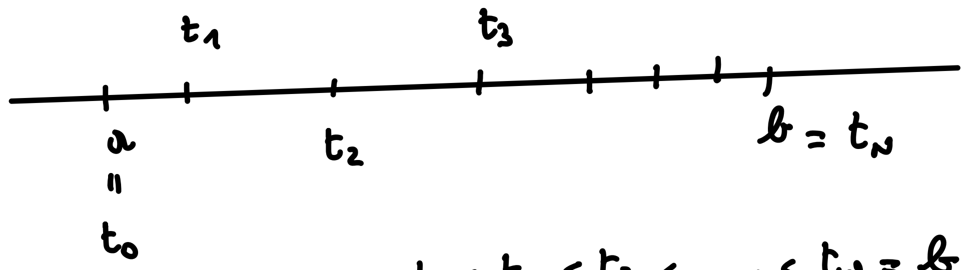
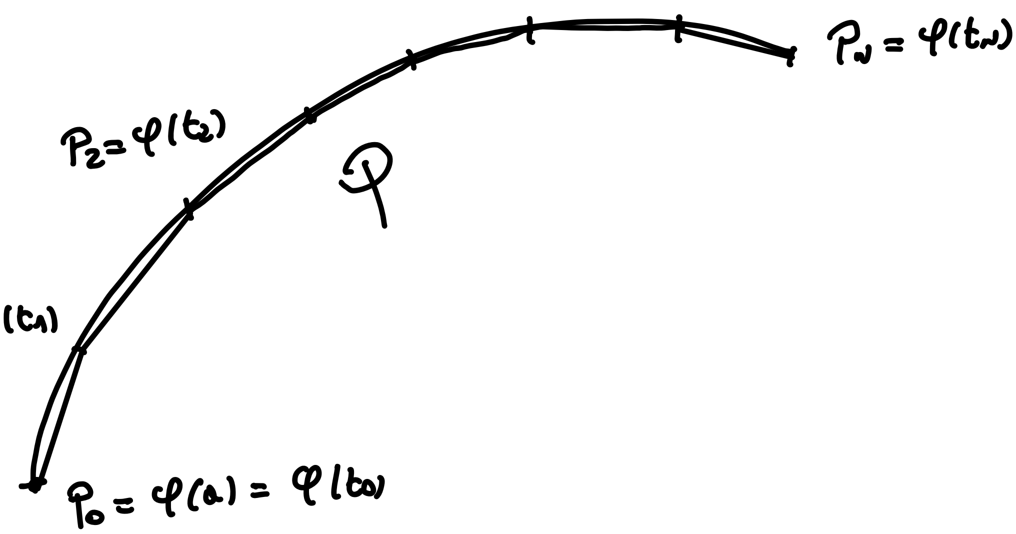
Lunghezza di un arco di curva

$\varphi : [a, b] \longrightarrow \mathbb{R}^n$ curva



$a = t_0 < t_1 < t_2 < \dots < t_N = b$



$$\ell(P) = \|\varphi(t_1) - \varphi(t_0)\| + \|\varphi(t_2) - \varphi(t_1)\| + \dots + \|\varphi(t_N) - \varphi(t_{N-1})\|$$

$L(\varphi) = \sup\{\ell(P) : P \text{ poligonale inscritta nella curva}\}$

lunghezza di φ

$L(\varphi) \in [0, +\infty]$

Se $L(\varphi) < +\infty$, φ si dice rettificabile.

Teorema (RETTIFICABILITÀ DELLE CURVE DI CLASSE C^1)

$\varphi : [a, b] \longrightarrow \mathbb{R}^n$ di classe C^1, allora φ è rettificabile ed inoltre:

$$L(\varphi) = \int_a^b \| \varphi'(t) \| \, dt$$

ES. $\varphi(t) = (x(t), y(t))$

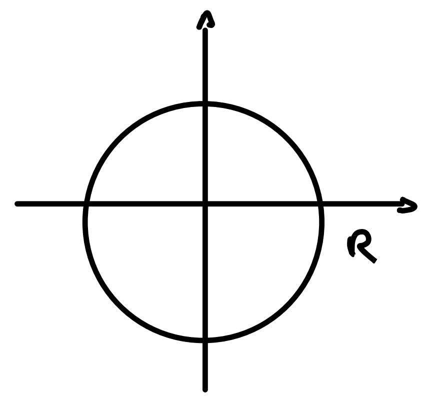
$\varphi'(t) = (x'(t), y'(t))$

$$L(\varphi) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

ES.

CIRCONFERENZA

$$\varphi = \begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \qquad t \in [\underbrace{0}_{a}, \underbrace{2\pi}_{b}]$$



$$\begin{cases} x' = -R\sin t \\ y' = R\cos t \end{cases} \qquad \sqrt{(x')^2 + (y')^2} = \sqrt{R^2} = R$$

$$L(\varphi) = \int_0^{2\pi} R \, dt = 2\pi R$$

$$\psi = \begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \qquad t \in [0, 4\pi]$$

$$L(\psi) = 4\pi R$$

OSS. $L(\varphi)$ = lunghezza della traiettoria.

Se φ è semplice, $L(\varphi)$ = lunghezza effettiva

OSS. Se φ e ψ sono curve equivalenti, $\varphi, \psi \in C^1$: se φ è rettificabile anche ψ è rettificabile e

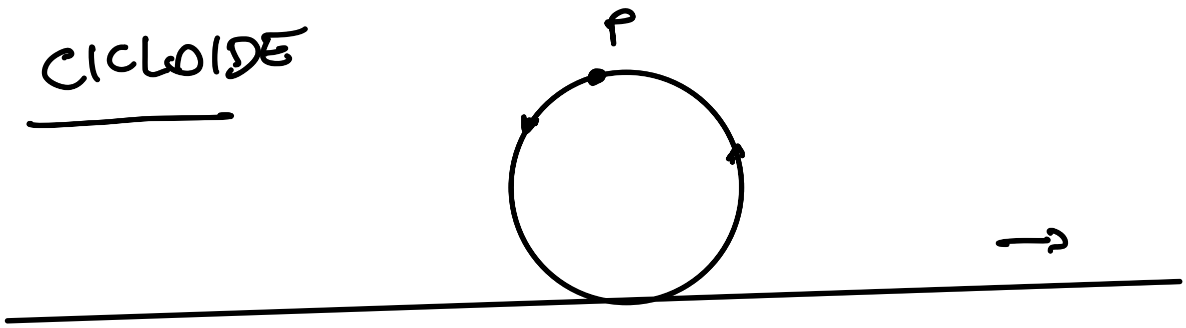
$$L(\varphi) = L(\psi).$$

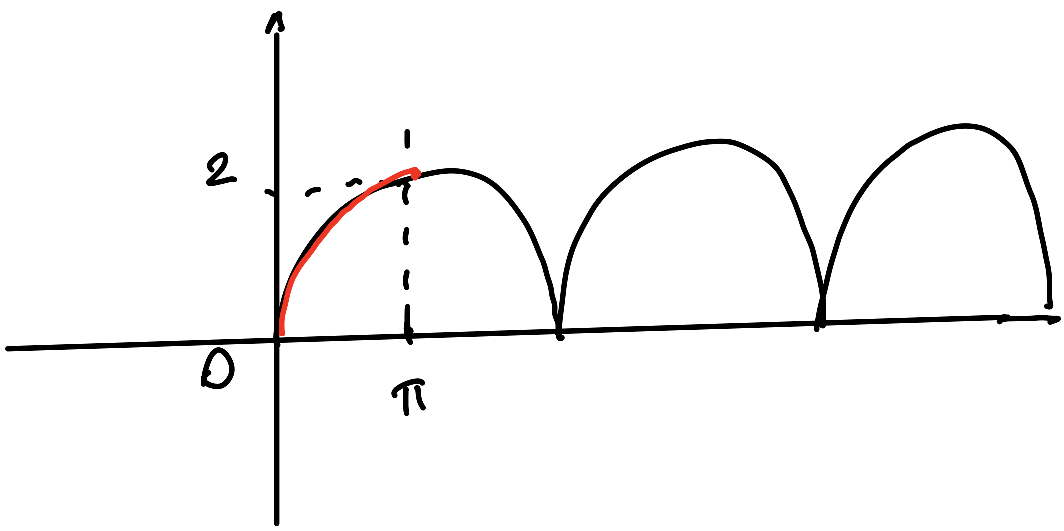
ES Calcolare la lunghezza di

$$\varphi \equiv \begin{cases} x \cdot \\ \quad\quad y \end{cases}$$

$$\varphi \equiv \begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases} \qquad t \in [0, \pi]$$

CICLOIDE





$$\begin{cases} x' = 1 - \cos t \\ y' = \sin t \end{cases}$$

$$(x')^2 + (y')^2 = 1 + \cos^2 t + \sin^2 t - 2\cos t$$
$$= 2 - 2\cos t$$

$$L(\varphi) = \int_0^{\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \sqrt{2} \int_0^{\pi} \sqrt{1 - \cos t}\, dt \quad .$$

$$\sin^2 t/2 = \frac{1-\cos t}{2} \ : \ \sin t/2 = \sqrt{\frac{1-\cos t}{2}}$$

$$\cos^2 t/2 = \frac{1+\cos t}{2}$$

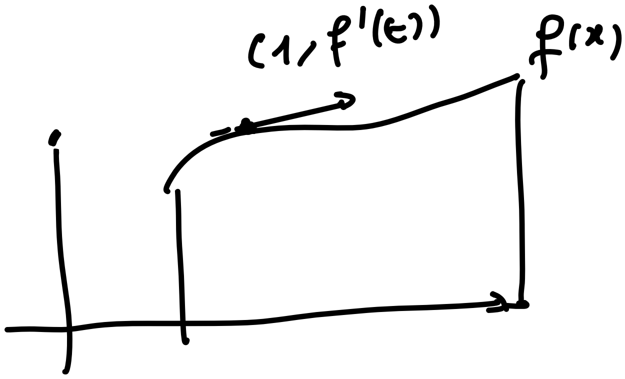
$$= \sqrt{2} \int_0^{\pi} \sqrt{2} \left(\sin \tfrac{t}{2}\right) dt =$$

$$= 2 \int_0^{\pi} \left(\sin \tfrac{t}{2}\right) dt =$$

$$= 4 \left(-\cos \tfrac{t}{2}\right)_0^{\pi} = -4 \left(\cos \tfrac{t}{2}\right)_0^{\pi} = 4 .$$

$y = f(x) \quad , x \in [a,b]$

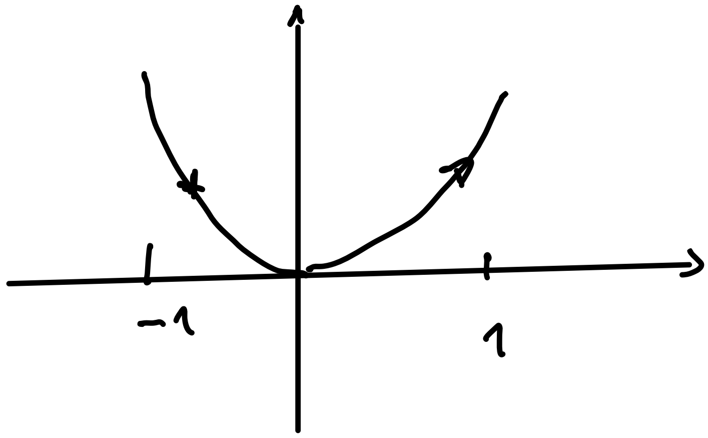
$$\varphi \equiv \begin{cases} x = t \\ y = f(t) \end{cases} \qquad \begin{cases} x' = 1 \\ y' = f'(t) \end{cases}$$



$$L(\varphi) = \int_a^b \sqrt{1 + (f'(t))^2}\, dt$$

ES. Calcolare la lunghezza del grafico della funzione $\quad y = x^2 \quad , \quad x \in [-1,1]$

$f'(x) = 2x$



$$\begin{cases} x = t \\ y = t^2 \end{cases}$$

$t \in [-1, 1]$

$$L(\varphi) = \int_{-1}^{1} \sqrt{1 + 4x^2}\, dx$$

$t = 2x$

$dx = \frac{dt}{2}$

$$\int \sqrt{1 + 4x^2}\, dx = \int \sqrt{1 + (2x)^2}\, dx$$

$$= \frac{1}{2} \int \sqrt{1 + t^2}\, dt .$$

$\int \sqrt{1+t^2} =$ (PER PARTI)

$f(t) = \sqrt{1+t^2}$ $\qquad g'(t) = 1$

$f' = \frac{t}{\sqrt{1+t^2}}$ $\qquad g(t) = t$

$$= t\sqrt{1+t^2} - \int \frac{t^2}{\sqrt{1+t^2}}\,dt$$

$$= t\sqrt{1+t^2} - \int \frac{1+t^2-1}{\sqrt{1+t^2}}\,dt$$

$$= t\sqrt{1+t^2} - \underbrace{\int \frac{1+t^2}{\sqrt{1+t^2}}\,dt}_{\int \sqrt{1+t^2}\,dt} + \underbrace{\int \frac{dt}{\sqrt{1+t^2}}}_{\text{NOTO}}$$

$$2\int \sqrt{1+t^2}\,dt = t\sqrt{1+t^2} + \int \frac{dt}{\sqrt{1+t^2}}$$

$$\int \sqrt{1+t^2}\,dt = \frac{t\sqrt{1+t^2}}{2} + \frac{1}{2}\underbrace{\int \frac{dt}{\sqrt{1+t^2}}}_{\log\left(t+\sqrt{1+t^2}\right)} \; ?$$

Lunghezza della cardioide:

$$\rho = a(1+\cos\theta), \quad \theta \in [-\pi, \pi]$$

$$\begin{cases} x = a(1+\cos\theta)\cos\theta \\ y = a(1+\cos\theta)\sin\theta \end{cases}, \quad \theta \in [-\pi, \pi]$$

$$(x')^2 + (y')^2$$

$$L(\varphi) = \int_{-\pi}^{\pi} \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta =$$

$$= ?$$