Lezione del 04/11/2022

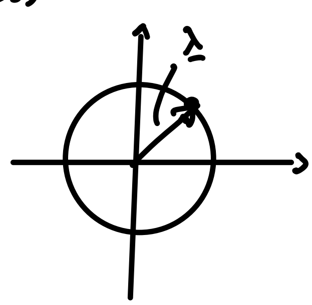
Formula del gradiente $\qquad \dfrac{\partial f}{\partial \lambda}(x_0, y_0) \overset{?}{=}$

$f = f(x,y)$, $f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$, A aperto. Se f è differenziabile in A, allora per ogni direzione $\underline{\lambda} \in \mathbb{R}^2$ esiste la derivata direzionale $\dfrac{\partial f}{\partial \underline{\lambda}}(x_0, y_0)$ per ogni $(x_0, y_0) \in A$ e si ha $\qquad \underline{\lambda} = (\lambda_1, \lambda_2)$

$$\frac{\partial f}{\partial \underline{\lambda}}(x_0, y_0) = \nabla f(x_0, y_0) \cdot \underline{\lambda}$$
$$= f_x(x_0, y_0)\lambda_1 + f_y(x_0, y_0)\lambda_2$$

Es. $f(x,y) = x^2 - 3x + 4xy + 5$: derivata direzionale in $\underbrace{(1,0)}_{(x_0, y_0)}$ rispetto alla direzione

$$\underline{\lambda} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$$



$f_x = 2x - 3 + 4y$

$f_y = 4x$

$f_x(1,0) = 2-3 = -1$

$f_y(1,0) = 4$

$$\frac{\partial f}{\partial \underline{\lambda}}(1,0) = \nabla f(1,0) \cdot \underline{\lambda} = (-1) \cdot \frac{\sqrt{2}}{2} + 4 \cdot \frac{\sqrt{2}}{2}$$
$$= -\frac{\sqrt{2}}{2} + 2\sqrt{2}$$

Dim (Formula del gradiente)

$$\frac{\partial f}{\partial \lambda}(x_0, y_0) =$$

$$= \lim_{t \to 0} \frac{f(x_0 + t\lambda_1, y_0 + t\lambda_2) - f(x_0, y_0)}{t} =$$

Funzione Ausiliaria

$$\varphi(t) = f(x_0 + t\lambda_1, y_0 + t\lambda_2) \quad \begin{cases} x(t) = x_0 + t\lambda_1 \\ y(t) = y_0 + t\lambda_2 \end{cases}$$

$$\varphi(0) = (t=0) = f(x_0, y_0) \quad \begin{cases} x' = \lambda_1 \\ y' = \lambda_2 \end{cases}$$

$$= \lim_{t \to 0} \underbrace{\frac{\varphi(t) - \varphi(0)}{t}}_{} = \varphi'(0) \ ? = *$$

$F(t) = f(x(t), y(t))$; $\quad F'(t) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$

f differenziabile

derivazione funzioni composte :

$$\varphi'(t) = (f \text{ è differenziabile}) = f_x(x_0 + t\lambda_1, y_0 + t\lambda_2)\,\lambda_1 + f_y(x_0 + t\lambda_1, y_0 + t\lambda_2)\,\lambda_2$$

$$\varphi'(0) = (t=0) = f_x(x_0, y_0)\,\lambda_1 + f_y(x_0, y_0)\,\lambda_2 \quad :$$

inserendo tale espressione in $(*)$:

$$\frac{\partial f}{\partial \lambda}(x_0, y_0) = f_x(x_0, y_0)\,\lambda_1 + f_y(x_0, y_0)\,\lambda_2$$

$$= \nabla f(x_0, y_0) \cdot \underline{\lambda}$$

C.V.D.

Verso di $\nabla f(x_0, y_0)$:

$$\left|\frac{\partial f}{\partial \lambda}\right| = \left|\nabla f(x_0, y_0) \cdot \underline{\lambda}\right| \leq \text{Cauchy-Schwartz}$$

$$\leq \|\nabla f(x_0,y_0)\| \underbrace{\|\underline{\lambda}\|}_{=1} = \|\nabla f(x_0,y_0)\|$$

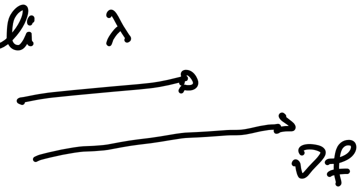
$$\Rightarrow \quad -\|\nabla f\| \leq \frac{\partial f}{\partial \lambda} \leq \|\nabla f\|$$

<u>Allora</u> $\frac{\partial f}{\partial \lambda}$ è massima quando:

$$\nabla f \cdot \lambda = \frac{\partial f}{\partial \lambda} = \|\nabla f\|$$

$$\Rightarrow \quad \|\nabla f\| \cos\theta = \|\nabla f\| \Rightarrow \cos\theta = 1 \; !$$

∇f e $\underline{\lambda}$ sono paralleli
e concordi



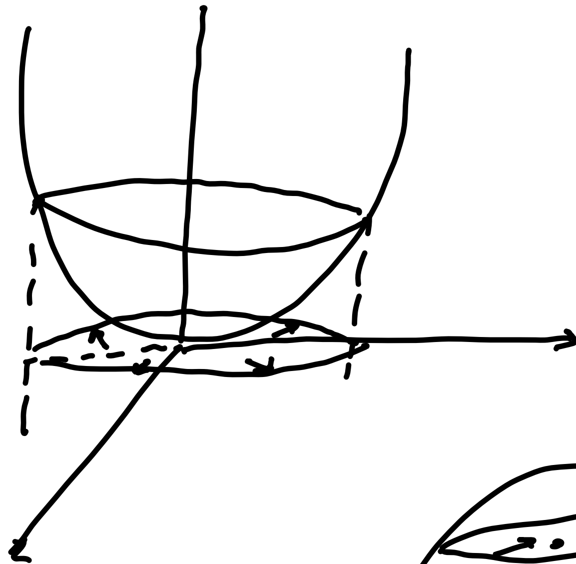
<u>Allora</u> $\underbrace{\underline{\lambda} = \frac{\nabla f}{\|\nabla f\|}}$

$\lambda = t \, \nabla f, \; t > 0$

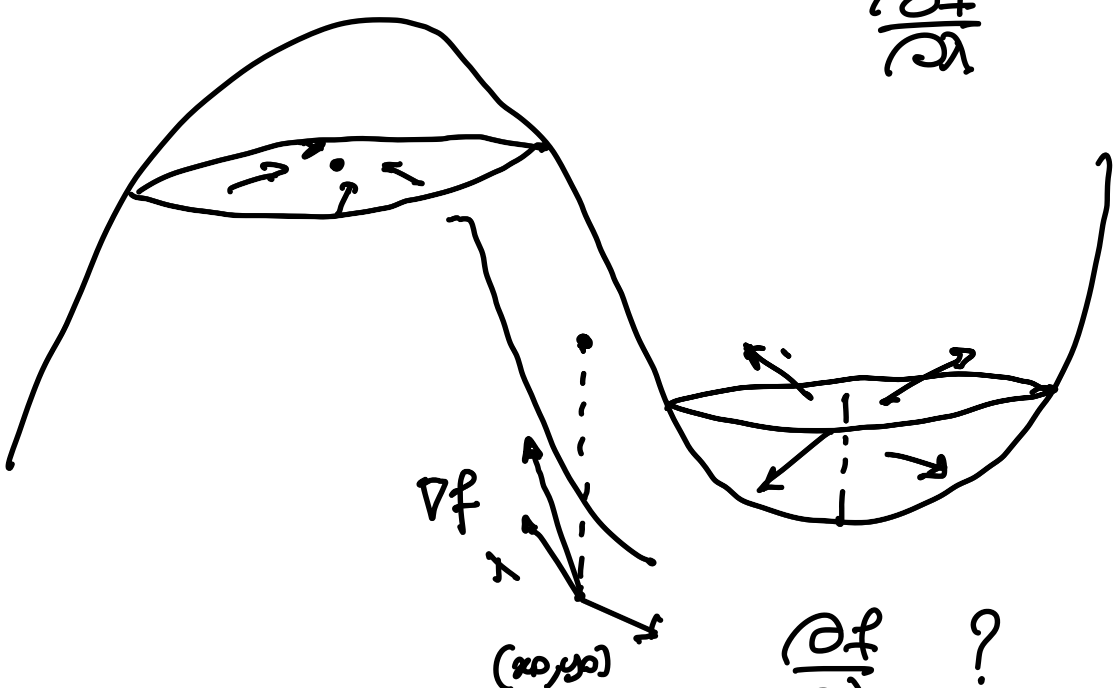
$t = \frac{1}{\|\nabla f\|}$

$\frac{\partial f}{\partial \underline{\lambda}}$ è minima quando $\frac{\partial f}{\partial \underline{\lambda}} = -\|\nabla f\|$

$$\Rightarrow \quad \underline{\lambda} = -\frac{\nabla f}{\|\nabla f\|}$$



$\frac{\partial f}{\partial \lambda}$



$\frac{\partial f}{\partial \lambda}$?

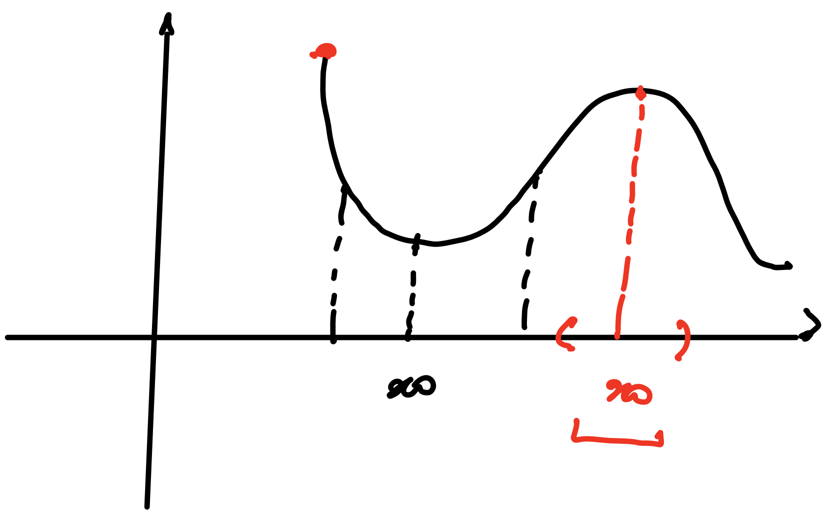
Estremi locali (o relativi)

$f = f(x) \qquad f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

$x_0 \in I$ minimo (massimo) relativo se

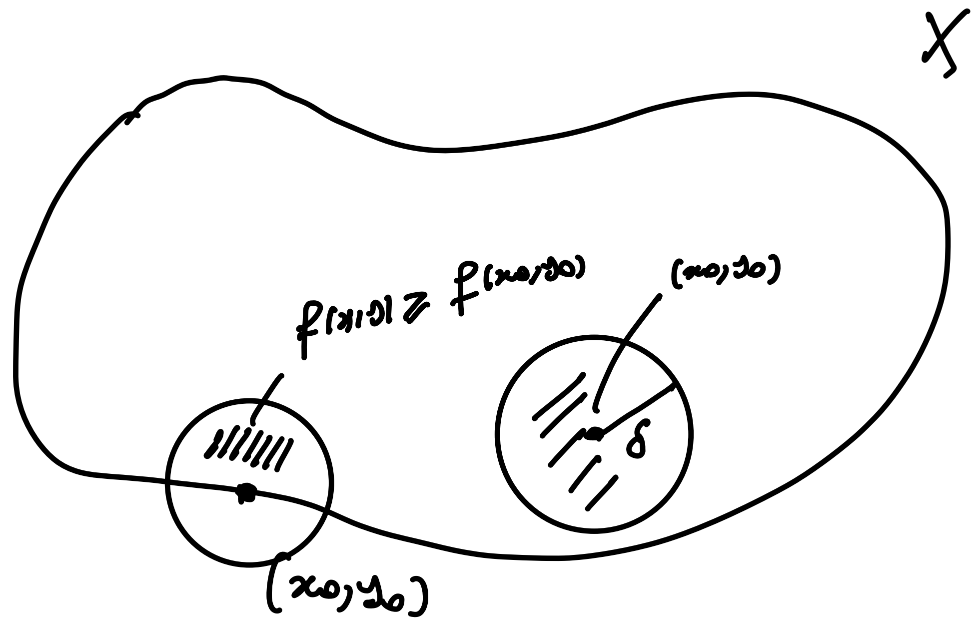
$\exists\, \delta > 0$ tale che

$$f(x) \geq f(x_0), \quad (\leq) \quad \forall x \in I \cap \overbrace{]x_0-\delta, x_0+\delta[}$$



Def $\quad f = f(x,y) \qquad f : X \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

$X \neq \emptyset$ Sia $(x_0, y_0) \in X$.

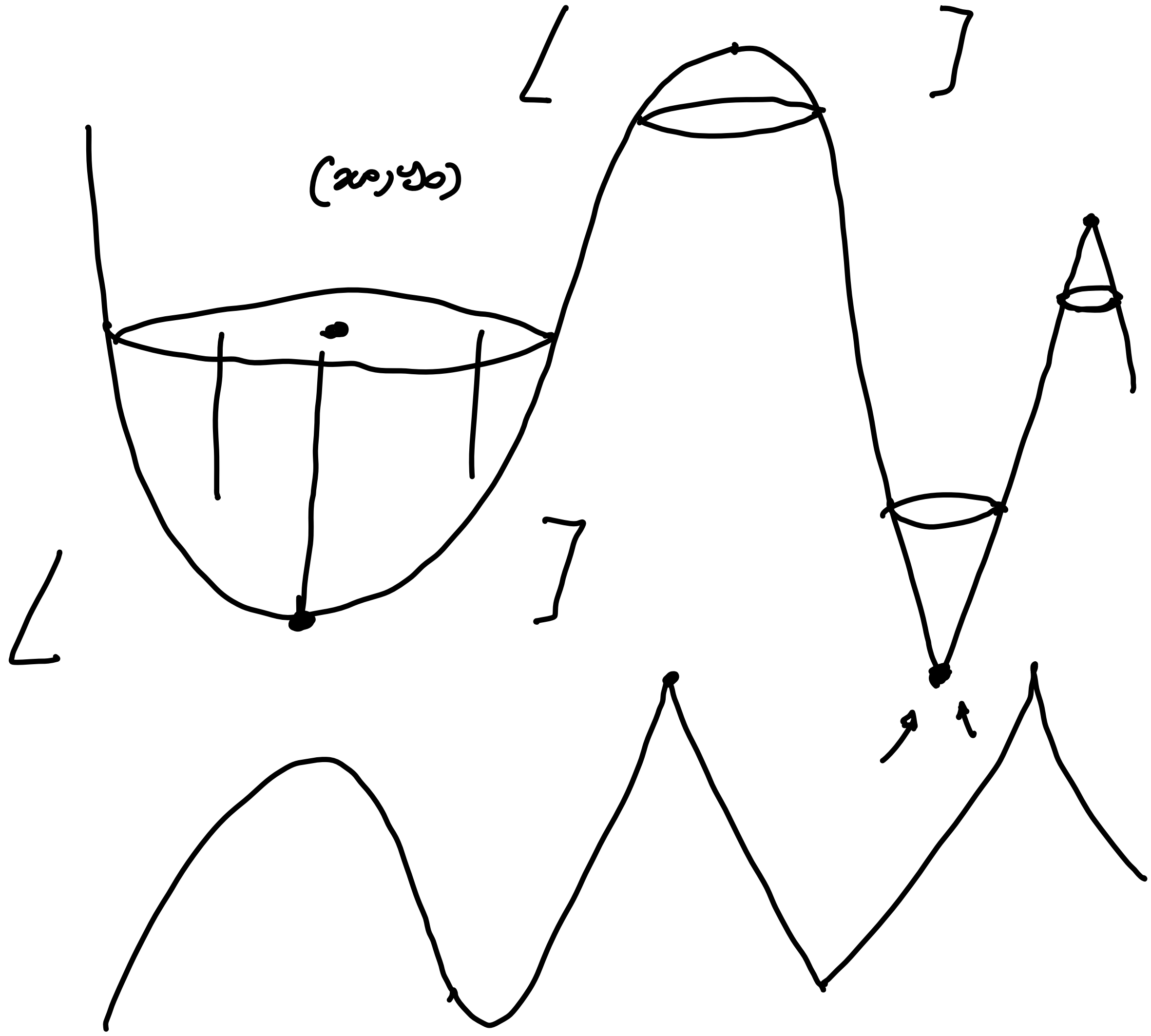


Si dice che (x_0, y_0) è di minimo (risp. massimo) relativo per f se

$\exists\ \delta > 0$ tale che

$$f(x,y) \underset{(\leq)}{\geq} f(x_0,y_0) \quad \forall (x,y) \in X \cap \overline{I_\delta(x_0,y_0)}$$

$I_\delta(x_0,y_0)$ intorno circolare



$$z = f(x_0,y_0) + f_x(x_0,y_0)\underbrace{(x-x_0) +}_{=0} f_y(x_0,y_0)(y-y_0)$$

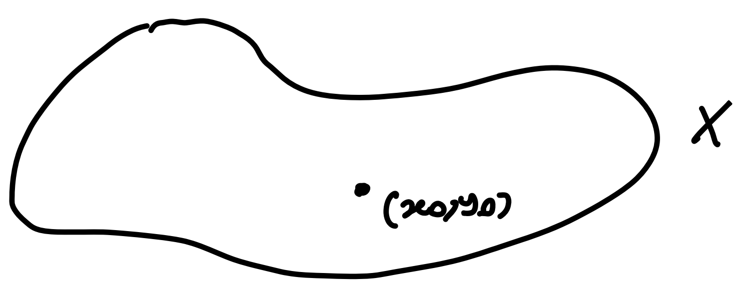
$$\begin{cases} f_x(x_0,y_0) = 0 \\ f_y(x_0,y_0) = 0 \end{cases}$$

Condizione necessaria del I° ordine

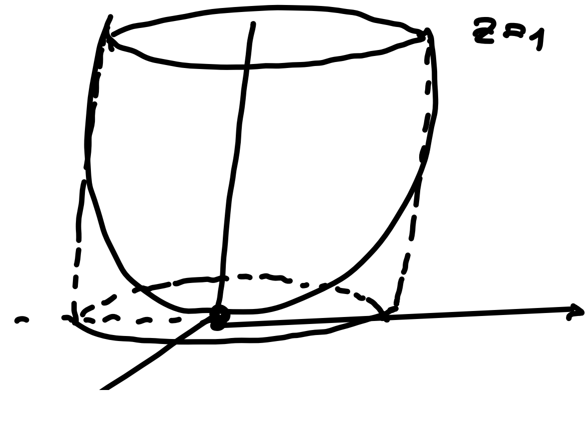
$f: X \subseteq \mathbb{R}^2 \to \mathbb{R}$, $(x_0,y_0) \in \mathring{X}$ (interno ad X)

f derivabile in (x_0,y_0). Se (x_0,y_0) è un estremo locale per f <u>allora</u> $\nabla f(x_0,y_0) = 0$

$$\begin{cases} f_x(x_0,y_0) = 0 \\ f_y(x_0,y_0) = 0 \end{cases}$$



$f(x,y) = x^2+y^2$ $\qquad X = \{(x,y) : \underbrace{x^2+y^2 \leq 1}_{\text{cerchio di raggio 1}}\}$



$\nabla f(0,0) = 0$

Se $(x,y) \in X$, $\quad f(x,y) = x^2+y^2 \leq 1$

Se $(x_0,y_0) \in$ Circonferenza : $x_0^2+y_0^2=1$

$$f(x,y) \leq 1 = f(x_0,y_0)$$

$\nabla f(x,y) = (2x, 2y) \qquad (x_0,y_0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$\nabla f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = (\sqrt{2}, \sqrt{2}) \neq (0,0)$$

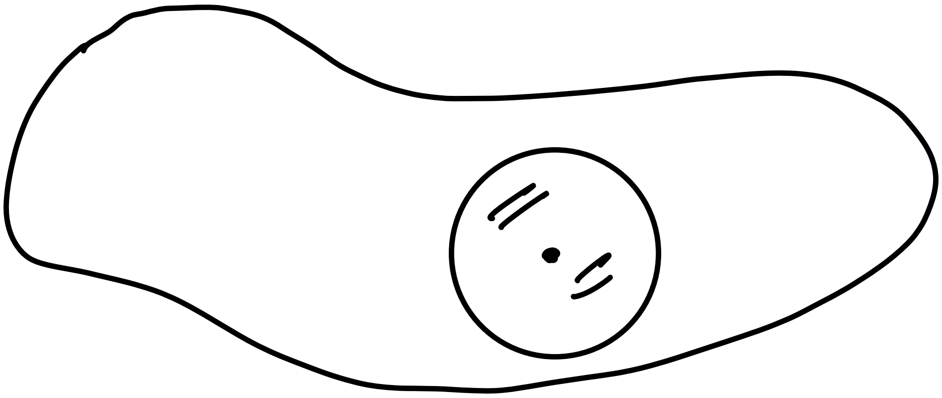
Dim. (Condizione necessaria)

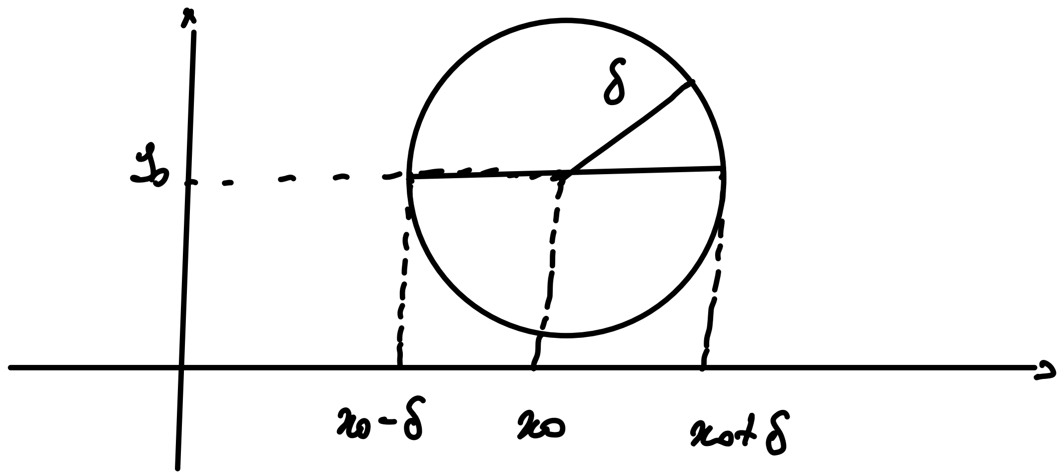
(x_0,y_0) di minimo per ipotesi.

Poiché $(x_0,y_0) \in \mathring{X}$, possiamo trovare $\delta > 0$

tale che $\quad I_\delta(x_0,y_0) \subseteq X \quad$ e

① $\quad f(x,y) \geq f(x_0,y_0), \quad \forall (x,y) \in I_\delta(x_0,y_0)$





$f_x(x_0,y_0)=0$ $\qquad$ (x, y_0)

$\rightsquigarrow \quad F(x) = f(x,y_0) \quad , \quad \forall x \in \,]x_0-\delta, x_0+\delta[$

- poiché (x_0,y_0) è di minimo, da (1) si ha

$$F(x) = f(x,y_0) \geq f(x_0,y_0) = F(x_0)$$

$$\forall x \in \,]x_0-\delta, x_0+\delta[$$

$\Rightarrow$ x_0 di minimo per $F(x)$ $\Rightarrow$

da Fermat, $\underbrace{F'(x_0)}=0$

$$0 = \lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = f_x(x_0, y_0)$$