Lezione del 21/09/2022

ELEMENTI DI ANALISI MATEMATICA II

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Liguori editore

ESERCIZI DI ANALISI MATEMATICA II

(STESSI AUTORI) VOL. I - II

ES. DI ANALISI MATEMATICA II

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1) Serie numeriche $\{a_n\}_{n \in \mathbb{N}}$ $a_n \in \mathbb{R}$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \qquad \sum_{n=1}^{\infty} \frac{1}{n}$$

Serie di funzioni $\sum_{n=1}^{\infty} f_n(x)$ $f_1(x), f_2(x), \dots, f_n(x), \dots$ funzioni

$$\sum_{n=1}^{\infty} x^n \qquad f_n(x) = x^n$$

2) Funzioni di più variabili

$$f(x), \quad x \in I \subseteq \mathbb{R}$$

$$f(x, y) \qquad (x, y) \in X \subseteq \mathbb{R}^2$$

differenziabilità, derivabilità, etc...

3) Equazioni differenziali $\qquad y''(x) + y(x) = 0$

4) Calcolo integrale $\qquad \int_a^b f(x)dx \quad ?$

$$\iint_D f(x,y)\,dx\,dy \quad ?$$

5) Probabilità ?

$\mathbb{R}$

<u>Serie numeriche</u> $\qquad a_1 \quad a_2 \quad a_3 \quad \dots \quad a_n$

$$a_1 + a_2 + a_3 + \dots + a_n \in \mathbb{R}$$

$a_n = 1/n \qquad 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots \longrightarrow 0$

$\frac{1}{2} > \frac{1}{3} > \dots > \frac{1}{n} > \frac{1}{n+1} \dots$ succ. decrescente

$a_n \geq a_{n+1}$ (decrescente)

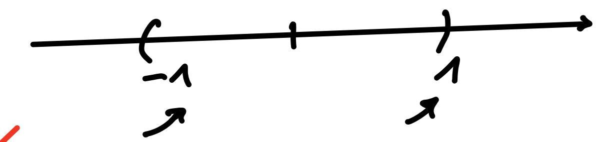
$a_n \le a_{n+1}$ (crescente)

$a_n = n \qquad 1, 2, 3, \ldots, n, \ldots$

Successioni oscillanti : non ammettono limite

$\nexists\, \ell : \quad a_n \to \ell$

$a_n = (-1)^n$

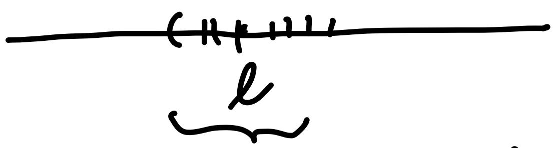


$\{a_n\}$ convergente $\overset{\not\Leftarrow}{\Rightarrow}$ $\{a_n\}$ limitata

$\exists\, \ell \in \mathbb{R} : \ a_n \xrightarrow[n\to\infty]{} \ell$

$\Leftrightarrow \forall \varepsilon > 0 \ \exists\, \nu \in \mathbb{N} : \ n > \nu \Rightarrow |a_n - \ell| < \varepsilon$

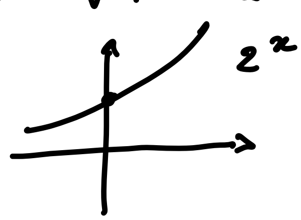
$\ell - \varepsilon < a_n < \ell + \varepsilon$



$(-1)^n$ succ. limitata : $|(-1)^n| = 1$

$\lim_{n\to\infty} a_n = +\infty \Leftrightarrow \forall M > 0 \ \exists\, \nu \in \mathbb{N} : n > \nu \Rightarrow a_n > M$

$\lim_{n\to\infty} 2^n = +\infty$



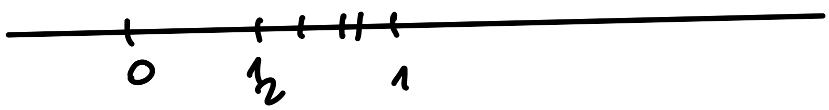
$$\lim_{n\to\infty} a_n = -\infty \iff$$

$$\forall M>0 \ \exists \nu \in \mathbb{N} : n>\nu \Rightarrow a_n < -M$$

$\{a_n\}$ crescente : $\displaystyle\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n$

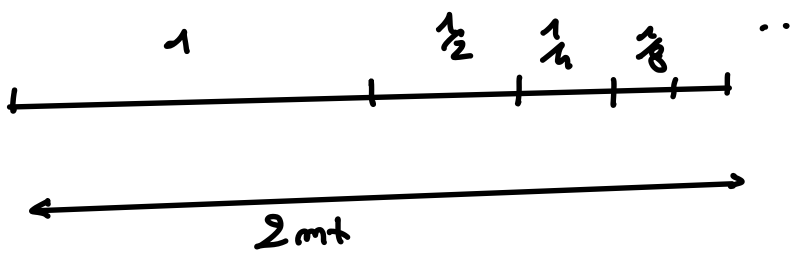
$a_n = \dfrac{n}{n+1}$: $\displaystyle\lim_{n\to\infty} a_n = 1 = \sup_{n\in\mathbb{N}} a_n$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \dots, \frac{n}{n+1} \longrightarrow 1$$



$\{a_n\}$ decrescente $\Rightarrow \displaystyle\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} a_n$

$a_n = 1/n$ $\qquad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \longrightarrow 0$

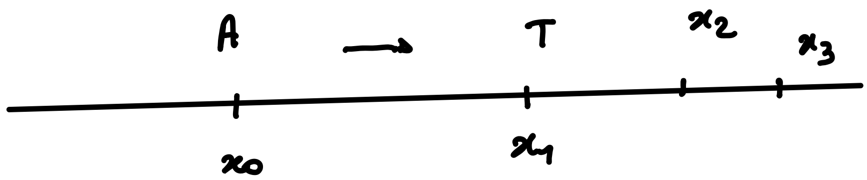


$$1 + \frac{1}{2} + \frac{1}{4} + \ + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = 2$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

Paradosso di Zenone (400 a.c)

ACHILLE E LA TARTARUGA



t_0 $\qquad x_0 < x_1$

$t_0 + t_1$ $\qquad x_1 < x_2$

$t_0 + t_1 + t_2$ $\qquad x_2 < x_3$

$\vdots$

$t_0 + t_1 + t_2 + \dots t_n + \dots = +\infty$

<u>**Def.**</u> (Serie numerica) $\quad \{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + a_3 + \dots + a_n$) somma parziale n-sima, oppure ridotta n-sima della serie

$\vdots$

$\{S_n\}_{n \in \mathbb{N}}$ è una nuova successione, che si chiama

serie di termine generale a_n

$$\sum_{n=1}^{\infty} a_n$$

$$a_n = 1/n$$

$S_1 = 1 \qquad S_2 = 1 + \frac{1}{2}, \quad S_3 = 1 + \frac{1}{2} + \frac{1}{3} \dots$

$\sum_{n=1}^{\infty} \frac{1}{n}$ serie armonica

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ serie geometrica

Si dice che $\sum_{n=1}^{\infty} a_n$ converge (diverge pos., neg.) oscillante se converge (risp. div. pos., neg., oscillante) la successione delle somme parziali $\{S_n\}$: in tal caso, il limite

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

si chiama somma della serie, e scriveremo

$$S = \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \quad \text{più avanti!}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)} + \dots$$

$S_1 = 1/2$

$S_2 = 1/2 + 1/6$

$\vdots$

$S_n = \frac{1}{2} + \frac{1}{6} + \dots + \frac{1}{n(n+1)}$

$$\frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{\cancel{n+1}}{n(\cancel{n+1})} - \frac{\cancel{n}}{\cancel{n}(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{m+1} \xrightarrow[m \to \infty]{} 1$$

$$\sum_{m=1}^{\infty} \frac{1}{m(m+1)} = 1$$

$$\sum_{m=1}^{\infty} (-1)^m = -1 + 1 - 1 + 1 \; \ldots$$

$$S_1 = -1, \quad S_2 = -1+1 = 0$$

$$S_3 = -1+1-1 = -1$$

$$\vdots$$

$$-1, 0, -1, 0, \ldots$$

Condizione necessaria per la convergenza delle serie numeriche

Se $\displaystyle\sum_{m=1}^{\infty} a_m = S \in \mathbb{R} \implies \lim_{m \to \infty} a_m = 0$

$$\not\Leftarrow$$

$$\sum_{m=0}^{\infty} \frac{1}{2^m} = 2 \qquad \frac{1}{2^m} \xrightarrow[m \to \infty]{} 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \qquad \frac{1}{n(n+1)} \xrightarrow[n\to\infty]{} 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty \quad !!$$

serie armonica

$$\frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

Dim

$$a_n = S_n - S_{n-1}$$

$$= \cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_{n-1}} + a_n$$

$$- \cancel{a_1} - \cancel{a_2} - \dots - \cancel{a_{n-1}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0$$

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = +\infty \qquad \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

$$\Downarrow$$

non converge

<u>OSS.</u> Se $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ <u>non converge</u>

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ serie armonica ARMONICA

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

$$= +\infty$$

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e \in (2,3)$$

f.i. $1^{+\infty}$

$$\left(1 + \frac{1}{n}\right)^n < e \quad \forall n \in \mathbb{N}$$

$$n \log\left(1 + \frac{1}{n}\right) < \log e = 1$$

$$\log\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$

$$\log\left(\frac{n+1}{n}\right) < \frac{1}{n}$$

$$\parallel$$

$$\log(n+1) - \log n < \frac{1}{n} \quad \forall n \in \mathbb{N}$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} > \cancel{\log 2} - \overset{=0}{\log 1} +$$

$$\sum \frac{1}{n} \qquad + \cancel{\log 3} - \cancel{\log 2}$$

$$+ \log 4 - \cancel{\log 3}$$

$$+ \dots + \log(n+1) - \cancel{\log n}$$

$$= \log(n+1)$$

$$S_n > \log(n+1) \xrightarrow[n\to\infty]{} +\infty$$

Per il criterio del confronto,

$$S_n \xrightarrow[n\to\infty]{} +\infty$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

<u>Cr. confronto</u>

1) $a_n \leq c_n \leq b_n$, $a_n \to \ell$, $b_n \to \ell$ $\Rightarrow$ $c_n \to \ell$

2) $a_n \leq b_n$, $a_n \to +\infty$ $\Rightarrow$ $b_n \xrightarrow[n\to\infty]{} +\infty$

3) $a_n \leq b_n \xrightarrow[n\to\infty]{} -\infty \Rightarrow a_n \to -\infty$