Lezione del 03/10/2022

Serie assolutamente convergenti

Def. $\sum_{n=1}^{\infty} a_n \qquad a_n \in \mathbb{R}$: si dice che la serie

è assolutamente convergente se

$$\sum_{n=1}^{\infty} |a_n| = \overset{\geq 0}{|a_1|} + \overset{\geq 0}{|a_2|} + \dots + \overset{\geq 0}{|a_n|} + \dots$$

(serie a termini non negativi)

è convergente.

Prop. $\sum_{n=1}^{\infty} a_n$ è assolutamente convergente allora

è convergente.

ASSOLUTA CONVERGENZA $\Rightarrow$ CONVERGENZA

$\not\Leftarrow$

ES $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \underset{+\,-}{(-1)^{n-1}} \frac{1}{n}$

$\underbrace{\text{convergente}}$ ad S

$$\sum_{n=1}^{\infty} \left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

Es $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

serie esponenziale $= e^x \quad \forall x \in \mathbb{R}$

$x \geq 0$

$$x = -1 \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + -$$

$\underline{x<0}$

$\forall x \in \mathbb{R}$ $$\sum_{n=0}^{\infty} \left|\frac{x^n}{n!}\right| = \sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

Criterio rapporto: $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{\cancel{n!}}{|x|^n}$$

$$(n+1)! = (n+1)\,n!$$

$$= |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{|x|^n}{n!} \text{ è convergente}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ è assolutamente convergente}$$

$$\Rightarrow \qquad '' \text{ è convergente } \underline{\forall x \in \mathbb{R}}$$

<u>ES</u> $$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$ Criterio della radice

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^{n^{\cancel{2}}}\right]^{\frac{1}{\cancel{n}}} = (*)$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \qquad \lim_{x\to\pm\infty} \left(1+\frac{1}{x}\right)^x = e$$

$$\{a_n\} \quad a_n \to \pm\infty \quad \lim_{n\to\infty} \left(1+\frac{1}{a_n}\right)^{a_n} = e$$

$$= (*) \lim_{n \to \infty} \left(\frac{(n+1)-1}{n+1} \right)^n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1} \right)^n$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{-(n+1)} \right)^n = \qquad 2 < e < 3$$

$$\lim_{n \to \infty} \left[\left(1 + \frac{1}{-(n+1)} \right)^{-(n+1)} \right]^{-\frac{n}{n+1}} = e^{-1} < 1$$

$\Rightarrow$ la serie converge

$$2) \sum_{n=1}^{\infty} e^{1/n} = +\infty \;\Big|\; \lim_{n \to \infty} \left(e^{1/n} \right)^{1/n} = \lim_{n \to \infty} e^{\left(\frac{1}{n^2}\right)^{\to 0}} = 1$$

$$\lim_{n \to \infty} e^{1/n} = 1 \neq 0$$

$\Rightarrow$ il termine generale non è infinitesimo

$\Rightarrow$ la serie diverge positivamente

$$3) \sum_{n=0}^{\infty} \frac{3^{n+1}}{9^{n+2}} = \sum_{n=0}^{\infty} \frac{3^n}{9^n} \cdot \left(\frac{3}{9^2} \right)$$

$$= \frac{3}{16} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n =$$

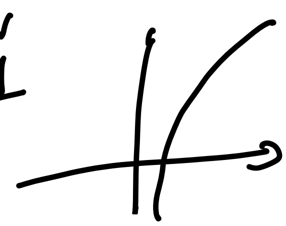
$$\sum_{n=0}^{\infty} h^n \text{ s. geometrica } = \begin{cases} \frac{1}{1-h} & \text{se } |h|<1 \\ +\infty & \text{se } h \geq 1 \\ \text{oscilla} & \text{se } h \leq -1 \end{cases}$$

$$= \frac{3}{16} \cdot \frac{1}{1-\frac{3}{4}} = \frac{3}{4}$$

$$\sum_{n \geq 1}^{\infty} (-1)^n \underbrace{\log\left(1+ \sin \frac{1}{n}\right)}_{a_n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \log\Big(\underbrace{1+ \sin \frac{1}{n}}_{\downarrow \atop 1}\Big) = 0$$

$$a_n \geq a_{n+1} \iff$$



$$\log_e\left(1+\sin\frac{1}{n}\right) \geq \log_e\left(1+\sin\frac{1}{n+1}\right)$$

$$\iff \cancel{1}+\sin\frac{1}{n} \geq \cancel{1}+\sin\frac{1}{n+1}$$

$$\iff \sin\frac{1}{n} \geq \sin\frac{1}{n+1}$$

seno crescente in $\left[0,\frac{\pi}{2}\right]$ $\iff$

$$\iff \frac{1}{n} \geq \frac{1}{n+1} \quad \text{VERO}$$

Converge per il criterio di LEIBNIZ

$$\sum_{n=1}^{\infty} \frac{e^n}{n!} \quad ; \quad \text{rapporto} \qquad e^{n+1} = e \cdot e^n$$

$$\lim_{n\to\infty} \frac{e^{n+1}/(n+1)!}{e^n/n!} = \lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n}$$

$$= e \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1 \to$$ la serie converge

$$\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$$

$$\frac{\sqrt[n]{e^n}}{\sqrt[n]{n!}} = \frac{e}{\sqrt[n]{n!}} \to 0 < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{n+1}{n}\right)^{n^2} = +\infty$$

$$\left(1+\frac{1}{n}\right)^{n^2} \sim e^n$$

Criterio radice: $\lim_{n\to\infty} \sqrt[n]{a_n} =$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{1}{2^n} \cdot \left(\frac{n+1}{n}\right)^{n^2}} =$$

$$= \lim_{n\to\infty} \frac{1}{2} \sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2}} =$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n} = \frac{e}{2} > 1$$

$\Rightarrow$ la serie diverge

(ES. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\sqrt{n}}}{n!}$)

$$\sum_{n=1}^{\infty} \cos n \, \frac{3^n}{4^{n+1}}$$

$$\left| \cos n \, \frac{3^n}{4^{n+1}} \right| = |\cos n| \frac{3^n}{4^{n+1}}$$

$$\leq \frac{3^n}{4^{n+1}} \quad : \sum \frac{3^n}{4^{n+1}} < +\infty$$

$< +\infty$

$$\sum_{n=1}^{\infty} \frac{3^n}{4^{n+1}} = \frac{1}{4} \sum_{n=1}^{\infty} \underbrace{\left(\frac{3}{4}\right)}_{}^{n}$$

$h = \frac{3}{4} < 1$

$\Rightarrow$ criterio confronto $\Rightarrow \sum \left| \cos n \, \frac{3^n}{4^{n+1}} \right| < +\infty$

$\Longrightarrow$ la serie iniziale è assolutamente convergente.

$$\sum_{n=1}^{\infty} \frac{n\sqrt{1+\sin^2 n}}{3^n}$$

$$|\sin n| \leq 1$$

$$\sin^2 n \leq 1$$

$$1+\sin^2 n \leq 1+1 = 2$$

$$\sqrt{1+\sin^2 n} \leq \sqrt{2}$$

$$\frac{n\sqrt{1+\sin^2 n}}{3^n} \leq \sqrt{2}\,\frac{n}{3^n} \quad (*)$$

$$\lim_{n\to\infty} \frac{(n+1)/3^{n+1}}{n/3^n} = \lim_{n\to\infty} \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n}$$

$$= \lim_{n\to\infty} \frac{3^n}{3^{n+1}} = \frac{1}{3}$$

< 1

$\Rightarrow$ la serie converge

Dalla (*), e dal criterio del confronto, la serie iniziale converge.

$$II \sum_{n=1}^{\infty} \left(\frac{n}{2} \sin \frac{1}{n} \right)^{\frac{n^2+1}{n+2}}$$

esponente!