Lezione del 28/10/2022

$f(x,y,z)$ $\qquad$ $f_x(x,y,z)$, $f_y(x,y,z)$, $f_z(x,y,z)$

$$D^2 f = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

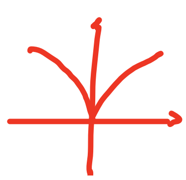
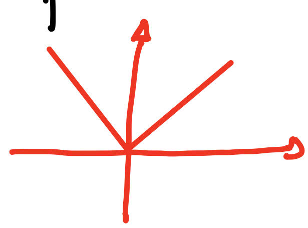
$(*)$ $\quad f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \quad f_{yz} = f_{zy}$

Teorema di Schwartz $\qquad$ $f_{xy}, f_{yx}, f_{xz}, f_{zx}, f_{yz}, f_{zy}$ continue

$\Rightarrow$ vale la $(*)$.

$f = f(x)$, derivabile in $x_0 \in I$ $\Rightarrow$ f è continua in x_0

$f: I \to \mathbb{R}$

 $\not\Leftarrow$ 

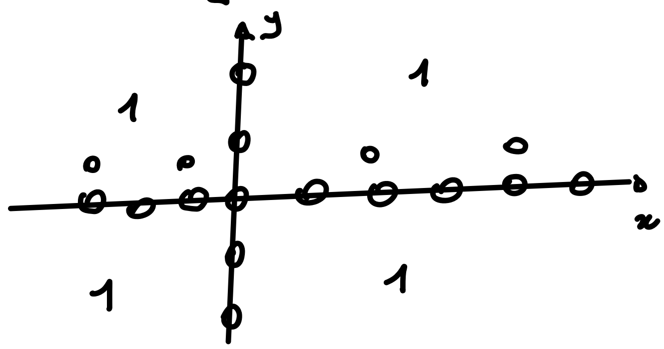
$f(x,y)$ derivabile in $(x_0,y_0) \in A \subseteq \mathbb{R}^2$ aperto

$$\not\Downarrow$$

$f(x,y)$ è continua in (x_0,y_0) ?

<u>Esempio</u>

$$f(x,y)=\begin{cases} 0 & \text{se } xy=0 \\ 1 & \text{se } xy\neq 0 \end{cases}$$



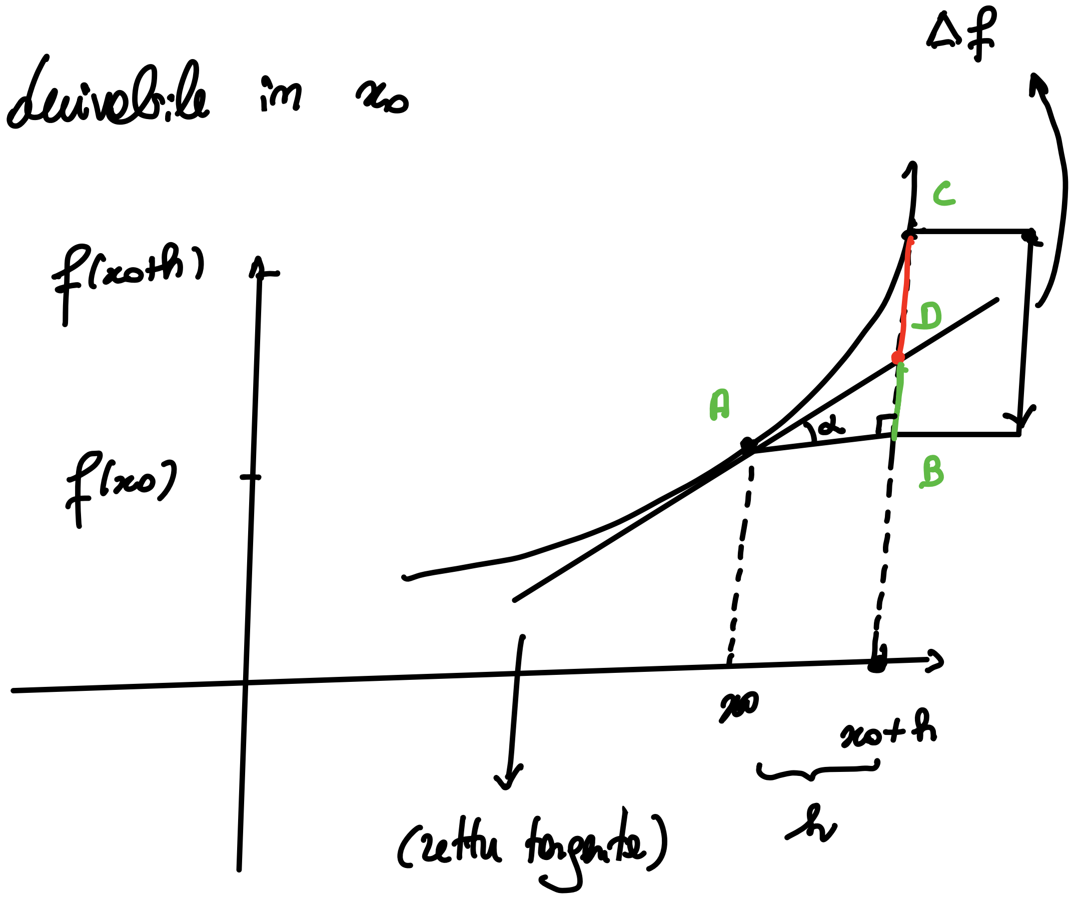
f <u>non</u> è continua, in particolare in $(0,0)$

$$\frac{\overbrace{f(h,0)}^{=0} - \overbrace{f(0,0)}^{=0}}{h} = 0 \quad : \quad f_x(0,0)=0$$

$$\frac{f(0,h) - f(0,0)}{h} = 0 \quad : \quad f_y(0,0)=0$$

<u>Differenziabilità</u>

$f = f(x)$ derivabile in x_0



$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$\Delta f = f(x_0+h) - f(x_0) = \overline{BD} + \overline{CD}$$

$$= \overline{AB}\, tg\alpha + \overline{CD}$$

$$\boxed{df(x_0)\, h = f'(x_0)\, h}$$

differenziale di f in x_0

$$= h\, tg\alpha + \overline{CD}$$

$$= f'(x_0) h + \overline{CD}$$

$$(*) \quad \Delta f = df(x_0)\, h + \underbrace{\overline{CD}}$$

$\overline{CD}$ infinitesimo per $h \to 0$

$= o(h)$

/

$$\overline{CD} = \Delta f - f'(x_0)h = f(x_0+h) - f(x_0) - f'(x_0)h$$

$$\frac{\overline{CD}}{h} = \frac{f(x_0+h)-f(x_0)}{h} - f'(x_0)$$

<u>Per</u> $h \to 0$ $\quad \frac{\overline{CD}}{h} \to 0$, quindi $\overline{CD} = o(h)$

(*)

$$\Delta f = df(x_0)h + o(h), \quad \text{per } h \to 0$$

$$h = x - x_0 \quad \Leftrightarrow \left[\lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{h} = 0 \right]$$

$$(*) \Leftrightarrow \quad f(x) - f(x_0) = f'(x_0)(x-x_0) + o(x-x_0)$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$$

<u>Def.</u> (Vettore gradiente) $\quad f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$

derivabile in $(x_0, y_0) \in A$: si dice <u>gradiente</u> di f,

e si indica con

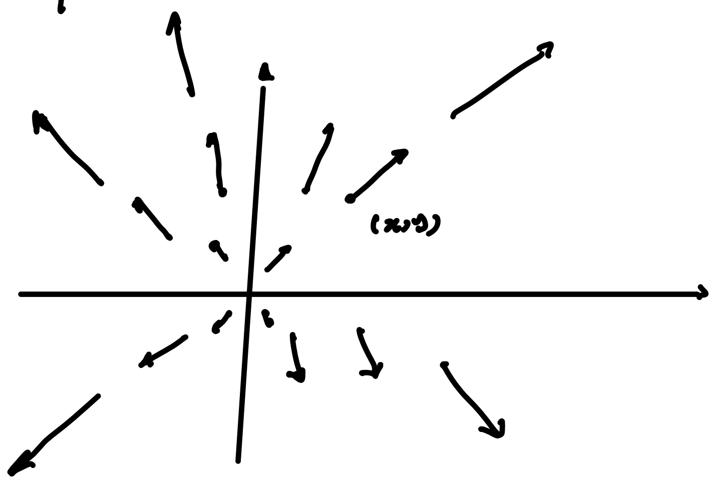
$$\nabla f(x_0, y_0) = \left(f_x(x_0,y_0), f_y(x_0,y_0) \right)$$

"nabla"

<u>Es.</u> $\quad f(x,y) = \frac{x^2}{4} + \frac{y^2}{4} \qquad f_x = \frac{x}{2}$

$g(x,y) = -\left(\frac{x^2}{4} + \frac{y^2}{4}\right) \quad f_y = \frac{y}{2} \quad \nabla g = -\frac{1}{2}(x,y)$

$\nabla f(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right) = \frac{1}{2}(x,y)$



Def. $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - \overbrace{f'(x_0)h}^{df(x_0)h}}{h} = 0$

$f(x,y)$ $\quad f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ derivabile in $(x_0,y_0) \in A$:
$(\exists \nabla f(x_0,y_0))$

Si dice che f è differenziabile in (x_0,y_0) se

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \nabla f(x_0,y_0) \cdot (h,k)}{\sqrt{h^2+k^2}} = 0$$

(h,k)

$$\underbrace{df(x_0,y_0)(h,k)}_{\text{differenziale di } f(x,y) \text{ in } (x_0,y_0)} = \nabla f(x_0,y_0) \cdot (h,k)$$

$$= f_x(x_0,y_0)h + f_y(x_0,y_0)k$$

$$\Delta f = f(x_0+h, y_0+k) - f(x_0,y_0)$$

$$(x_0,y_0) \xrightarrow{(h,k)} (x_0+h, y_0+k)$$

$$\lim_{(h,k)\to(0,0)} \frac{\Delta f - df(x_0,y_0)(h,k)}{\sqrt{h^2+k^2}} = 0$$

$$\Delta f - df(x_0,y_0)(h,k) = o\left(\sqrt{h^2+k^2}\right)$$

$$\Delta f = df(x_0,y_0)(h,k) + o\left(\sqrt{h^2+k^2}\right) \quad (*)$$

$$\Delta f \approx df(x_0,y_0)(h,k) \qquad (h,k)\to(0,0)$$

$$y=f(x) \; : \quad y = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{\text{eq. retta tangente}}$$

Riscriviamo la $(*)$ ponendo $h = x-x_0$, $k = y-y_0$

$$f(x,y) - f(x_0,y_0) =$$

$$= f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$+\, o\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$$

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + o\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right) \quad \begin{matrix} x \to x_0 \\ y \to y_0 \end{matrix}$$

1) $$z = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

equazione del piano tangente al grafico di f in $(x_0, y_0, f(x_0,y_0))$

$$\begin{cases} x_0 = 3 \\ y_0 = 1 \end{cases}$$

$f(x,y) = 3x^2 + 4xy + 5y$ piano tangente in $(3,1) =: (x_0,y_0)$

$$f(3,1) = 3\cdot 9 + 4\cdot 3 + 5 = 27 + 12 + 5 = 44$$

$$f_x = 6x + 4y \qquad f_x(3,1) \underset{\substack{x=3 \\ y=1}}{=} 6\cdot 3 + 4 = 22$$

$$f_y = 4x + 5 \qquad f_y(3,1) = 4\cdot 3 + 5 = 17$$

$$\Rightarrow \quad z = 44 + 22(x-3) + 17(y-1)$$

Eq. $f(x,y) = \operatorname{arctg}(x+2y)$ $\qquad (x_0,y_0) = (1,0)$

$$f(1,0) = \operatorname{arctg} 1 = \frac{\pi}{4}$$

$$f_x = \frac{1}{1+(x+2y)^2} \cdot 1 \quad , \quad f_x(1,0) = \frac{1}{2}$$

$$f_y = \frac{1}{1+(x+2y)^2} \cdot 2 \quad , \quad f_y(1,0) = \frac{2}{2} = 1$$

$$z = \frac{\pi}{4} + \frac{1}{2}(x-1) + y$$

$$f(x,y) = \frac{1}{(x^2+y^2)^2} \quad , \quad (x_0,y_0) = (\sqrt{2}, 0)$$

Prop. Se $f(x,y)$ è differenziabile in (x_0,y_0), allora essa è continua in (x_0,y_0).

Dim. Vogliamo dimostrare che f è continua in (x_0,y_0)

cioè
$$\lim_{(h,k)\to(0,0)} f(x_0+h, y_0+k) = f(x_0,y_0)$$

$$f(x,y) \xrightarrow[\substack{x\to x_0 \\ y\to y_0}]{} f(x_0,y_0)$$

$$|f(x_0+h, y_0+k) - f(x_0,y_0)| = |\nabla f(x_0,y_0)\cdot(h,k) + o(\sqrt{h^2+k^2})| \quad ②$$

$$f(x_0+h, y_0+k) - f(x_0,y_0) = \nabla f(x_0,y_0)\cdot(h,k) + o(\sqrt{h^2+k^2})$$

$$\leq \left| \nabla f(x_0,y_0) \cdot (h,k) \right| + o\left(\sqrt{h^2+k^2}\right) \leq$$

$\left(|\underline{x} \cdot \underline{y}| \leq \|x\| \, \|y\| \right.$ dis. Cauchy - Schwartz $\left. \right)$

$$\leq \|\nabla f(x_0,y_0)\| \, \underbrace{\|(h,k)\|}_{\sqrt{h^2+k^2}} + o\left(\sqrt{h^2+k^2}\right)$$

<u>Quindi</u> :

$$|f(x_0+h, y_0+k) - f(x_0,y_0)| \leq$$

$$\leq \|\nabla f(x_0,y_0)\| \, \underbrace{\|(h,k)\|}_{\sqrt{h^2+k^2}} + o\left(\sqrt{h^2+k^2}\right) \xrightarrow[(h,k)\to(0,0)]{} 0$$

$$\Rightarrow \left[\lim_{(h,k)\to(0,0)} |f(x_0+h, y_0+k) - f(x_0,y_0)| = 0 \right]$$

Ma allora $\displaystyle\lim_{(h,k)\to(0,0)} f(x_0+h, y_0+k) = f(x_0,y_0)$.

<u>Teorema del differenziale</u> $f(x,y)$, $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^2$ aperto, f derivabile in A. <u>Se</u> f_x, f_y sono continue

in $(x_0,y_0) \in A$, allora f è differenziabile in (x_0,y_0).

Def "f è di classe C^1" $\quad f \in C^1(A)$

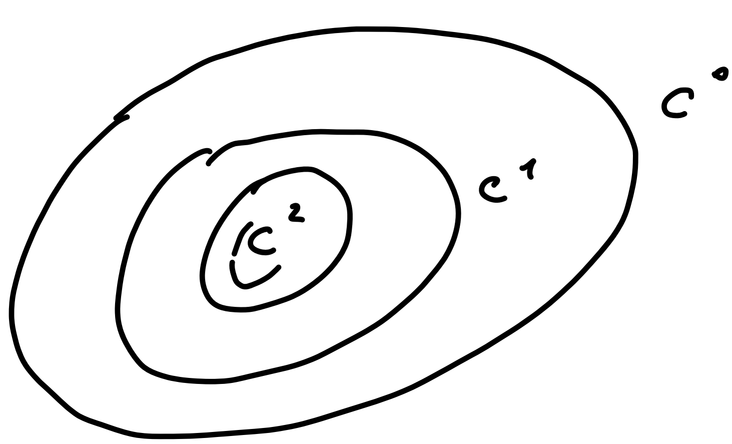
$$C^1(A) = \{ f: A \to \mathbb{R} \text{ derivabili}, \ f_x, f_y \text{ continue in } A \}$$

$$C^0(A) = \{ f: A \to \mathbb{R} \text{ continue in } A \}$$

$f \in C^1(A) \Rightarrow f_x, f_y$ continue $\Rightarrow f$ è differenziabile in $A \Rightarrow f$ continua $\Rightarrow f \in C^0(A)$

$$C^2(A) = \{ f \text{ derivabili due volte in } A, \text{ con derivate seconde continue in } A \}$$

$f \in C^2(A) \Rightarrow f_x, f_y$ continue in $A \Rightarrow f \in C^1(A)$
$\Rightarrow f \in C^0(A)$



$$f \in C^k(A) \Rightarrow f \in C^{k-1}(A)$$

$$C^\infty(A) = \left\{ f \text{ derivabili infinite volte con tutte le derivate continue in } A \right\}$$