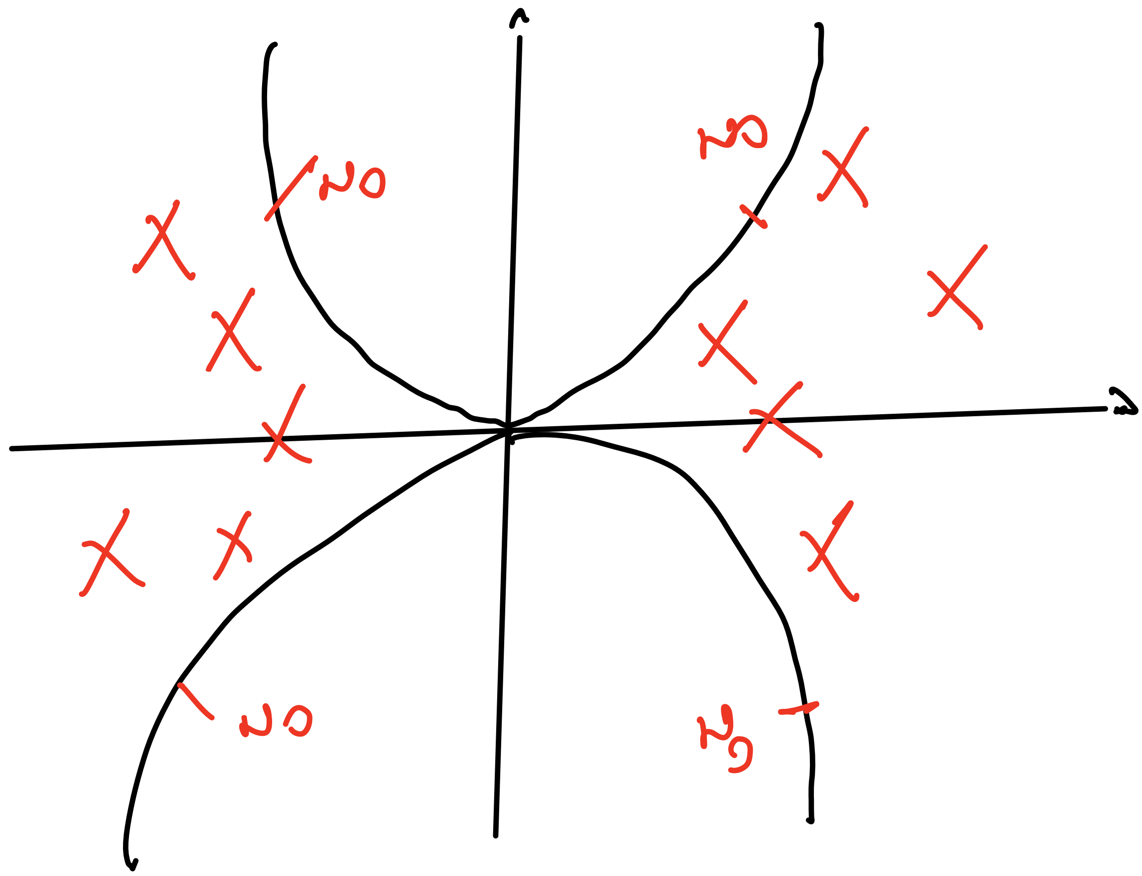
$y = -x^2$



$x^2 - y > 0 \iff y \underset{=}{<} x^2$



$y = 1 < 0 = x^2$



$$f(x,y) = \log(x^2 - y^2)$$

$$f(x,y) = \log(y^4 - x^2)$$