Lezione del 28/11/2022

$$y'' + a y' + b y = f(x)$$

$f(x) = \cos x, \quad \sin x$

$$\underbrace{y'' + y = \cos x}\ ?$$

$y'' + y = 0 \qquad \lambda^2 + 1 = 0 \qquad \lambda = \pm i$

$\alpha = \operatorname{Re}\lambda = 0$

$\beta = \operatorname{Im}\lambda = 1$

$y = c_1 \cos x + c_2 \sin x \quad , c_1, c_2 \in \mathbb{R}$

polinomi

$\rightarrow f(x) = e^{\lambda x}\left[P_m(x)\cos(\mu x) + Q_k(x)\sin(\mu x)\right]$

$\lambda \in \mathbb{R} \qquad \mu \in \mathbb{R}$

$f(x) = \cos x = e^{0\cdot x}\left[1\cdot\cos(1\cdot x) + 0\cdot\sin(1\cdot x)\right]$

$\lambda = 0$

$P_m = 1$

$Q_k = 0$

$\mu = 1$

$$f(x) = e^{-x}\left[\underset{\smile}{x}\cos(2x) - x^2\sin(2x)\right]$$

$\lambda = -1$ $\qquad \downarrow \qquad \downarrow$ $\qquad \mu = 2$

$p_m(x) \qquad r_k = -x^2$

1a) $P(\lambda \pm i\mu) \neq 0$

$$y'' + y' = \cos x = e^{\overset{\lambda}{\overset{\shortparallel}{0}}\cdot x}\left[1\cdot\cos(\overset{\mu}{\overset{\shortparallel}{1}}\cdot x) + 0\,\sin(1\cdot x)\right]$$

$$\lambda^2 + \lambda = 0 \iff \lambda = -1,\ \lambda = 0$$

$$\lambda \pm i\mu = \pm i \ : \ \bar{y} = \underbrace{a\cos x + b\sin x}$$

Allora: un integrale particolare è della forma

$$\bar{y} = e^{\lambda x}\left[\underset{\sim}{q_{\bar{m}}}(x)\cos(\mu x) + \underset{\sim}{S_{\bar{m}}}(x)\sin(\mu x)\right]$$

polinomi di grado

$\bar{m} = \max\{m, k\}$

$$\bar{y}' = -a\sin x + b\cos x$$

$$\bar{y}'' = -a\cos x - b\sin x$$

inseriamo nell' equazione completa: $y'' + y' = \cos x$

$$-\underbrace{a\cos x} - b\sin x - a\sin x + \underbrace{b\cos x} = \cos x$$

$$(b-a)\cos x - (b+a)\sin x = \cos x$$

$$\begin{cases} b - a = 1 \\ \underline{b+a} = 0 \end{cases} \iff \begin{cases} -2a = 1 \iff a = -\frac{1}{2} \\ b = -a \iff b = \frac{1}{2} \end{cases}$$

$$\bar{y} = -\frac{1}{2}\cos x + \frac{1}{2}\sin x$$

$$y'' + y' = 0$$

$$\lambda^2 + \lambda = 0 \qquad \lambda = -1, \ \lambda = 0$$

Int. generale: $y = c_1 e^{-x} + c_2$

Int. generale completa:

$$y = c_1 e^{-x} + c_2 + \frac{1}{2}(\sin x - \cos x)$$

$1_B)$ $P(\lambda \pm i\mu) = 0$.

Un integrale particolare è della forma

$$\bar{y} = e^{\lambda x} \cdot x \left(q_{\bar{m}}(x) \cos \mu x + s_{\bar{m}}(x) \sin \mu x \right)$$

<u>ES.</u> $y'' + y = \sin x$

$y'' + y = 0$

$\lambda^2 + 1 = 0 \quad : \quad \lambda = \pm i$

$f(x) = \sin x = e^{0 \cdot x} [0 \cdot \cos(1 \cdot x) +$
$+ 1 \cdot \sin(1 \cdot x)]$

$\lambda = 0 \qquad \mu = 1$

$P_m = 0 \ , \ r_k = 1$

$\lambda \pm i\mu = \pm i$ è radice della caratteristica!

$\bar{y} = 1 \cdot x \cdot [a \cos x + b \sin x]$
$= x [a \cos x + b \sin x]$

$\bar{y} = -\frac{1}{2} x \cos x$

$$y' = a\cos x + b\sin x + x\left[-a\sin x + b\cos x\right]$$

$$y'' = -a\sin x + b\cos x - a\sin x + b\cos x + x\left[-a\cos x - b\sin x\right]$$

$$= -2a\sin x + 2b\cos x - x\left[a\cos x + b\sin x\right]$$

$$-2a\sin x + 2b\cos x \cancel{- x\left[a\cos x + b\sin x\right]} \cancel{+ x\left[a\cos x + b\sin x\right]} = \sin x$$

$$\begin{cases} -2a = 1 \\ 2b = 0 \end{cases} \iff \begin{cases} a = -\frac{1}{2} \\ b = 0 \end{cases}$$

L'integrale generale della completa è

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$$

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases} \quad ?$$

$$0 = y(0) = c_1 \quad . \quad c_1 = 0$$

$$y' = -c_1 \sin x + c_2 \cos x$$

$$- \frac{1}{2} \cos x + \frac{1}{2} x \sin x$$

$$- \frac{1}{2} \cos x + \frac{1}{2}$$

$$y'(0) = c_2 - \frac{1}{2} = 1$$

$$\Rightarrow \quad c_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$y = \frac{3}{2} \sin x - \frac{1}{2} x \cos x$$

<u>**ES.**</u> $y'' + y = 2x \sin x$

$y'' + y = 0 \qquad \lambda^2 + 1 = 0 \Leftrightarrow \lambda = \pm i$

$$f(x) = e^{0 \cdot x} \left[0 \cdot \cos(1 \cdot x) + \right.$$
$$\left. + 2x \sin(1 \cdot x) \right]$$

$$\lambda = 0 \quad , \quad \mu = 1$$

$$P_m(x) = 0 \, , \quad r_n(x) = 2x$$

$\lambda \pm i\mu = \pm i$ è radice!

$$\bar{y} = x\left[(ax+b)\cos x + (cx+d)\sin x\right]$$

... calcolando le derivate ...

e inserendo nell'equazione, ordinando in seno e coseno, si ha:

$$(2a - 4cx - 2d)\sin x +$$

$$+ (4ax + 2b + 2c)\cos x =$$

$$= 2x \sin x$$

$$\begin{cases} 2a - 4cx - 2d = 2x \\ 4ax + 2b + 2c = 0 \end{cases}$$

$$\begin{cases} -4cx + (2a - 2d) = 2x \\ 4ax + (2b + 2c) = 0 \end{cases}$$

Principio di identità:

$$\begin{cases} -4c = 2 \\ 2a - 2d = 0 \\ 4a = 0 \\ 2b + 2c = 0 \end{cases}$$

$$\begin{cases} c = -\frac{1}{2} \\ a = d \\ a = 0 \\ b = -c \end{cases} \Leftrightarrow \begin{cases} a = 0 \\ b = \frac{1}{2} \\ c = -\frac{1}{2} \\ d = 0 \end{cases}$$

$$\bar{y} = x\left[(ax+b)\cos x + (cx+d)\sin x\right] =$$

$$= x\left[\frac{1}{2}\cos x - \frac{1}{2}\sin x\right]$$

$$y'' - 2y' = x - \cos x \quad \odot$$

$$y'' - 2y' = 0 \qquad \lambda^2 - 2\lambda = 0$$

$$\lambda = 0, \ \lambda = 2 \quad \Delta > 0$$

$$\updownarrow$$

$$y = c_1 + c_2 e^{2x} \quad , c_1, c_2 \in \mathbb{R}$$

Per la $\odot$:

$$y'' - 2y' = x \longrightarrow \bar{y}$$

$$y'' - 2y' = -\cos x \longrightarrow \bar{\bar{y}}$$

Un integrale particolare di $\odot$ sarà

$$y_0 = \bar{y} + \bar{\bar{y}}$$

$$y'' - 2y' = x = e^{0\cdot x} \underbrace{x}_{p(x)} \quad (1)$$

$$\lambda = 0$$

$\lambda = 0$ è radice di molt. 1

$$\bar{y} = x e^{0\cdot x} [ax + b]$$

$$= ax^2 + bx$$

$$\bar{y}' = 2ax + b$$

: inserendo in (1)

$$\bar{y}'' = 2a$$

$$2a - 4ax - 2b = x$$

$$-4ax + (2a - 2b) = x$$

$$\begin{cases} -4a = 1 \\ a = b \end{cases} \iff \begin{cases} a = -\frac{1}{4} \\ b = -\frac{1}{4} \end{cases}$$

$$\bar{y} = -\frac{x}{4}\left[x + 1\right]$$

$$y'' - 2y' = -\cos x \quad =$$

$$= e^{0\cdot x}\left[(-1)\cos(1\cdot x) + 0\cdot\sin(1\cdot x)\right]$$

$$\lambda = 0 \qquad \mu = 1 \qquad P_m = -\underline{1}$$

$$R_n = 0$$

$\lambda \pm i\mu = \pm i$ <u>NON</u> $\bar{\in}$ radice

della caratteristica

$$\bar{\bar{y}} = a\cos x + b\sin x$$

$$\bar{\bar{y}}' = -a\sin x + b\cos x$$

$$\bar{\bar{y}}'' = -a\cos x - b\sin x$$!

inserendolo in

$$y'' - 2y' = -\cos x$$

$$-a\cos x - b\sin x + 2a\sin x$$

$$-2b\cos x = -\cos x$$

$$-(a+2b)\cos x + (2a-b)\sin x$$

$$= -\cos x$$

$$\begin{cases} a+2b=1 \\ 2a-b=0 \end{cases} \qquad \begin{cases} 5a=1 \\ b=2a \end{cases}$$

$$\begin{cases} a=1/5 \\ b=2/5 \end{cases}$$

$$\bar{\bar{y}} = \frac{1}{5} \cos x + \frac{2}{5} \sin x$$

Un integrale particolare di

$$y'' - 2y' = x - \cos x \quad \text{è}$$

$$y_0 = \bar{y} + \bar{\bar{y}} =$$

$$= -\frac{x}{4}(x+1) +$$

$$+ \frac{1}{5} \cos x + \frac{2}{5} \sin x$$

L'int. generale della completa

$$y = c_1 + c_2 e^{2x} + y_0$$

Metodo di variazione delle costanti (Lagrange)

$$y'' + a(x)\,y' + b(x)\,y = f(x)$$

$$y'' + a(x)\,y' + b(x)\,y = 0$$

y_1, y_2 int. lin. indipendenti

Int. generale dell'omogenea è

$$y = c_1\,y_1(x) + c_2\,y_2(x)$$

$\forall c_1, c_2 \in \mathbb{R}$

Un integrale particolare della completa è

$$\bar{y} = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$$

dove $c_1'(x)$, $c_2'(x)$ risolvono

$$\begin{cases} c_1'(x)\, y_1(x) + c_2'(x)\, y_2(x) = 0 \\ c_1'(x)\, y_1'(x) + c_2'(x)\, y_2'(x) = f(x) \end{cases}$$

Dalla regola di Cramer:

$$C_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ f(x) & y_2'(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} \qquad \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = W(x) \neq 0$$

$$C_2'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & f(x) \end{vmatrix}}{W(x)}$$

$$y'' + y = \frac{1}{\sin x},$$

$$y'' + y = 0 \qquad \lambda^2 + 1 = 0, \quad \lambda = \pm i$$

$$y = C_1 \cos x + C_2 \sin x$$

$$\underbrace{}_{y_1} \qquad \underbrace{}_{y_2}$$

$$\bar{y} = c_1(x)\cos x + c_2(x)\sin x \ : \ c_1', c_2'$$

risolviamo

$$\begin{cases} c_1'(x)\underbrace{\cos x} + c_2'(x)\underbrace{\sin x} = 0 \\ -c_1'(x)\sin x + c_2'(x)\cos x = \frac{1}{\sin x} \end{cases}$$

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$c_1' = \begin{vmatrix} 0 & \sin x \\ \frac{1}{\sin x} & \cos x \end{vmatrix} = -1$$

Integrando, $\boxed{c_1(x) = \underbrace{-x}}$

$$C_2' = \begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{1}{\sin x} \end{vmatrix} = \frac{\cos x}{\sin x}$$

$$C_2(x) = \int \frac{\cos x}{\sin x}\,dx = \log|\sin x|$$

y_2

$$\bar{y} = C_1(x)\cos x + C_2(x)\sin x =$$

$$= -x\cos x + \left(\log|\sin x|\right)\sin x$$

Int. generale:

$$y = C_1\cos x + C_2\sin x$$
$$-x\cos x + \left(\log|\sin x|\right)\sin x$$

$$\begin{cases} y(\pi/2) = 0 \\ y'(\pi/2) = 2 \end{cases} \quad ?$$

$$y'' + 2y' + y = \frac{\log x}{e^x}$$