Lezione del 26/10/2022

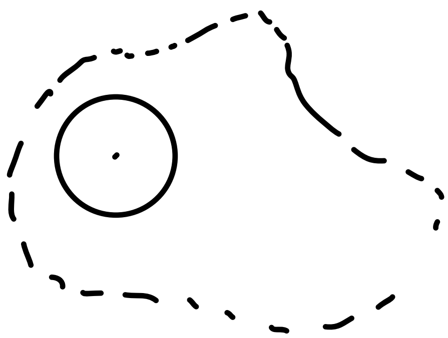
Derivate parziali

$f = f(x)$

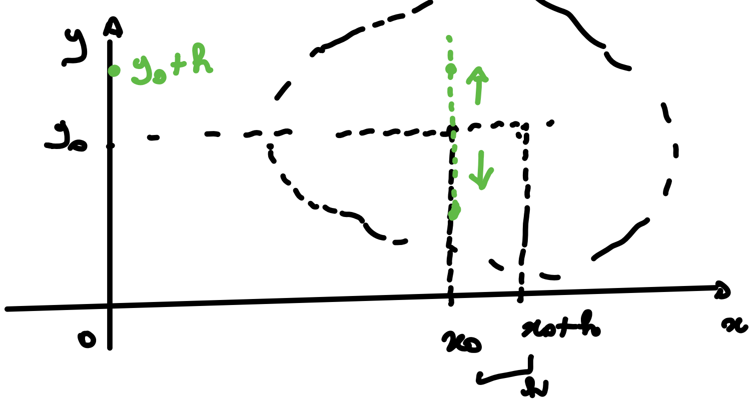
$f = f(x,y)$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$A \subseteq \mathbb{R}^2$ aperto $\qquad f : A \to \mathbb{R}$



Sia $(x_0, y_0) \in A$



Se $\exists \displaystyle\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$ finito,

f si dice derivabile parzialmente rispetto ad x, nel punto (x_0, y_0) e

$$\frac{\partial f}{\partial x}(x_0,y_0) = f_x(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

$\|$ si dice derivata parziale di f rispetto a x, nel punto (x_0,y_0)

$\cancel{f'_x}$ ~~No~~ No !!

f'

Se $\exists \displaystyle\lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$ finito,

f si dirà derivabile parzialmente rispetto ad y, in (x_0,y_0)

Derivata parziale di f rispetto ad y, in (x_0,y_0)

$$\frac{\partial f}{\partial y}(x_0,y_0) = f_y(x_0,y_0) := \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

Esempi

$$f(x,y) = x^2 y$$

$$f_x(x,y) = y \frac{\partial}{\partial x}(x^2) = 2xy$$

$$f_y(x,y) = x^2 \frac{\partial}{\partial y}(y) = x^2$$

$f(x,y) = x \quad , \quad f_x = 1 \qquad f_y = 0$

$$f(x,y) = \sin(xy) \qquad (g \circ f)'(x) =$$

$$= g'(f(x)) \cdot f'(x)$$

$$f_x = \cos(xy) \cdot y$$

$$f_y = \cos(xy) \cdot x$$

$$f(x,y) = tg^2(x^2 - y^2) \qquad D\,tg\,t = \frac{1}{\cos^2 t}$$

$$f_x = 2\,tg(x^2-y^2) \cdot \frac{1}{\cos^2(x^2-y^2)} \cdot 2x$$

$$f_y = 2\,tg(x^2-y^2) \cdot \frac{1}{\cos^2(x^2-y^2)} \cdot (-2y)$$

$$f(x,y) = x \cos\left(\frac{x}{y}\right)$$

$$f_x = \cos\left(\frac{x}{y}\right) + x\left[-\sin\left(\frac{x}{y}\right) \cdot \frac{1}{y}\right]$$

$$f_y = x\left[-\sin\left(\frac{x}{y}\right) \cdot x \cdot \left(-\frac{1}{y^2}\right)\right]$$

$$= \frac{x^2}{y^2} \sin\left(\frac{x}{y}\right)$$

$$f(x,y) = \log(x^4 - y^2) \qquad D\log t = \frac{1}{t}$$

$$f_x = \frac{1}{x^4 - y^2} \cdot (4x^3) \qquad f_y = \frac{1}{x^4-y^2}(-2y)$$

$$f(x,y) = \frac{x-y}{x^2+y^2} \qquad f_x = \frac{x^2+y^2-(x-y)\cdot 2x}{(x^2+y^2)^2}$$

$$D\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

$$f_y = \frac{-(x^2+y^2) - (x-y)\cdot 2y}{(x^2+y^2)^2}$$

$$f(x,y) = e^{x^2/y}$$

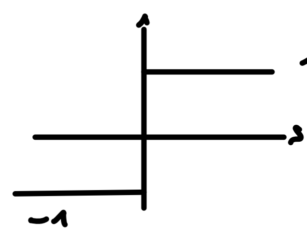
$$f_x = e^{x^2/y} \cdot \left[\frac{2x}{y}\right]$$

$$f_y = e^{x^2/y}\left[-\frac{x^2}{y^2}\right]$$

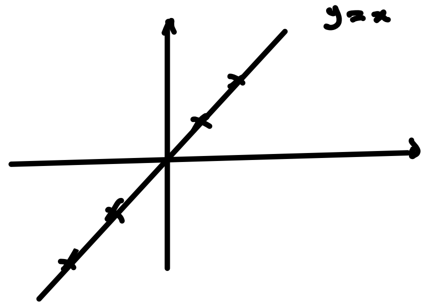
$$f(x,y) = a^{x^2/y} \qquad f_x = a^{x^2/y} \log a \left[\frac{2x}{y}\right]$$

$$f_y = \quad '' \quad '' \quad \left[-\frac{x^2}{y^2}\right]$$

$$D|t| = \begin{cases} 1 & \text{se } t>0 \\ -1 & \text{se } t<0 \end{cases} = \frac{t}{|t|}$$



$$f(x,y) = |x-y| \qquad f_x = \frac{x-y}{|x-y|}\cdot 1$$



$$f_y = \frac{x-y}{|x-y|}(-1) = \frac{y-x}{|x-y|}$$

$y = x$

(x,x)
$=$

$$\frac{f(x+h,\, x) - f(x,x)}{h} =$$

$f = |x-y|$

$$= \frac{|x+h-x| - |x-x|}{h} = \frac{|h|}{h}$$

$$\lim_{h\to 0} \text{---} = \lim_{h\to 0} \frac{|h|}{h} \quad \underline{\text{non esiste}}$$

$$f(x,y,z) = \log(x^2 y + z^3)$$

$$f_x = \frac{1}{x^2 y + z^3} \cdot 2xy$$

$$f_y = \frac{1}{x^2 y + z^3} \cdot x^2$$

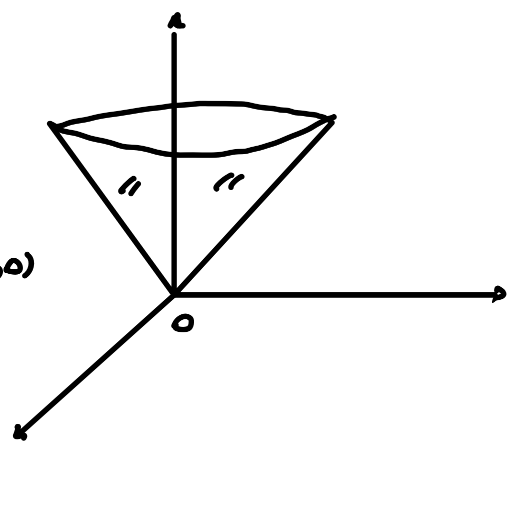
$$f_z = \frac{1}{x^2 y + z^3} \cdot 3z^2$$

$$f(x,y) = \sqrt{x^2+y^2} \qquad D = \mathbb{R}^2 \qquad D\sqrt{t} = \frac{1}{2\sqrt{t}}$$

$$z = \sqrt{x^2+y^2}$$

$$f_x = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x$$

$O = (0,0)$



$$f_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt{h^2}}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

$f(0,0)=0$

non esiste

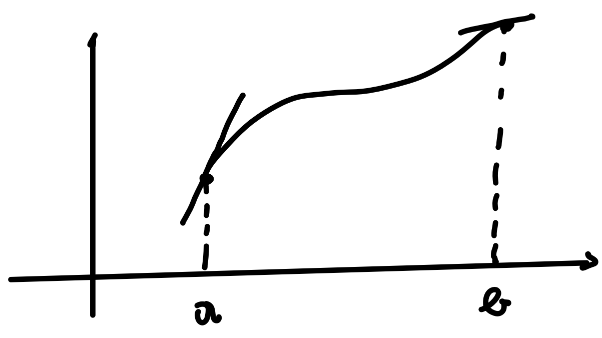
Def Se $f: A \to \mathbb{R}$ è derivabile rispetto ad x, y in $(x_0,y_0) \in A$, si dice che f è derivabile in (x_0,y_0).

Si dice che f è derivabile in A, se è derivabile in ogni punto di A.

$f_x = f_x(x,y) \qquad f_x : A \to \mathbb{R}$

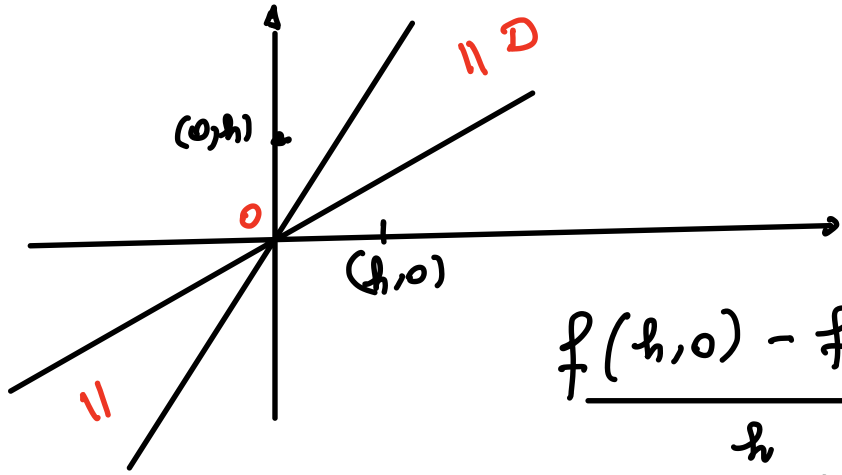
$f_y = f_y(x,y) \qquad f_y : A \to \mathbb{R}$

$f = f(x)$



$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

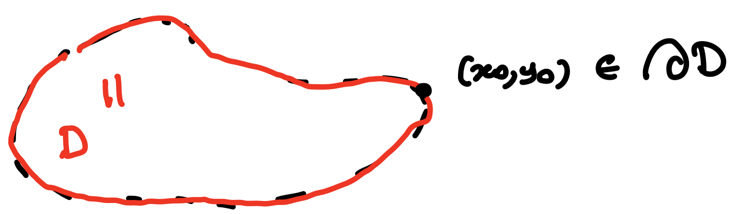
$$f'_-(b) = \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$$



$$\frac{f(h,0)-f(0,0)}{h}$$

$$\frac{f(0,h)-f(0,0)}{h}$$

D dominio di $\mathbb{R}^2$, ossia $D = \bar{A}$, A aperto limitato



$f = f(x,y)$ $\quad$ $f : D \to \mathbb{R}$ derivabile in $\mathring{D}$

f_x, f_y continue in $\mathring{D}$

Se $(x_0,y_0) \in \partial D$, $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f_x(x,y)$: se

$\exists$ finito il limite, $\displaystyle f_x(x_0,y_0) = \lim_{(x,y)\to(x_0,y_0)} f_x(x,y)$

$\exists$ finito il limite $\lim_{(x,y)\to(x_0,y_0)} f_y(x,y)$

$$f_y(x_0,y_0) = \lim_{(x,y)\to(x_0,y_0)} f_y(x,y)$$

$$\lim_{x\to x_0^+} f'(x) = f'_+(x_0)$$

$$\lim_{x\to x_0^-} f'(x) = f'_-(x_0)$$

Derivate seconde

$f(x) \qquad f'(x)$

$$f''(x) = (f')'(x)$$

$f = f(x,y) \qquad f: A \longrightarrow \mathbb{R}, \quad A \subseteq \mathbb{R}^2$ aperto

derivabile in A

$\forall (x,y)\in A \quad \exists \ f_x(x,y), \ f_y(x,y)$

$f_x : A \longrightarrow \mathbb{R} \qquad f_x = f_x(x,y)$

$f_y : A \longrightarrow \mathbb{R} \qquad f_y = f_y(x,y)$

$f_{xx} = \frac{\partial}{\partial x}(f_x) \qquad \frac{\partial}{\partial y}(f_x) = f_{xy}$

$f_{yx} = \frac{\partial}{\partial x}(f_y) \qquad \frac{\partial}{\partial y}(f_y) = f_{yy}$

$$D^2 f(x,y) = \begin{pmatrix} f_{xx} = \dfrac{\partial^2 f}{\partial x^2} & f_{xy} = \dfrac{\partial^2 f}{\partial x \partial y} \\ f_{yx} = \dfrac{\partial^2 f}{\partial y \partial x} & f_{yy} = \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

matrice Hessiana di $f(x,y)$

Es.

$f(x,y) = x^2 y^2 \qquad f_x = 2xy^2 \,, \quad f_y = 2x^2 y$

$$D^2 f = \begin{pmatrix} f_{xx} = 2y^2 & f_{xy} = 4xy \\ f_{yx} = 4xy & f_{yy} = 2x^2 \end{pmatrix}$$

$$f_{xy} = f_{yx}$$

Teorema di Schwartz $\quad f: A \to \mathbb{R}$ derivabile due volte in A ($\exists$ tutte le derivate seconde in ogni punto di A).

Se $(x_0,y_0) \in A$ e f_{xy}, f_{yx} sono continue in A

allora

$$f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0) \qquad]$$