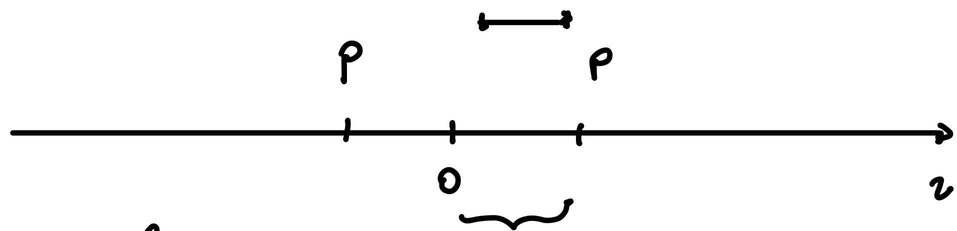
Lezione del 03/12/2022

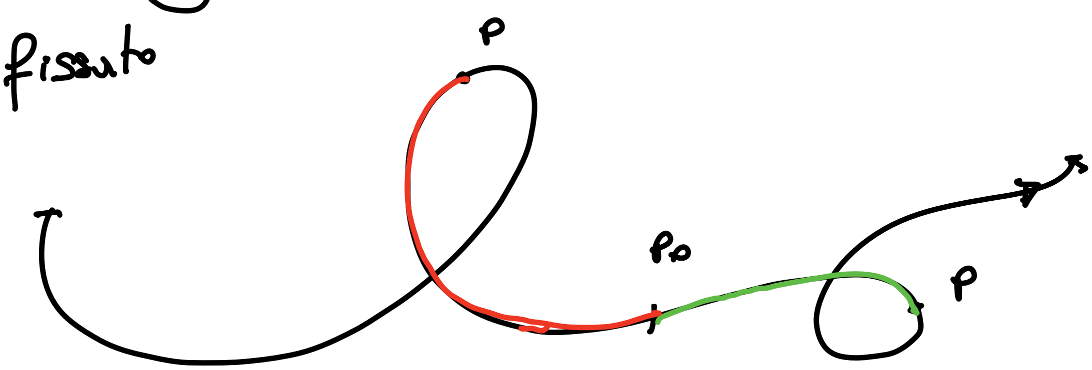
Ascissa curvilinea o lunghezza d'arco



γ curva regolare

$\Gamma \subseteq \mathbb{R}^n$ sostegno di γ

$P_0 \in \Gamma$ fissato



$\underline{P \in \Gamma}$: $P \longrightarrow s(P) = \begin{cases} |\widehat{P_0P}| & \text{se } P \text{ segue } P_0 \\ -|\widehat{P_0P}| & \text{se } P \text{ precede } P_0 \end{cases}$

$\Gamma \longleftrightarrow [a,b]$

Sia φ una rappresentazione parametrica di γ

$\varphi : [a, b] \longrightarrow \mathbb{R}^n$ curva regolare

$t_0 \in [a, b]$, $P_0 = \varphi(t_0)$ origine delle ascisse curvilinee.

$P \in \Gamma$: $P = \varphi(t)$

$$\left[\quad s(t) = \int_{t_0}^{t} \| \varphi'(\tau) \| d\tau \quad \right]$$

$\forall t \in [a, b]$: ascissa curvilinea

o lunghezza d'arco.

Se P <u>segue</u> P_0, <u>allora</u>

dev'essere <u>$t > t_0$</u> e

$$s(t) = \int_{t_0}^{t} \|\varphi'(\tau)\| d\tau =$$

lunghezza dell'arco di curva $\widehat{P_0 P}$

Se P <u>precede</u> P_0, <u>allora</u>

dev'essere $t < t_0$

$$s(t) = -\int_t^{t_0} \|\varphi'(\tau)\| d\tau$$

$$= -\left(\text{lunghezza di } \widehat{P_0 P}\right)$$

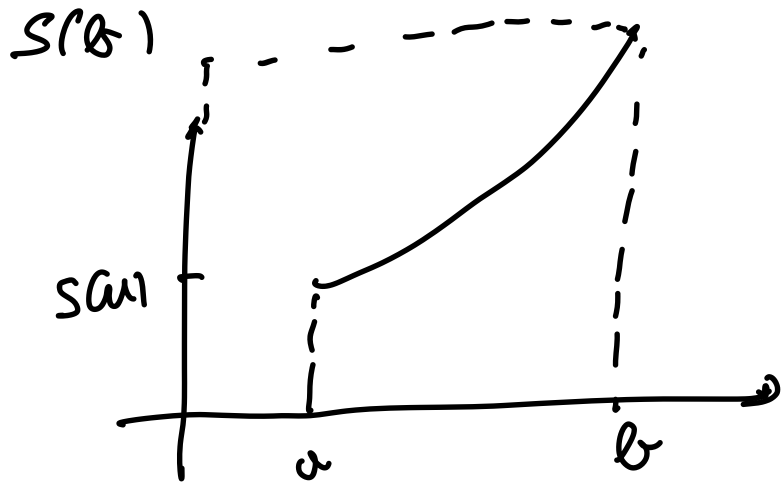
<u>OSS.</u> Se calcoliamo $s'(t)$, si ha

$s'(t) =$ (per il teorema fondamentale del calcolo integrale)

$$= \|\varphi'(t)\| > 0 \quad \forall t \in (a, b)$$

$\Rightarrow$ $s = s(t)$ f.ne strett. crescente in $[a,b]$

$$s(t) : t \in [a,b] \longrightarrow s = s(t) \in [\underset{\sim}{s(a)}, \underset{\sim}{s(b)}]$$



$s = g(t)$, $g \in C^1$

$g' \neq 0$:

$s = s(t)$ è

un cambiamento ammissibile!

Quindi $s = s(t)$ è invertibile e

$t = t(s)$, funzione inversa

$$t : s \in [s(a), s(b)] \longrightarrow t(s) \in [a,b] .$$

Se poniamo : $\qquad s'(t) = \|\varphi'(t)\|$

$$\gamma(s) = \varphi(t(s)), \quad s \in [s(a), s(b)]$$

$$\gamma \sim \varphi$$

$$\|\gamma'(s)\| = 1$$: infatti,

$$\gamma'(s) = \varphi'(t(s)) \cdot \frac{dt}{ds}(s)$$

$$= \varphi'(t(s)) \cdot \frac{1}{\underbrace{s'(t(s))}_{}} =$$

$$= \frac{\varphi'(t(s))}{\|\varphi'(t(s))\|}$$

HA NORMA 1

ES. ELICA CILINDRICA

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$$

$t \in [0, 2\pi]$

$P_0 = (0,0,0) = O$

$t = 0 \longrightarrow P_0$

$$s(t) = \int_0^t \sqrt{(x')^2 + (y')^2 + (z')^2}\, d\tau$$

$$= \sqrt{2} \int_0^t dt = \sqrt{2}\, t$$

$$s = \sqrt{2}\, t : \quad t = s/\sqrt{2}$$

$s \in [0, 2\pi\sqrt{2}]$

$$s(2\pi) = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}$$

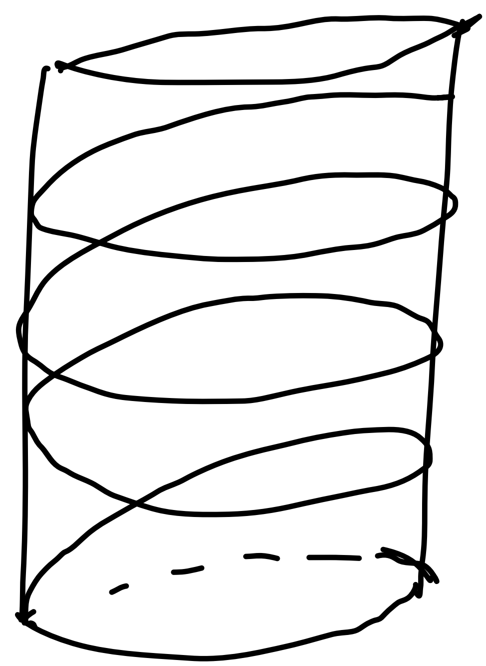
$$\begin{cases} x = \cos\left(s/\sqrt{2}\right) \\ y = \sin\left(s/\sqrt{2}\right) \\ z = s/\sqrt{2} \end{cases}, \quad s \in [0, 2\pi\sqrt{2}]$$

$$x'(s) = -\sin\left(s/\sqrt{2}\right) \cdot 1/\sqrt{2}$$

$$y'(s) = \cos\left(s/\sqrt{2}\right) \cdot 1/\sqrt{2}$$

$$z' = 1/\sqrt{2}$$

$$(x')^2+(y')^2+(z')^2 = \frac{1}{2}+\frac{1}{2}=1$$



Integrale curvilineo di una funzione

γ curva di $\mathbb{R}^n$, regolare, di sostegno $\Gamma \subseteq \mathbb{R}^n$.

$f = f(x) = f(x_1, .., x_n)$, $\quad f: \Gamma \to \mathbb{R}$ continua.

Se $\varphi: [a,b] \to \mathbb{R}^n$ è una rapp. par. di γ,

definiamo: $\left[\int_\gamma f \, d\underset{\sqcup}{s} = \int_a^b f(\varphi(t)) \, \|\varphi'(t)\| \, dt\right]$

integrale curvilineo di f esteso a γ

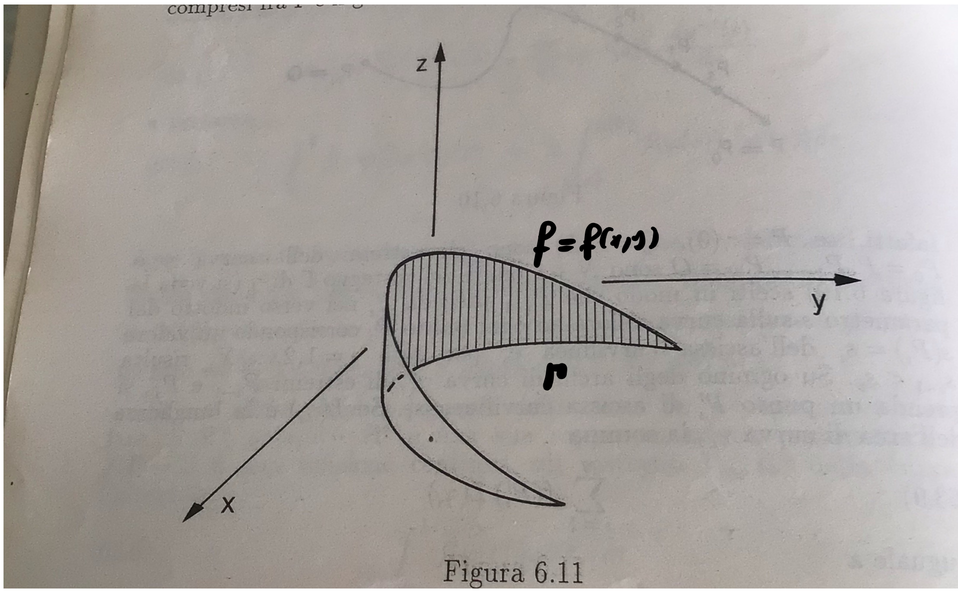


Figura 6.11

$\int_\gamma f\,ds$?

Es. $n=2$, $\varphi(t) = (x(t), y(t))$

$f = f(x,y)$

$$\int_\gamma f\,ds = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

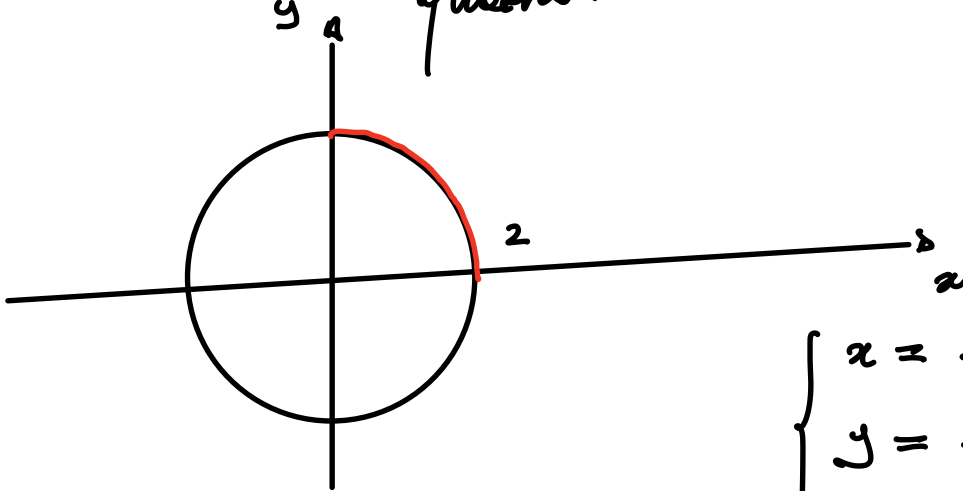
Prop. 1). Se $\varphi \sim \psi$, allora $\int_\gamma f\,ds$ è lo stesso, ossia è invariante per rappresentazioni equivalenti

2.) $f \equiv 1$: $\int_\gamma f\,ds = \int_a^b \|\varphi'(t)\|\,dt = L(\gamma)$

3.) $\int_\gamma (\alpha f + \beta g)\,ds = \alpha \int_\gamma f\,ds + \beta \int_\gamma g\,ds$

4.) $\gamma = \gamma_1 \cup \dots \cup \gamma_m$: $\displaystyle\int_\gamma f\,ds = \sum_{i=1}^{m} \int_{\gamma_i} f\,ds$

$\displaystyle\int_\gamma xy\,ds$, $\gamma \equiv$ quarto di circonferenza $x^2+y^2=4$, nel primo quadrante.

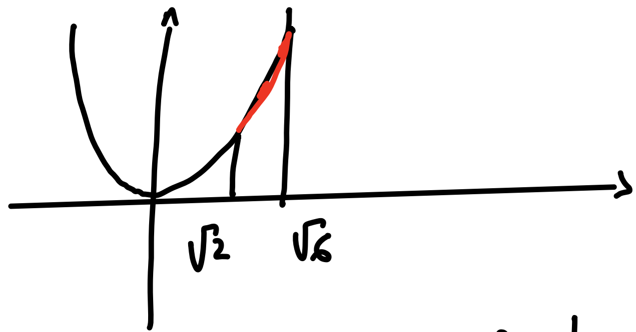


$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \qquad t \in [\underbrace{0}_{a}, \underbrace{\tfrac{\pi}{4}}_{b}]$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{4} = 2 \qquad \begin{cases} x' = -2\sin t \\ y' = 2\cos t \end{cases}$$

$$\int_\gamma xy\,ds = \int_0^{\frac{\pi}{4}} (2\cos t)(2\sin t)\cdot 2\,dt =$$
$$= 8\int_0^{\frac{\pi}{4}} \cos t \sin t\,dt = 4\left(\sin^2 t\right)_0^{\frac{\pi}{4}}$$
$$= 4\cdot \frac{1}{2} = 2.$$

$$\int_\gamma \frac{x}{y}\,ds \qquad \gamma:\ y=x^2,\ x\in[\sqrt{2},\sqrt{6}]$$



$$\begin{cases} x=t \\ y=t^2 \end{cases} \qquad t\in[\sqrt{2},\sqrt{6}]$$

$$\begin{cases} x'=1 \\ y'=2t \end{cases}$$

$$\sqrt{(x')^2+(y')^2} = \sqrt{1+4t^2}$$

$$\int_\gamma \frac{x}{y}\,ds = \int_{\sqrt{2}}^{\sqrt{6}} \frac{\cancel{t}}{t^{\cancel{2}}}\cdot\sqrt{1+4t^2}\,dt$$

$$= \int_{\sqrt{2}}^{\sqrt{6}} \frac{\sqrt{1+4t^2}}{t}\,dt \quad ; \qquad u=\sqrt{1+4t^2}$$

$$u^2 = 1+4t^2$$

$$4t^2 = u^2-1$$

$$t^2 = \frac{u^2-1}{4}$$

$$t = \frac{\sqrt{u^2-1}}{2}$$

$$dt = \frac{1}{\cancel{2}}\cdot\frac{1}{2\sqrt{u^2-1}}\cdot 2u$$

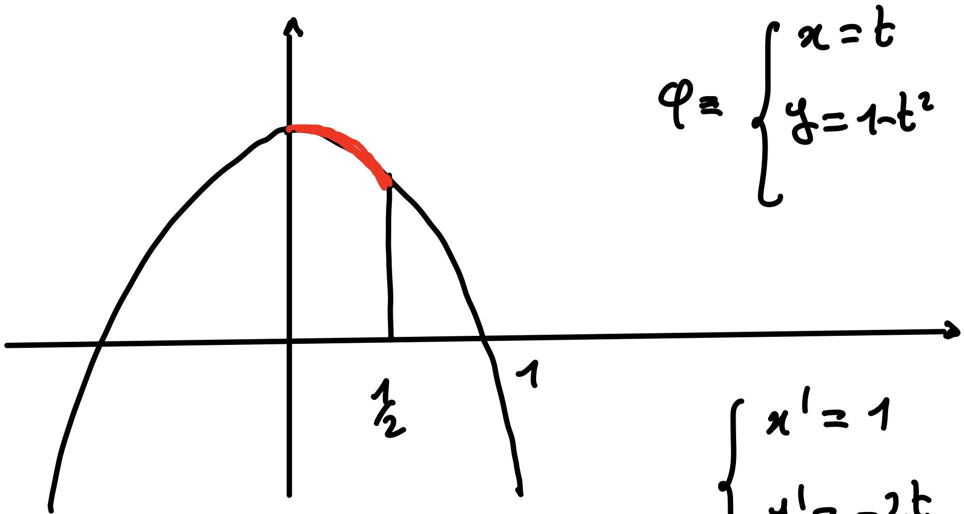
$$t=\sqrt{2} \to u=\sqrt{1+8}=3$$

$$t=\sqrt{6} \to u=\sqrt{1+4\cdot 6}=5$$

$$= \int_3^5 \frac{u}{\frac{\sqrt{u^2-1}}{\cancel{2}}}\cdot\frac{u}{\cancel{2}\sqrt{u^2-1}}\,du$$

$$= \int_3^5 \frac{u^2}{u^2-1}\,du \quad = \quad \text{FRATTI SEMPLICI}$$

$$\int_\gamma \frac{xy+1}{y\sqrt{1+4x^2}}\,ds \qquad \gamma: \underbrace{y = 1-x^2}_{\text{parabola}}, \; x \in [0, \tfrac{1}{2}]$$



$$\varphi = \begin{cases} x = t \\ y = 1-t^2 \end{cases}$$

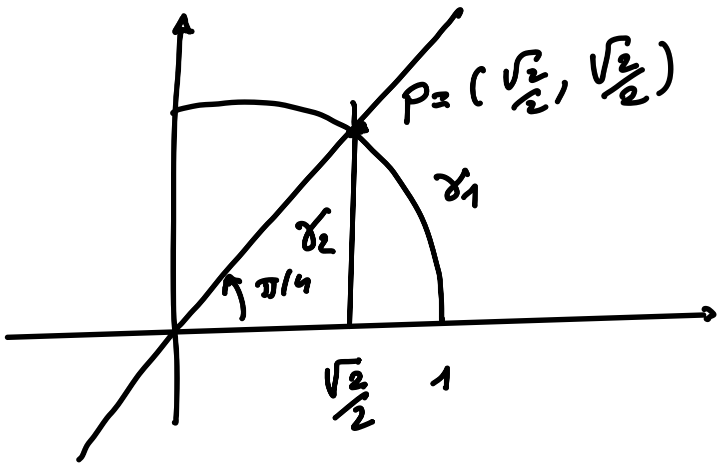
$$\begin{cases} x' = 1 \\ y' = -2t \end{cases}$$

$$= \int_0^{1/2} \frac{t\cdot(1-t^2)+1}{(1-t^2)\cancel{\sqrt{1+4t^2}}} \cdot \cancel{\sqrt{1+4t^2}}\,dt$$

$$= \int_0^{1/2} \frac{t-t^3+1}{1-t^2}\,dt = \int_0^{1/2} \frac{t^3-t-1}{t^2-1}\,dt$$

$$\int_\gamma \frac{ds}{x^2+y^2}$$

$$\gamma = \gamma_1 \cup \gamma_2$$



$$\int_\gamma - = \int_{\gamma_1} - + \int_{\gamma_2} -$$

$$\gamma_1 \equiv \begin{cases} x = \cos t \\ y = \sin t \\ t \in [0, \frac{\pi}{4}] \end{cases}$$

$$\int_{\gamma_1} \frac{ds}{x^2+y^2} = \int_0^{\frac{\pi}{4}} \frac{1}{1} \cdot 1 \, dt = \frac{\pi}{4}$$

$$\gamma_2 \equiv \begin{cases} x = \frac{\sqrt{2}}{2} \\ y = t \end{cases} \qquad t \in [0, \frac{\sqrt{2}}{2}] \qquad \begin{cases} x' = 0 \\ y' = 1 \end{cases}$$

$$\int_{\gamma_2} \frac{ds}{x^2+y^2} = \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\frac{1}{2} + t^2} \cdot 1 \, dt =$$

$$= \int_0^{\frac{\sqrt{2}}{2}} \frac{dt}{t^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$$

$$= \int_0^{\sqrt{2}/2} \cdots$$

$$\int_\gamma \left(\frac{2}{3}x + 4z\right) ds \qquad \gamma : \begin{cases} x = 3t \\ y = \frac{3}{2}t^2 \\ z = t^3 \end{cases}$$

$$t \in [0,1]$$

$$\begin{cases} x' = 3 \\ y' = 3t \\ z' = 3t^2 \end{cases} \qquad \sqrt{\underbrace{(x')^2 + (y')^2}_{} + (z')^2} = \sqrt{9 + 9t^2 + 9t^4}$$

$$= 3\sqrt{1 + t^2 + t^4}$$

$$\int_\gamma - \quad = \int_0^1 \left(\frac{2}{\cancel{3}} \cdot \cancel{3}t + 4t^3\right) \cdot 3\sqrt{1+t^2+t^4}\, dt$$

$$= 3\int_0^1 (2\underbrace{t + 4t^3}_{})\sqrt{\underbrace{t^4 + t^2}_{} + 1}\, dt =$$

$$= \cancel{3} \cdot \frac{2}{\cancel{3}} \left[(t^4 + t^2 + 1)^{\frac{3}{2}}\right]_0^1 =$$

$$= 2\left[3\sqrt{3} - 1\right]$$

ES $\int_{\gamma} x \, ds$

$\rho = 1 + \cos\vartheta, \quad \vartheta \in [-\pi, \pi]$

$$\begin{cases} x = \rho(\vartheta)\cos\vartheta = \ldots \\ y = \rho(\vartheta)\sin\vartheta = \ldots \end{cases}$$

$x' + y'$