Lezione del 19/10/2022

$$f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$

$f = f(x)$

$x_0 \in \mathbb{R}$ p.to di accumulazione per X

$$\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$$

$\overset{\text{def.}}{\iff} \forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in X, \ x \neq x_0$
$|x - x_0| < \delta$

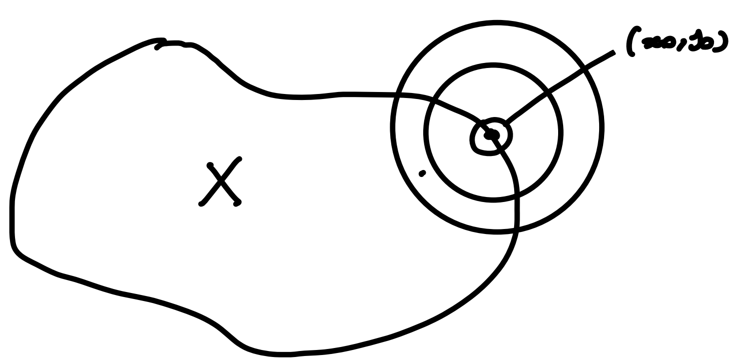
$\Downarrow$

$U =]\ell - \varepsilon, \ell + \varepsilon[\qquad |f(x) - \ell| < \varepsilon$

$\iff f(x) \in U$

$f(x,y) \qquad f : X \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x_0, y_0) \in \mathbb{R}^2$ di accumulazione per X



<u>**Def.**</u> Diremo che $f(x,y)$ <u>tende</u> ad $\ell \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ per $(x,y) \to (x_0,y_0)$, e scriveremo

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$$

$$\overset{def.}{\iff} \quad \forall\, U \text{ intorno di } \ell \quad \exists\, \delta > 0 :$$

$$\forall (x,y) \in X,\ (x,y) \neq (x_0,y_0) \text{ t.c. }$$

$$\|(x,y) - (x_0,y_0)\| < \delta$$

$$\Downarrow$$

$$f(x,y) \in U$$

In particolare, se $\ell \in \mathbb{R}$,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$$

$$\iff \left[\forall \varepsilon > 0 \quad \exists\, \delta > 0 : \ \forall (x,y) \in X,\ (x,y) \neq (x_0,y_0) \text{ e } \|(x,y) - (x_0,y_0)\| < \delta \ \Downarrow \ |f(x,y) - \ell| < \varepsilon \right]$$

$\ell = +\infty$?

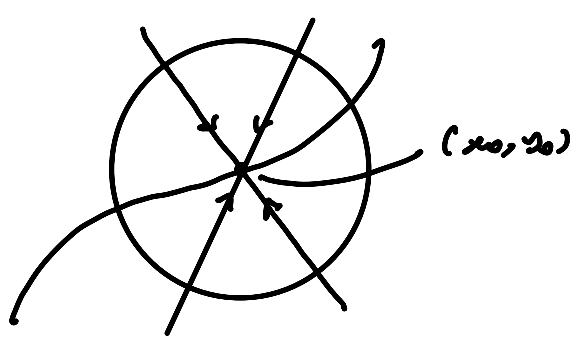
$$\forall K>0 \ \exists\, \delta>0 : \ \forall (x,y)\in X, \ (x,y)\neq(x_0,y_0)$$

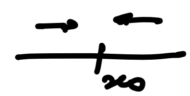
$$\text{e } \|(x,y)-(x_0,y_0)\|<\delta$$

$$\Downarrow$$

$$f(x,y) > K$$

$(\ell=-\infty \quad \dots \quad f(x,y) < -K)$

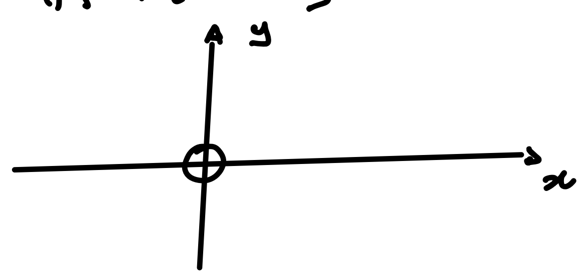




ES.

$$f(x,y) = \frac{x^2}{\sqrt{x^2+y^2}} \qquad x^2+y^2>0 \iff (x,y)\neq(0,0)$$

$$X = \mathbb{R}^2 \setminus \{(0,0)\}$$



$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0 \qquad \underline{\text{VERIFICA}}\ !$$

$$\frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \frac{\sqrt{(x^2+y^2)^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$$

$= \|(x,y)\|_{\mathbb{R}^2}$: <u>se</u>, fissato $\varepsilon > 0$,

prendo (x,y) : $\|(x,y)\| < \varepsilon =: \delta$

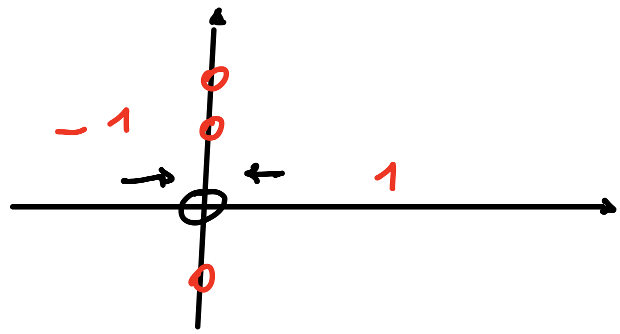
$\Longrightarrow$ dalle disuguag. precedente si ha

$$\underset{\substack{\overset{0}{\|}\\ \ell}}{|f(x,y)|} = \frac{x^2}{\sqrt{x^2+y^2}} < \varepsilon$$

$$|f(x,y) - \ell| = |f(x,y)| < \varepsilon$$

$$\ell = 0$$

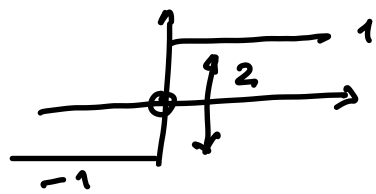
$$g(x,y) = \frac{x}{\sqrt{x^2+y^2}} \qquad \lim_{(x,y)\to(0,0)} g(x,y) \ \underline{\text{non esiste}}$$



Sull'asse delle x: $y=0$, $g(x,0) = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$

$$\lim_{x \to 0^+} g(x,0) = 1 \quad , \quad \lim_{x \to 0^-} g(x,0) = -1$$

$$g(x,0) = \begin{cases} 1 & \text{se } x > 0 \\ -1 & \text{se } x < 0 \end{cases}$$



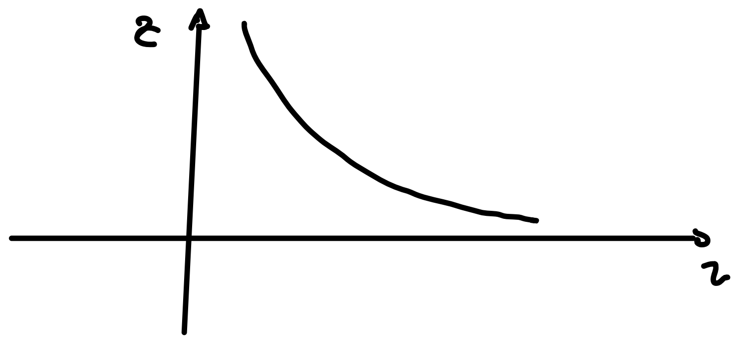
Sull'asse delle y, $x=0$

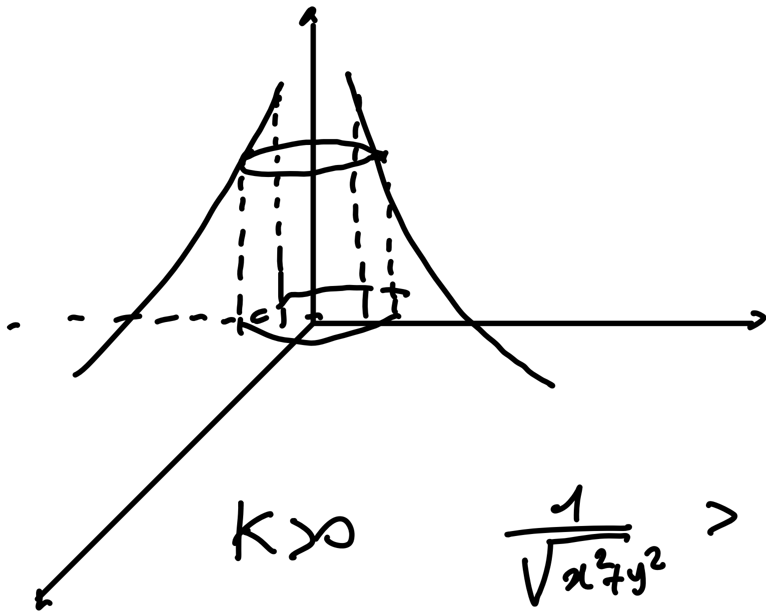
$$g(0,y) = 0 \quad , \quad \forall y \neq 0$$

ES.

$$f(x,y) = \frac{1}{\underbrace{\sqrt{x^2+y^2}}_{2}} \qquad (x,y) \to (0,0)$$

$$\longrightarrow \frac{1}{2}$$





$K>0 \qquad \frac{1}{\sqrt{x^2+y^2}} > K$

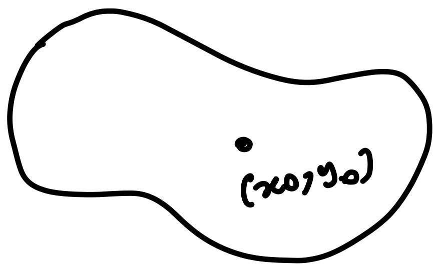
$\iff \sqrt{x^2+y^2} < 1/K =: \delta$

$$\lim_{(x,y)\to(0,0)} f(x,y) = +\infty$$

<u>Continua</u> in x_0 $\qquad \lim_{x\to x_0} f(x) = f(x_0)$

Def $f : X \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f = f(x,y)$

$(x_0,y_0) \in X$ di accumulazione per X



si dice che $f(x,y)$ è continua in (x_0,y_0)

se $$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Per convenzione $f(x,y)$ è continua in ogni punto isolato di X.

Quindi $f(x,y)$ è continua in (x_0,y_0)

$\iff \forall \varepsilon > 0 \ \exists \delta > 0 : \forall (x,y) \in X$ t.c.

$$\|(x,y)-(x_0,y_0)\| < \delta$$

$$\Downarrow$$

$$|f(x,y) - \underbrace{f(x_0,y_0)}_{\ell}| < \varepsilon$$

$$f(x,y) = \sin^2(xy)$$

$$f_1(x,y) = \sin(xy) \qquad f_2(t) = t^2$$

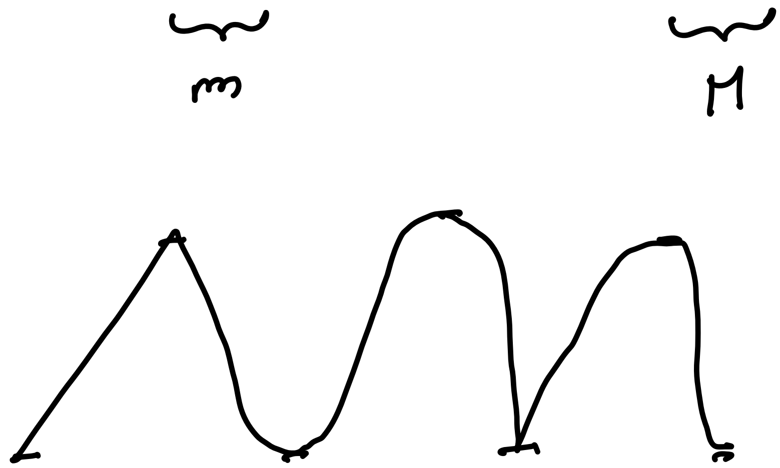
$$(f_2 \circ f_1)(x,y) = \sin^2(xy) = f(x,y)$$

Teorema di Weierstrass

$f = f(x)$, $\quad x \in [a,b]$, $\quad f : [a,b] \to \mathbb{R}$

continua : $\exists\ \bar{x}, \bar{\bar{x}} \in [a,b]$ tali che

$$f(\bar{x}) \leq f(x) \leq f(\bar{\bar{x}}) \ , \ \forall x \in [a,b]$$



$f = f(x,y)$, $f : K \subseteq \mathbb{R}^2 \to \mathbb{R}$

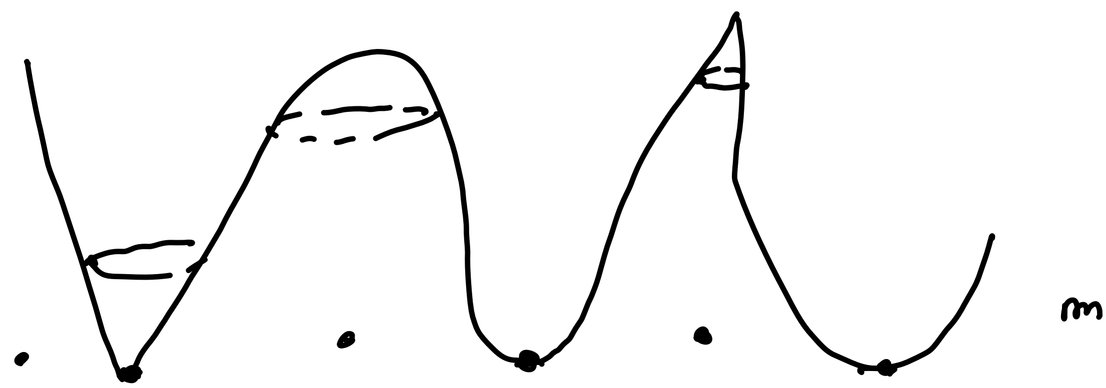
K <u>compatto</u> (chiuso e limitato). Allora,

se f è <u>continua</u> in K,

$\exists\, (\bar{x},\bar{y}), (\bar{\bar{x}},\bar{\bar{y}}) \in K$ tali che

$$\underbrace{f(\bar{x},\bar{y})}_{m} \leq f(x,y) \leq \underbrace{f(\bar{\bar{x}},\bar{\bar{y}})}_{M} \qquad \forall (x,y) \in K$$

m, M minimo e massimo del codominio di $f(x,y)$



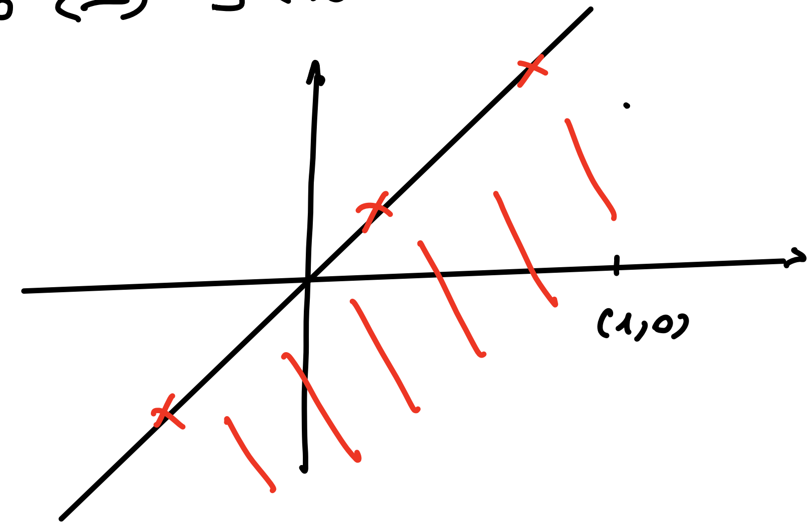
$$m = \min_{(x,y)\in K} f(x,y) \qquad M = \max_{(x,y)\in K} f(x,y)$$

$$f(x,y) = \log(x^2-y^2)$$

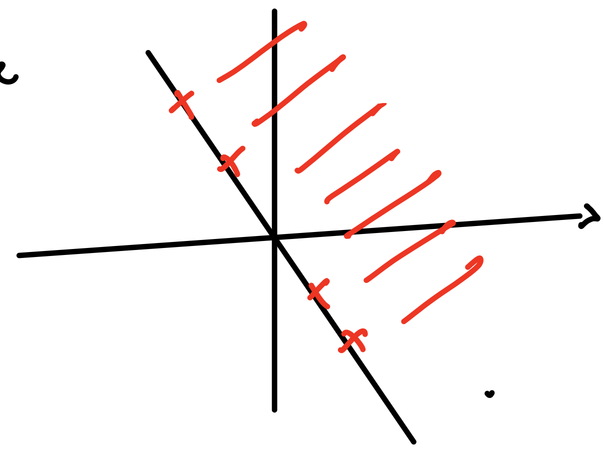
$$(x^2-y^2)>0 \iff (x-y)(x+y)>0$$

$$\odot \begin{cases} x-y>0 \\ x+y>0 \end{cases} \cup \begin{cases} x-y<0 \\ x+y<0 \end{cases}$$

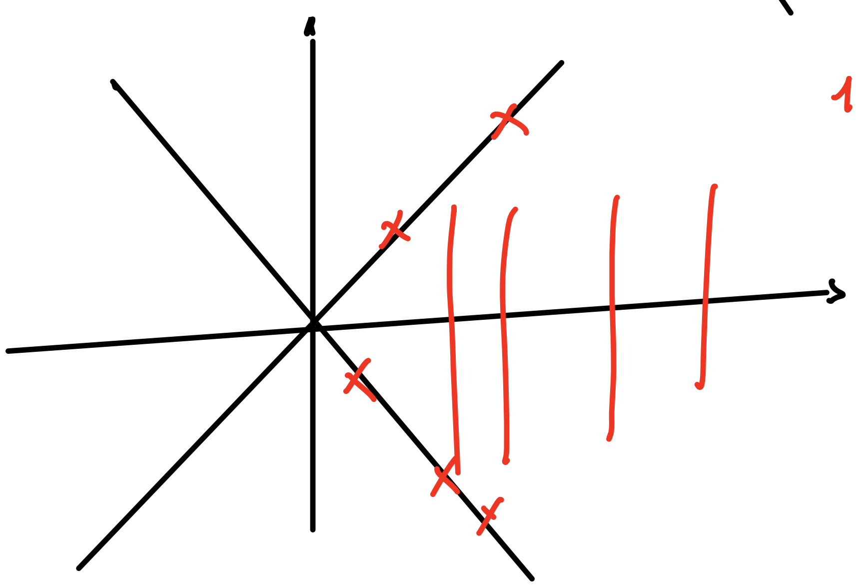
$x-y>0 \iff y<x$



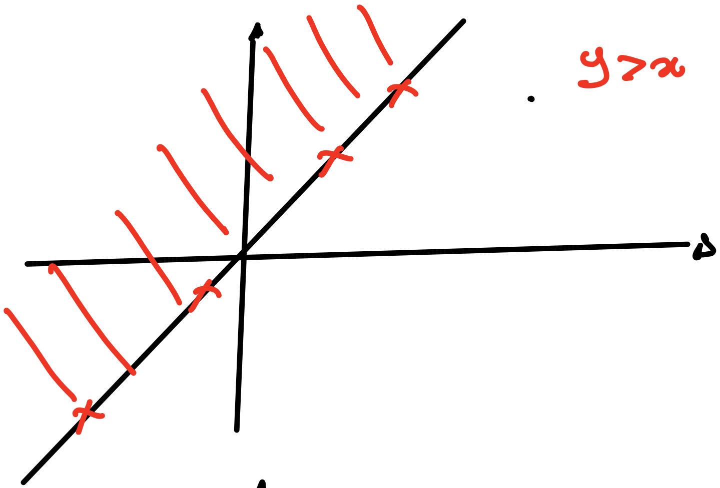
$x+y>0 \iff y>-x$



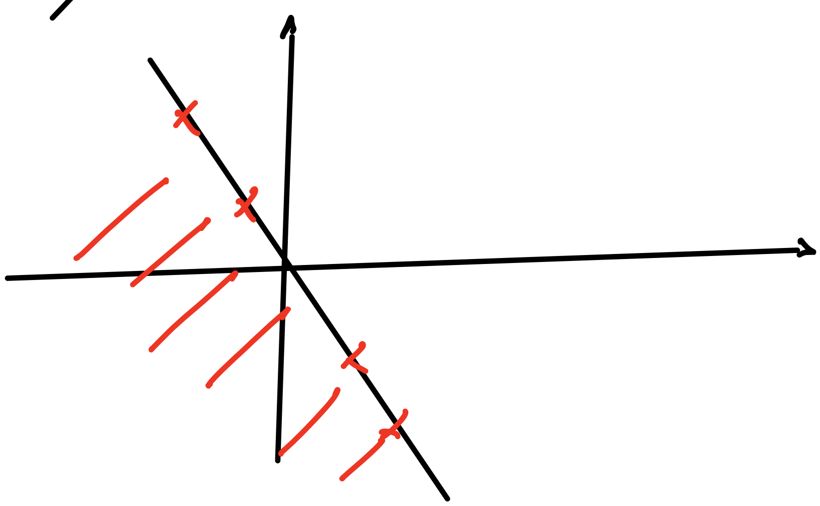
1º SISTEMA



$$\begin{cases} x - y < 0 \\ x + y < 0 \end{cases} \iff \begin{cases} y > x \\ y < -x \end{cases}$$



$y > x$



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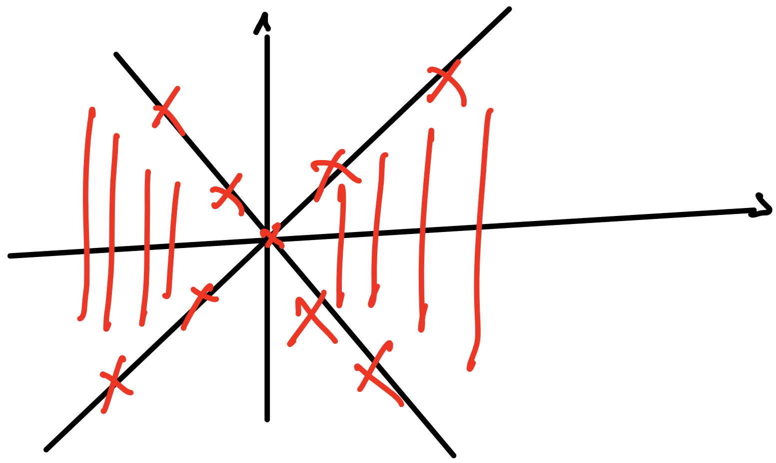


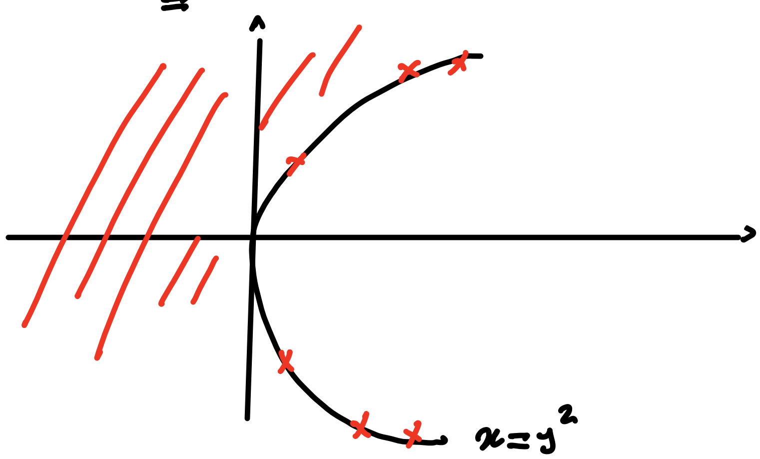
$f(x,y) = \log(y^4 - x^2)$

$y^4 - x^2 > 0 \iff (y^2 - x)(y^2 + x) > 0$

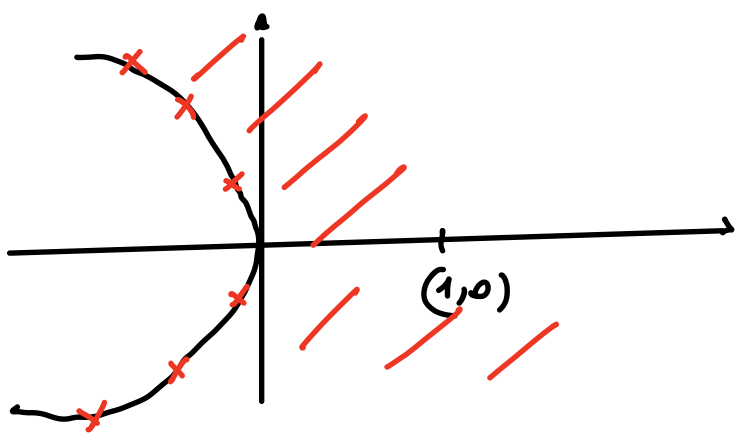
$\underbrace{\begin{cases} y^2 - x > 0 \\ y^2 + x > 0 \end{cases}} \cup \begin{cases} y^2 - x < 0 \iff x > y^2 > 0 \\ y^2 + x < 0 \iff x < -y^2 < 0 \end{cases}$

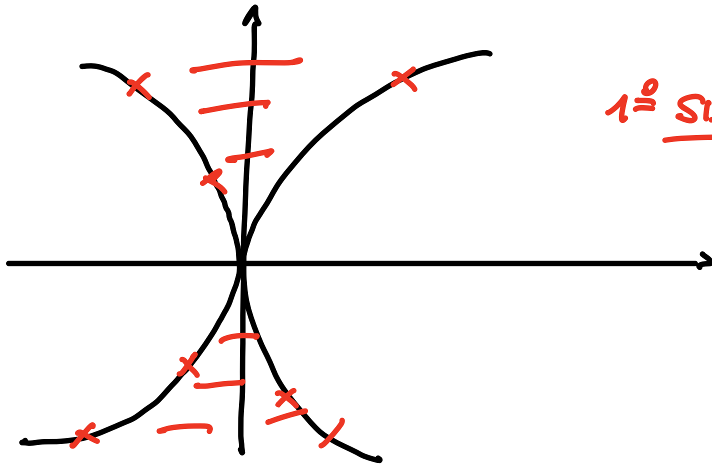
$\underbrace{}_{\emptyset}$

$y^2 - x > 0 \iff \underline{\underline{x < y^2}}$



$x > -y^2$





1^{o} SISTEMA

Insieme di definizione di:

$f(x,y) = \frac{1}{x-y}$, $f(x,y) = \frac{1}{\sqrt{x+y-1}}$

$f(x,y) = \arccos(x^2+y^2-4)$