Lezione del 02/11/2022

$f = f(x,y)$ differenziabile in $(x_0, y_0) \in A$ aperto

$f : A \longrightarrow \mathbb{R}$

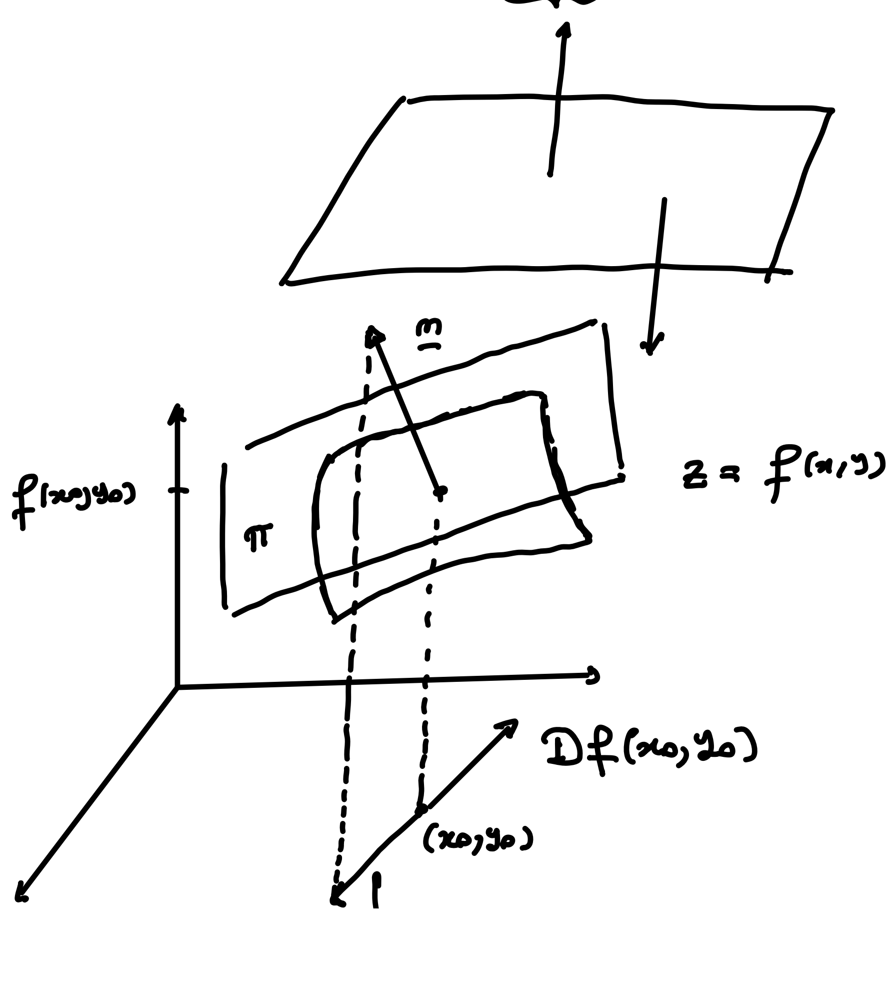
$ax + by + cz + d = 0$

$\underline{n} = (a, b, c)$

Eq. piano tangente : $1 \cdot z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

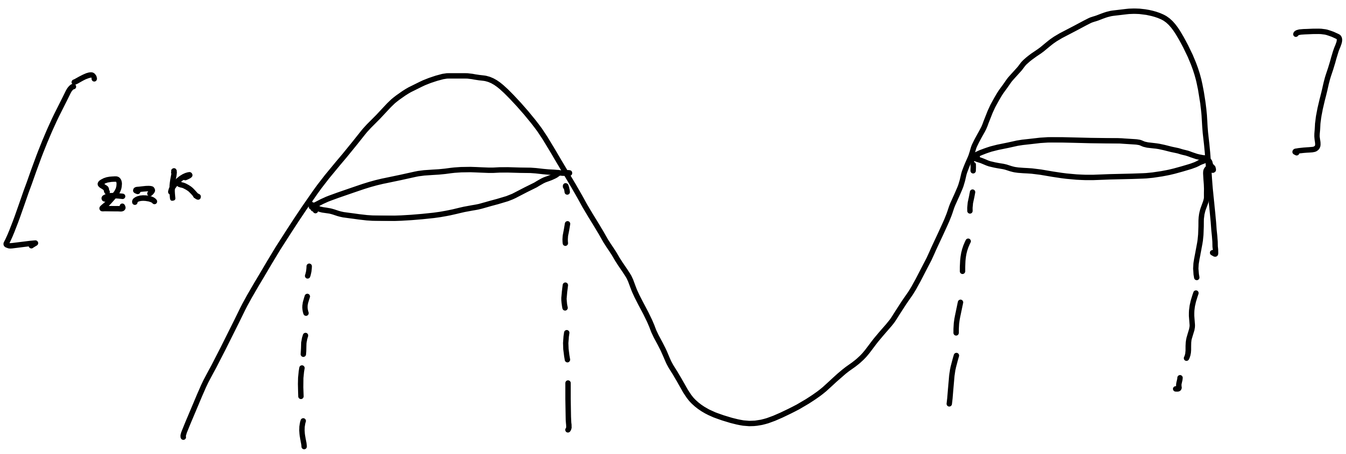
Un vettore normale al piano tangente è

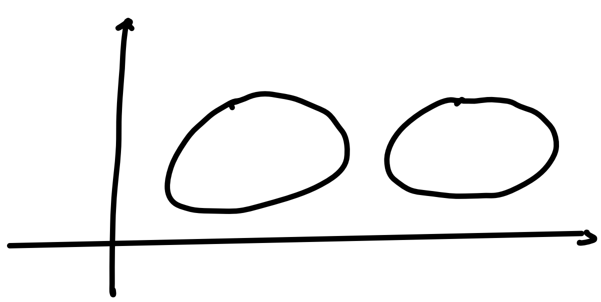
$$\underline{n} = (-f_x(x_0,y_0), -f_y(x_0,y_0), 1)$$

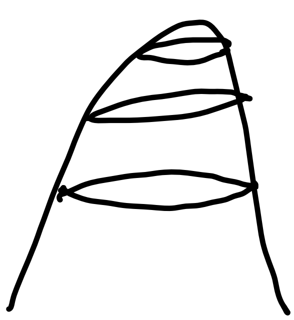


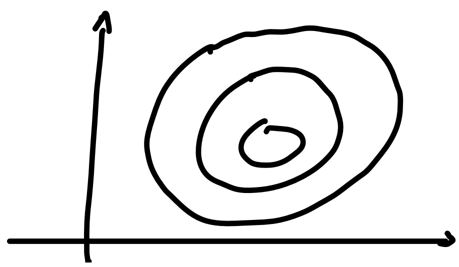
$-Df(x_0, y_0)$

Caratterizzazione di ortogonalità del gradiente alle curve di livello di una funzione differenziabile

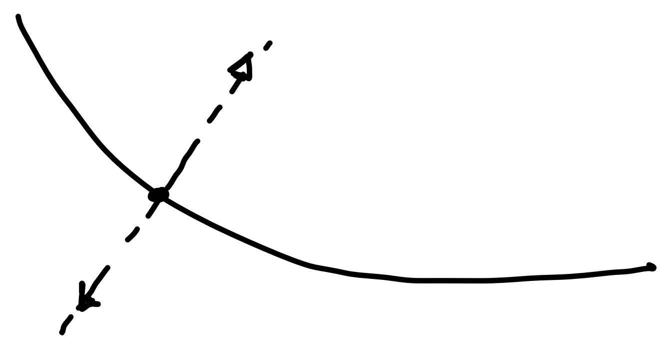






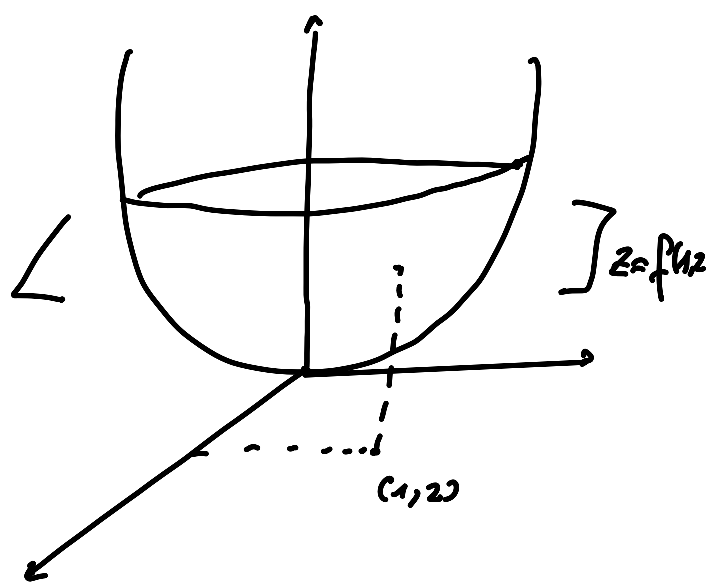


$$\underbrace{\{(x,y)\in A : f(x,y) = K\}}_{\text{curva di livello } k \text{ di } f(x,y)}$$



Es. $f(x,y) = x^2+y^2 = z$

Curva di livello passante per $(1,2)$

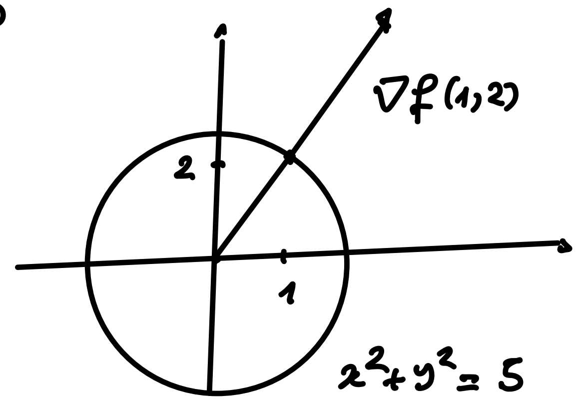


$f(x,y) = f(1,2)$

$f(1,2) = 5$

$x^2+y^2=5$

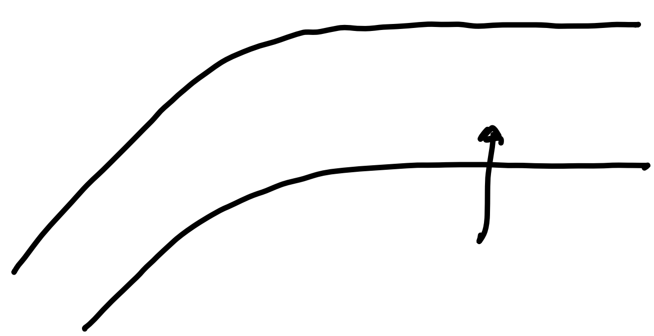
$\nabla f = (2x, 2y)$



$\nabla f(1,2) = (2,4) = 2(1,2)$

Teorema $\quad f: A \to \mathbb{R}$ differenziabile in (x_0,y_0)

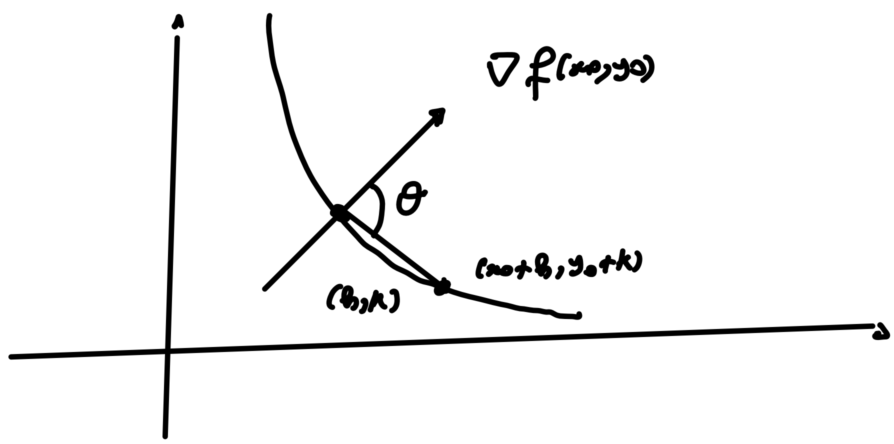
e $\quad \nabla f(x_0,y_0) \neq (0,0)$



Allora $\nabla f(x_0,y_0)$ è ortogonale alla curva di livello passante per $f(x_0,y_0)$.

Dim.

$$\underbrace{f(x,y) = f(x_0,y_0)}_{\text{curva di livello passante per } (x_0,y_0)}$$



$(h,k) \neq (0,0)$ $\qquad$ (x_0+h, y_0+k)

Tesi Mostriamo che per $(h,k) \to (0,0)$, $\theta \to \frac{\pi}{2}$

$\iff \cos\theta \to 0$ per $(h,k) \to (0,0)$

$$\nabla f(x_0,y_0) \cdot (h,k) = \|\nabla f(x_0,y_0)\| \sqrt{h^2+k^2} \cos\theta$$

$$\cos\theta = \frac{\nabla f(x_0,y_0) \cdot (h,k)}{\underbrace{\|\nabla f(x_0,y_0)\|}_{\nabla f \neq 0} \sqrt{h^2+k^2}}$$

$$\lim_{(h,k)\to(0,0)} \cos\theta = -\lim_{(h,k)\to(0,0)} \frac{-\nabla f(x_0,y_0)\cdot(h,k)}{\|\nabla f(x_0,y_0)\|\sqrt{h^2+k^2}} =$$

f continua in (x_0,y_0):

$$\lim_{(h,k)\to(0,0)} f(x_0+h, y_0+k) = f(x_0,y_0)$$

$=$ (f è continua in (x_0,y_0)) $=$

$$= -\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \nabla f(x_0,y_0)\cdot(h,k)}{\|\nabla f(x_0,y_0)\|\sqrt{h^2+k^2}}$$

$= 0$ perché f è differenziabile in (x_0,y_0)

c.v.d.

Derivazione di funzioni composte

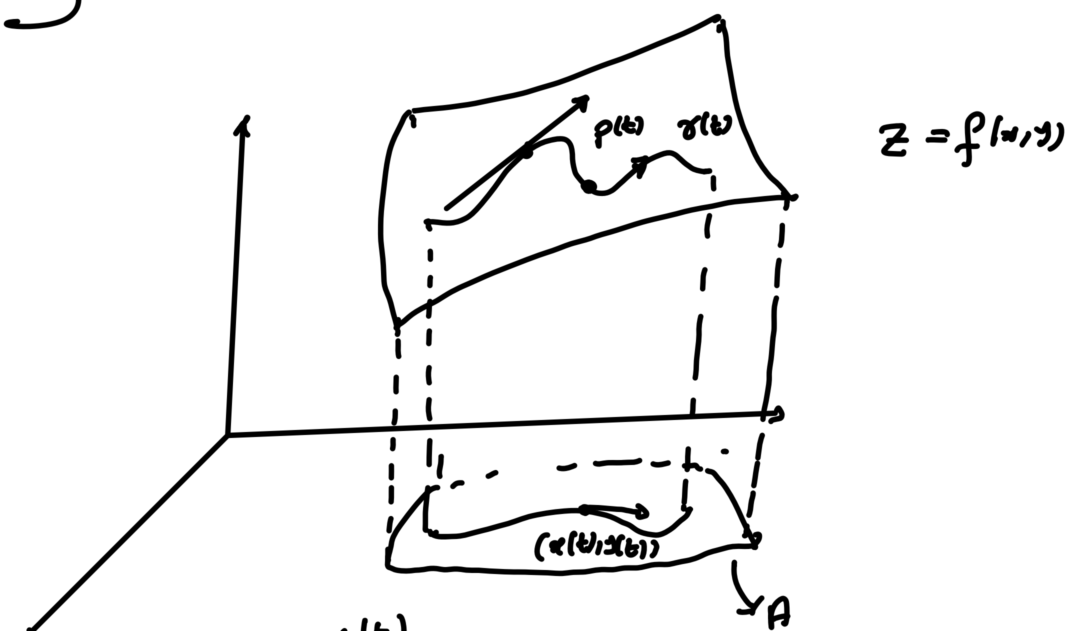
$f = f(x)$, $g = g(y)$

f, g derivabili

$$(g\circ f)'(x) = g'(f(x))\cdot f'(x)$$

$f(x,y)$ S) $z=f(x,y)$

Punto materiale $(x(t),y(t),z(t))$ che si muove lungo S)



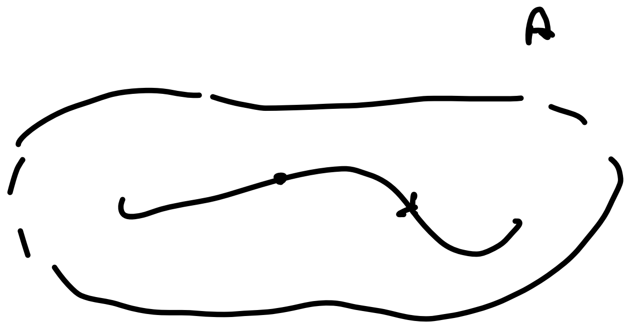
$$\underbrace{\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) = f(x(t),y(t)) \end{cases}}_{\text{equazioni del moto}}$$

$$\underline{V}(t) = \begin{cases} x'(t) \\ y'(t) \\ \frac{d}{dt} f(x(t),y(t)) \quad ? \end{cases}$$

Teorema (derivazione funzioni composte)

$f = f(x,y)$ $\quad f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ differenziabile in A

$\gamma : t \in I \longrightarrow (x(t), y(t)) \in A$, derivabile (le due componenti $x(t), y(t)$ sono derivabili)



Allora la funzione composta

$$F(t) = (f \circ \gamma)(t) = f(x(t), y(t))$$

della sola variabile t è derivabile e

$$F'(t) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$$

ES.

$$f(x,y) = \frac{x^2}{4} + \frac{y^2}{2} \qquad \gamma(t) = \begin{cases} x = 2\cos t \\ y = \sqrt{2}\sin t \end{cases} \quad t \in [0, 2\pi]$$

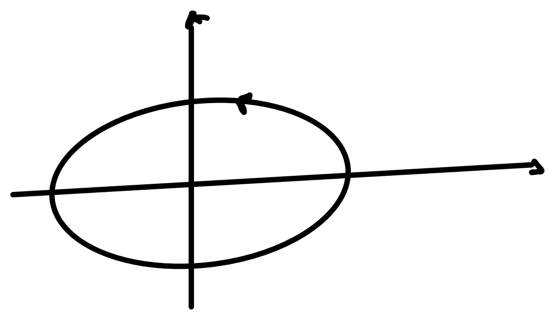
$\gamma(t)$ ha rappresentazione cartesiana

$$\begin{cases} x' = -2\sin t \\ y' = \sqrt{2}\cos t \end{cases}$$

$$\frac{x^2}{4} + \frac{y^2}{2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ ellisse di semiassi

$a = 2, \quad b = \sqrt{2}$



$$F(t) = f(\gamma(t)) = f(x(t), y(t)) =$$

$$= f(2\cos t, \sqrt{2}\sin t) =$$

$$= \frac{\cancel{4}\cos^2 t}{\cancel{4}} + \frac{\cancel{2}\sin^2 t}{\cancel{2}} = \cos^2 t + \sin^2 t = 1$$

$$F'(t) = 0$$

Applichiamo la formula di derivazione delle funzioni composte:

$f_x = \frac{x}{2}$ $\qquad f_y = y$

$$F'(t) = f_x(2\cos t, \sqrt{2}\sin t)\cdot(-2\sin t)$$
$$+ f_y(2\cos t, \sqrt{2}\sin t)\cdot \sqrt{2}\cos t$$
$$= \cos t(-2\sin t) + \sqrt{2}\sin t \cdot \sqrt{2}\cos t$$
$$= -2\sin t\cos t + 2\sin t\cos t = 0$$

ES. $\quad f(x,y) = x^2+y^2 \qquad \gamma \equiv \begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$

$$\begin{cases} x' = -3\sin t \\ y' = 2\cos t \end{cases}$$

ellisse $\quad a=3, \ b=2$

$$F(t) = f(x(t), y(t)) = f(\underbrace{3\cos t}, \underbrace{2\sin t}) =$$
$$= 9\cos^2 t + 4\sin^2 t = 9(1-\sin^2 t)$$
$$+ 4\sin^2 t = 9 - 9\sin^2 t + 4\sin^2 t =$$

$$= 9 - 5\sin^2 t$$

$$F'(t) = -10 \sin t \cos t$$

Der. funzioni composte:

$$f_x = 2x$$
$$f_y = 2y$$

$$F'(t) = f_x(3\cos t, 2\sin t)(-3\sin t) +$$
$$+ f_y(3\cos t, 2\sin t)(2\cos t) =$$
$$= 2\cdot 3\cos t(-3\sin t) + 2\cdot 2\sin t \cdot 2\cos t$$
$$= -18\sin t\cos t + 8\sin t\cos t$$
$$= -10\sin t\cos t$$

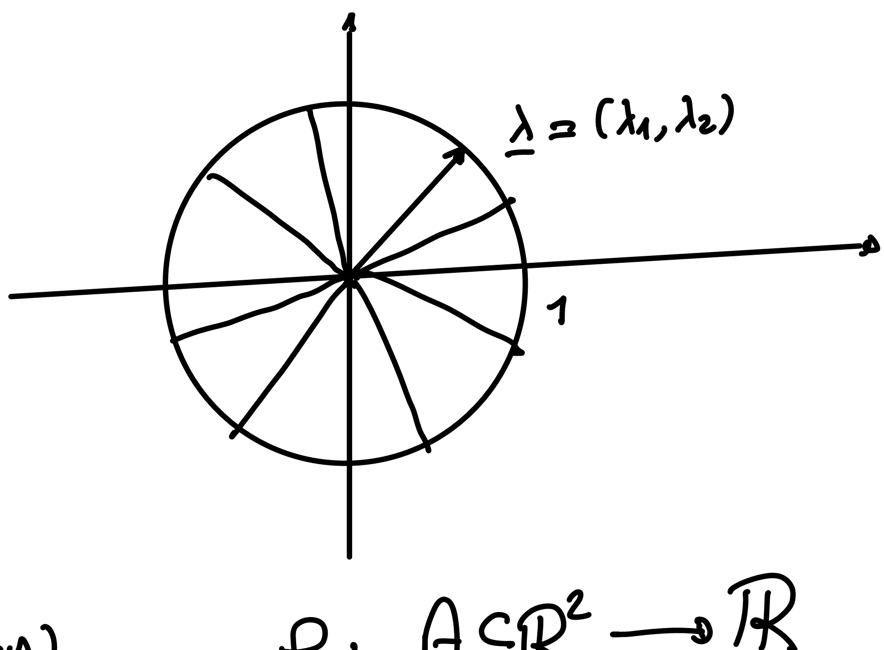
Derivate direzionali

$\mathbb{R}^2$: versore $\underline{\lambda} = (\lambda_1, \lambda_2)$ tale che

$$\|\underline{\lambda}\| = 1$$

$$\Leftrightarrow \quad \lambda_1^2 + \lambda_2^2 = 1$$

Una direzione in $\mathbb{R}^2$ è un qualsiasi versore



$f = f(x,y)$ $\qquad$ $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$

$(x_0, y_0) \in A$

$\underline{\lambda}$ direzione qualsiasi

$$\begin{cases} x = x_0 + \lambda_1 t \\ y = y_0 + \lambda_2 t \end{cases}$$

$$\lim_{t \to 0} \frac{f(x_0 + \lambda_1 t, y_0 + \lambda_2 t) - f(x_0, y_0)}{t} = \frac{\partial f}{\partial \lambda}(x_0, y_0)$$

se esiste finito, tale limite si dice derivata direzionale di f in (x_0, y_0) secondo la direzione $\underline{\lambda}$

Se $\underline{\lambda} = (1, 0) = \underline{e}_1$: $\frac{\partial f}{\partial e_1}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$

$$\underline{v} = (0,1) = \underline{e}_2 : \quad \frac{\partial f}{\partial \underline{e}_2}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) .$$