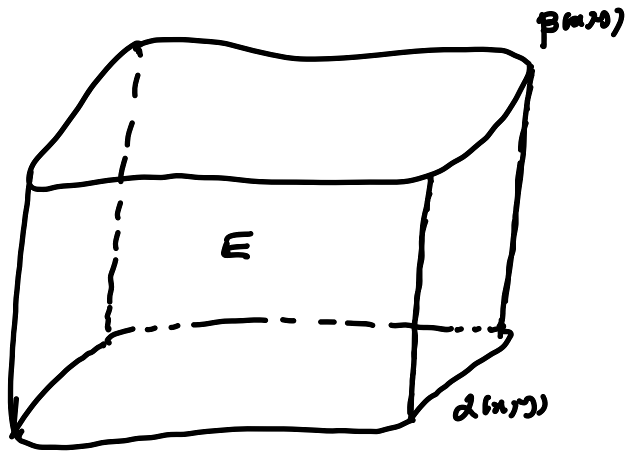
Lezione del 19/12/2022

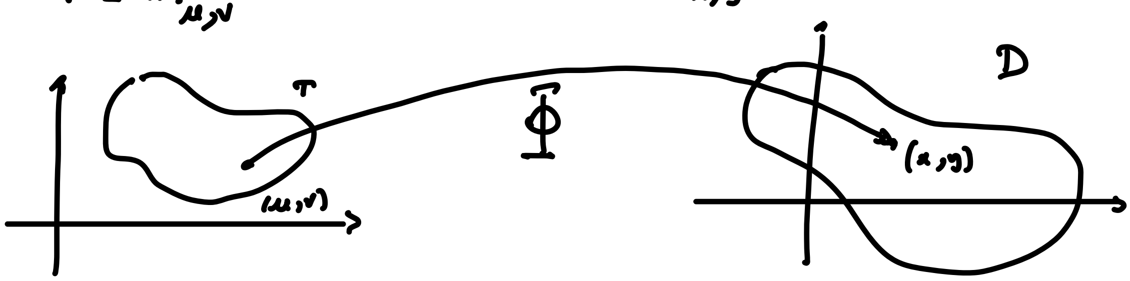
$$E = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in D \, , \; \alpha(x,y) \leq z \leq \beta(x,y) \}$$

regolare se $\alpha, \beta \in C^1(D)$.

Def $E \subseteq \mathbb{R}^3$: si dice che E è un dominio regolare se E è unione di un numero finito di domini normali regolari, a due a due privi di punti interni in comune.

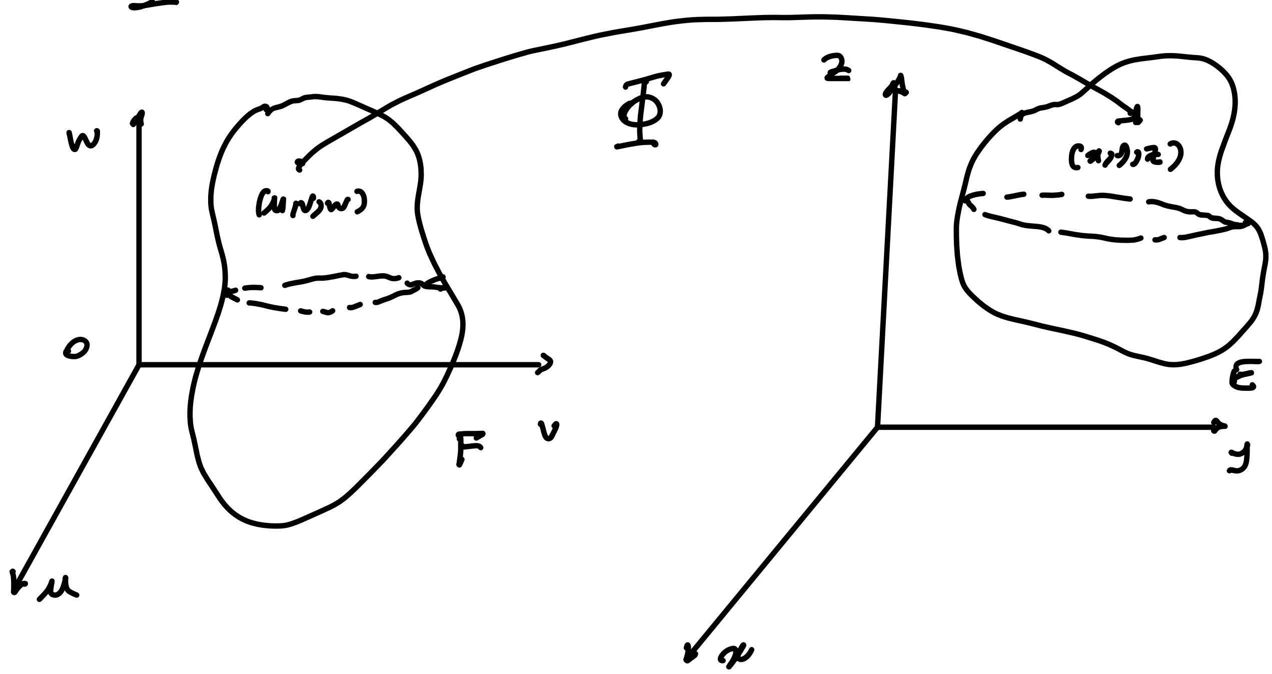


$$\Phi : T \subseteq \mathbb{R}^2_{u,v} \longrightarrow D \subseteq \mathbb{R}^2_{x,y} \qquad \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$



$E \subseteq \mathbb{R}^3_{x,y,z}$, $F \subseteq \mathbb{R}^3_{u,v,w}$ domini regolari

$$\Phi : (u,v,w) \in F \longrightarrow (x(u,v,w), y(u,v,w), z(u,v,w)) \in E$$



$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$

Se $\Phi \in C^1(F)$, definiamo

$$\det \frac{\partial(x,y,z)}{\partial(u,v,w)} = \text{det. jacobiano di } \Phi = |J_\Phi|$$

$$= \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \text{f.ne delle variabili } u,v,w$$

Formula del cambiamento di variabili negli integrali tripli

E, F domini regolari di $\mathbb{R}^3$, sia $\Phi : F \to E$ una trasformazione biunivoca, di classe C^1 in F e tale che

$$\det \frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0 \quad \text{in } F.$$

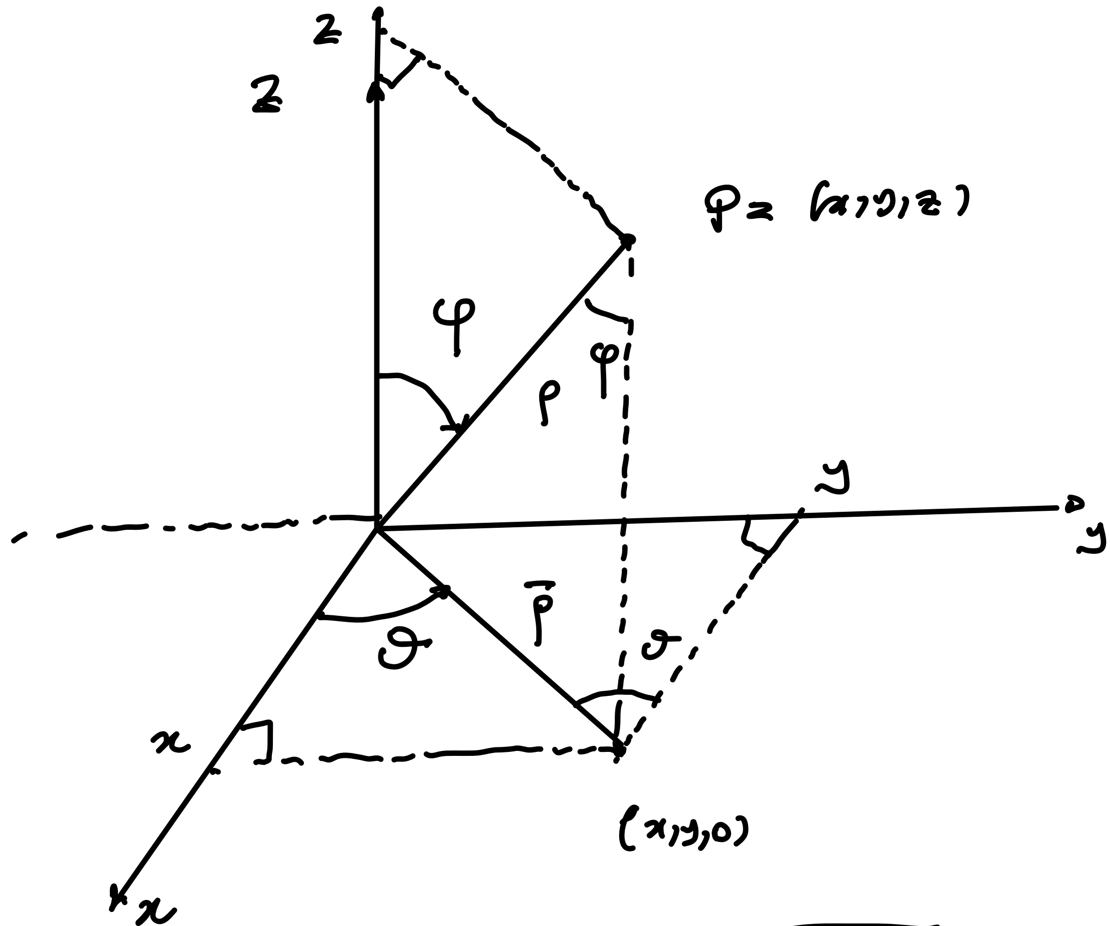
Allora se $f = f(x,y,z)$, $f : E \to \mathbb{R}$ continua

si ha

$$\iiint_E f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_F f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big) \cdot \left| \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

1 Trasformazione alle coordinate sferiche



$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\begin{cases} x = \bar{\rho} \cos\vartheta \\ y = \bar{\rho} \sin\vartheta \\ z = \rho \cos\varphi \end{cases}$$

$\vartheta =$ longitudine

$\varphi =$ colatitudine

$\frac{\pi}{2} - \varphi =$ latitudine

$$\bar{\rho} = \rho \sin\varphi$$

$$\Phi \equiv \begin{cases} x = \rho \cos\vartheta \sin\varphi \\ y = \rho \sin\vartheta \sin\varphi \\ z = \rho \cos\varphi \end{cases}$$

trasformazione alle coordinate sferiche

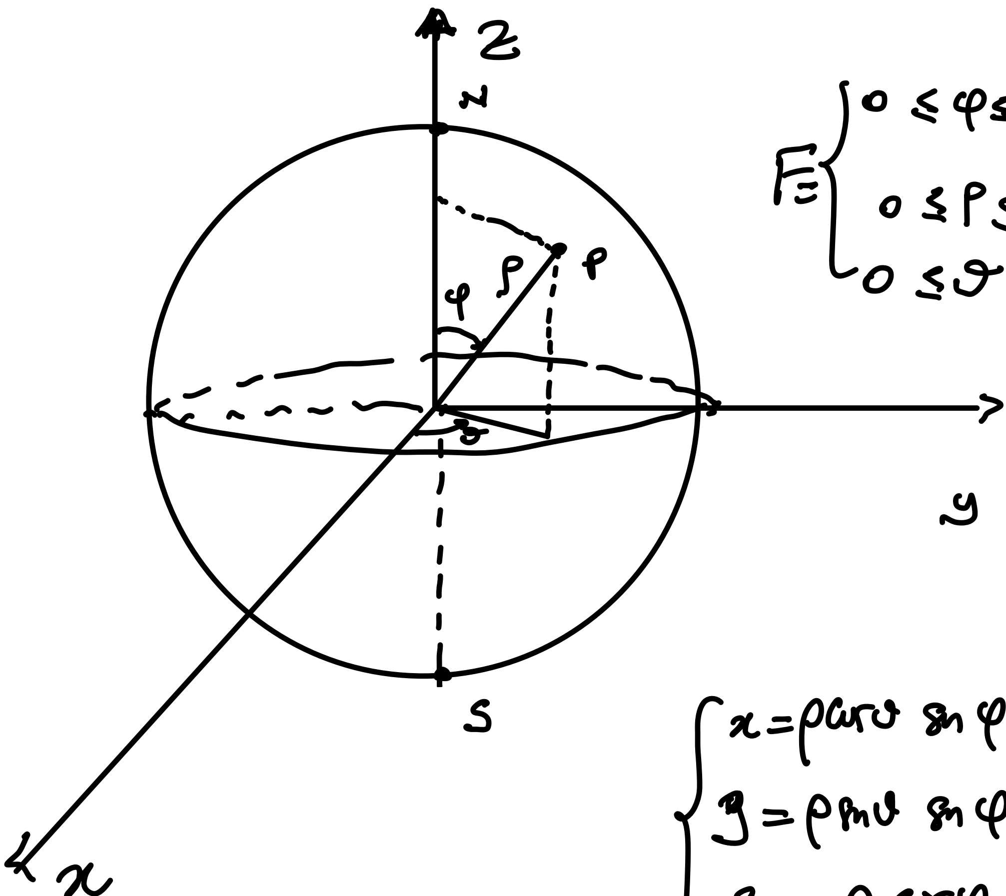
$$\det \frac{\partial(x,y,z)}{\partial(\rho,\vartheta,\varphi)} = \begin{vmatrix} \cos\vartheta \sin\varphi & -\rho \sin\vartheta \sin\varphi & \rho\cos\vartheta\cos\varphi \\ - & - - - - - & \\ - & - - - & \end{vmatrix}$$

$$= \rho^2 \sin\varphi \geq 0$$

ES

$$\iiint_E \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz$$

E = sfera centrata nell'origine, di raggio 1.



$$F = \begin{cases} 0 \le \varphi \le \pi \\ 0 \le \rho \le 1 \\ 0 \le \vartheta \le 2\pi \end{cases}$$

$$\begin{cases} x = \rho \cos\vartheta \sin\varphi \\ y = \rho \sin\vartheta \sin\varphi \\ z = \rho \cos\varphi \end{cases}$$

$$\iiint_E \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz =$$

$$F = \underset{\rho}{[0,1]} \times \underset{\vartheta}{[0,2\pi]} \times \underset{\varphi}{[0,\pi]}$$

$=$ (f.la cambiamento di variabili)

$$= \iiint_F \rho \cdot \rho^2 \sin\varphi \; d\rho \, d\theta \, d\varphi$$

$$= \iiint_F \rho^3 \sin\varphi \; d\rho \, d\theta \, d\varphi =$$

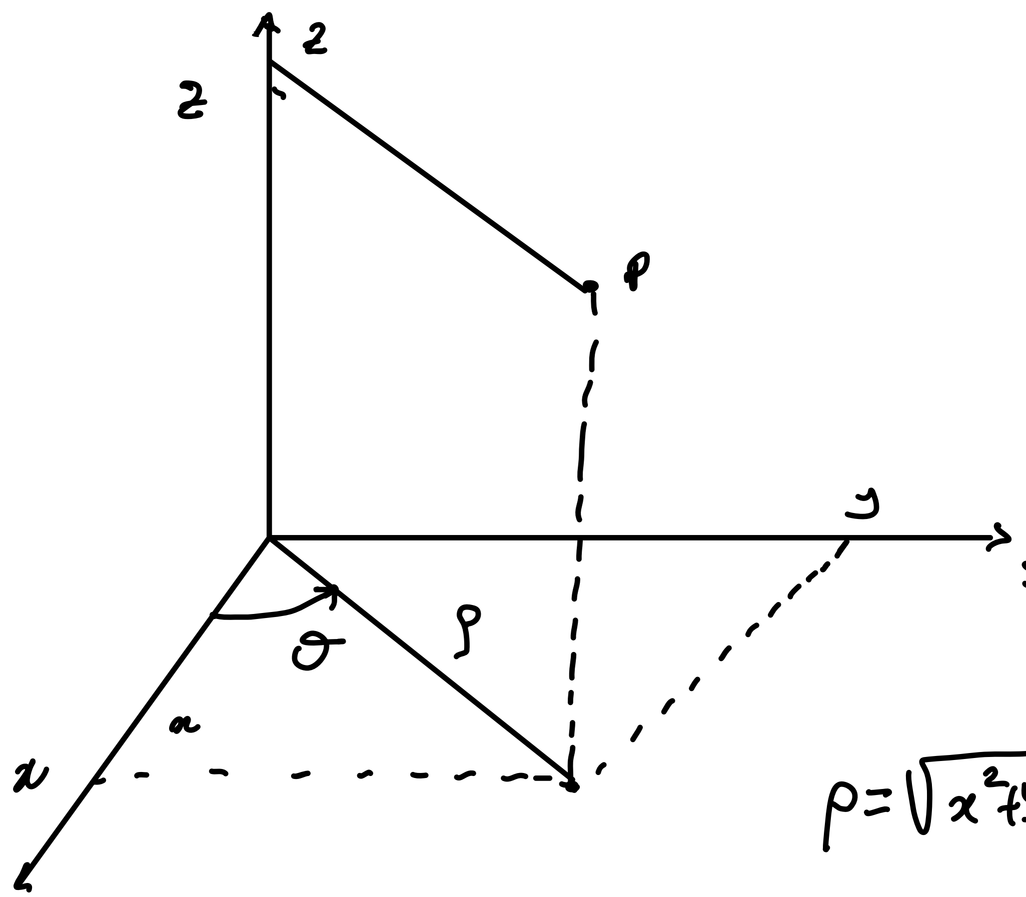
$$= \int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \rho^3 \sin\varphi \, d\varphi$$

$$= 2\pi \int_0^1 \rho^3 \, d\rho \int_0^{\pi} \sin\varphi \, d\varphi =$$

$$= 2\pi \left(\frac{\rho^4}{4} \right)\Big|_0^1 \underbrace{\left(-\cos\varphi \right)\Big|_0^{\pi}}_{2}$$

$$= 2\pi \cdot 1/4 \cdot 2 = \pi .$$

Trasformazione alle coordinate cilindriche



$$\rho = \sqrt{x^2+y^2}$$

$$\Phi = \begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \\ z = z \end{cases}$$

(ρ, θ, z) coordinate cilindriche

$x^2+y^2 = \rho^2$ cilindro di curva direttrice la circonferenza di raggio ρ

$$\det \frac{\partial(x,y,z)}{\partial(\rho,\vartheta,z)} = \begin{vmatrix} \cos\vartheta & -\rho\sin\vartheta & 0 \\ \sin\vartheta & \rho\cos\vartheta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \rho\cos^2\vartheta + \rho\sin^2\vartheta = \rho$$

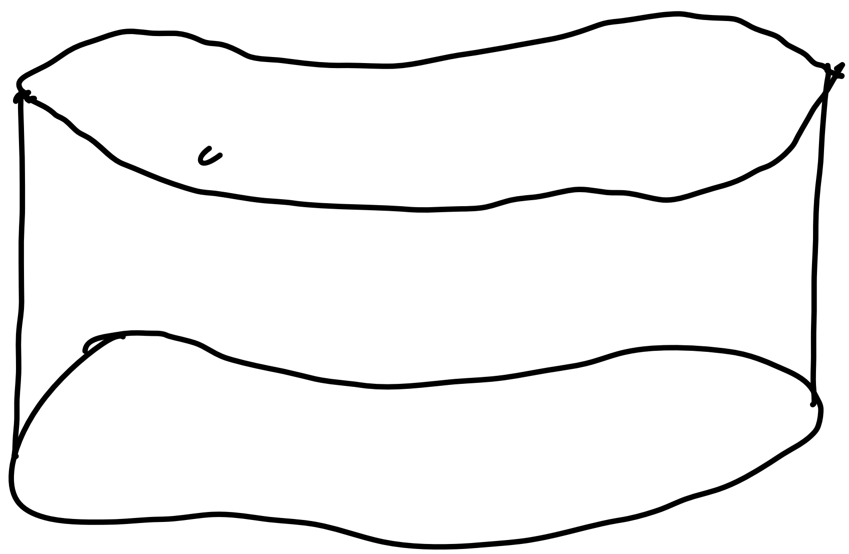
2)

Verificare che il volume della sfera di raggio R

è $\frac{4}{3}\pi R^3$ ..

$$Vol(E) = \iiint_E 1 \cdot dx\,dy\,dz$$

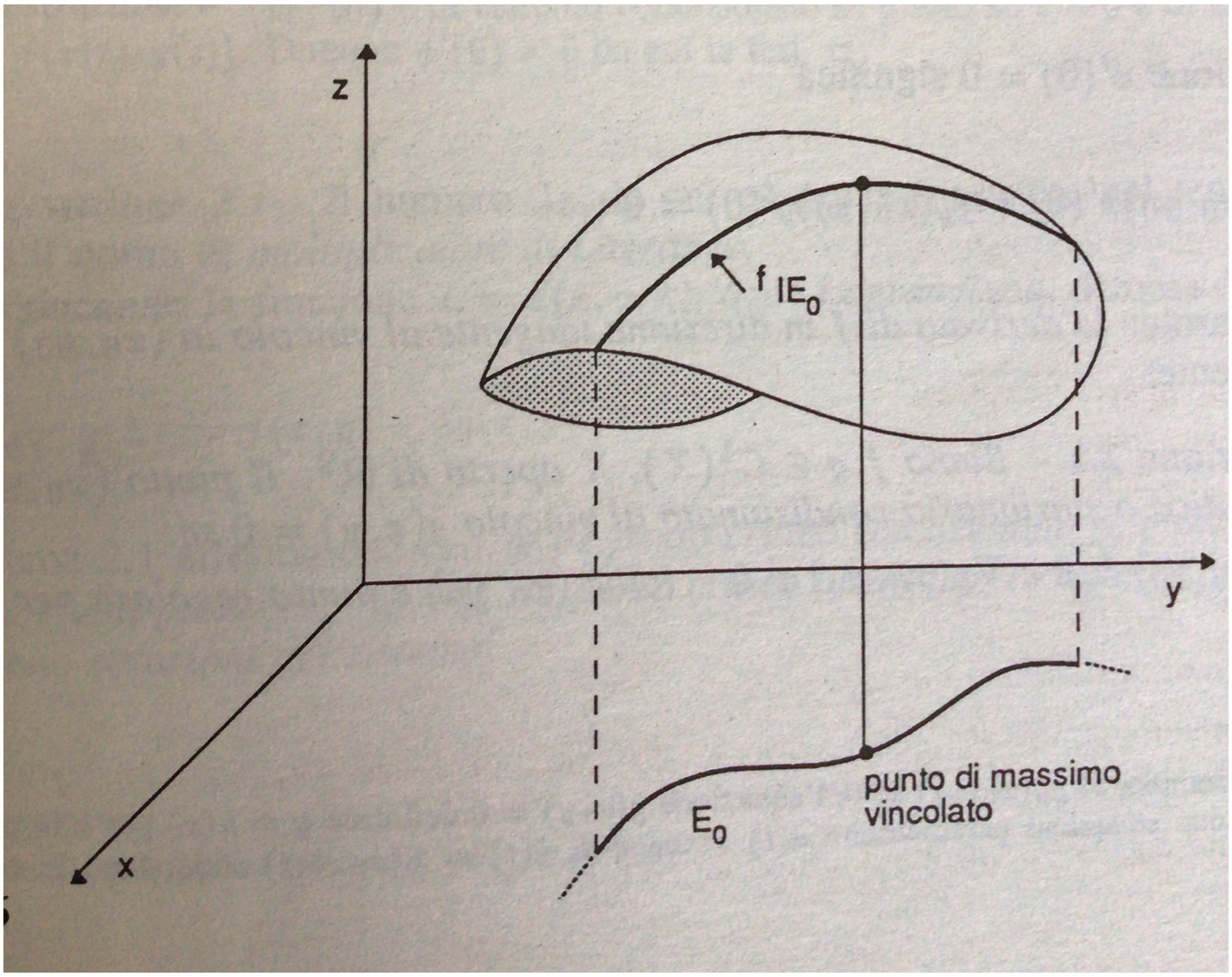
$E \subseteq \mathbb{R}^2 \qquad \text{Area}(E) = \iint_E 1 \cdot dx\,dy$



Massimi e minimi vincolati

$f = f(x,y)$, $f : A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad f, g \in C^1(A)$

$g : A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$



Problema Determinare gli estremi di $f = f(x,y)$ (funzione obiettivo), ristretta all'insieme

$$E_0 = \{ (x,y) \in A : g(x,y) = 0 \}$$

vincolo

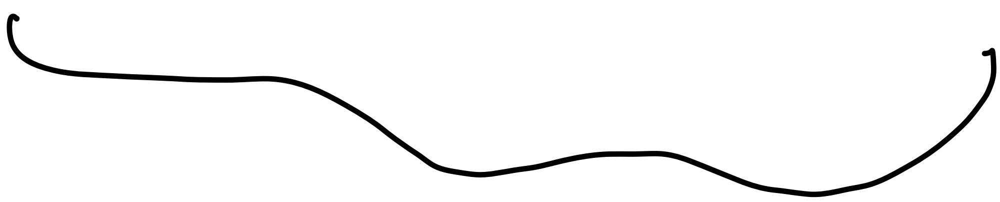
Es. $f(x,y)$ $\qquad g(x,y) = x^2+y^2-1$

$$\underbrace{g(x,y)}=0 \iff x^2+y^2=1 \quad \text{c.rf. di raggio } 1$$

Estremi assoluti di f su $E_0 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad t\in[0,2\pi]$$

$$\phi(t) = f(\cos t, \sin t) \quad , \forall t\in[0,2\pi]$$

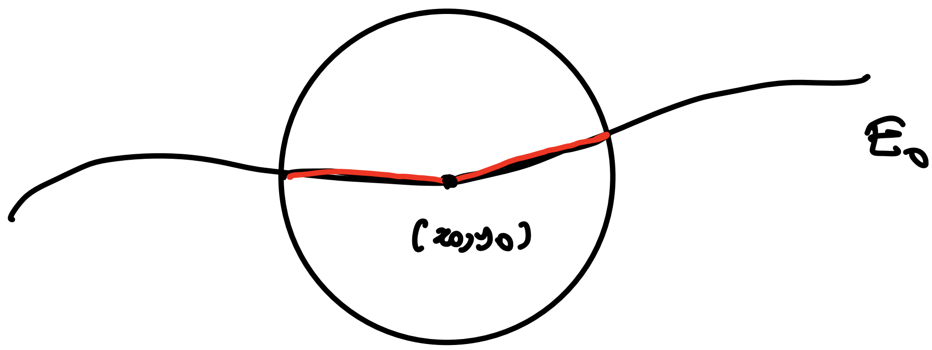


Def. $f: A \to \mathbb{R}$, $\quad f = f(x,y)$

$$E_0 = \{(x,y)\in\mathbb{R}^2 : g(x,y)=0\}$$

(x_0,y_0) p.to di $\overset{\text{(min.)}}{\text{max.}}$ relativo per f condizionato al vincolo E_0 se $\exists\, \delta>0$ t.c.

$$f(x,y) \underset{(\geq)}{\leq} f(x_0,y_0) \quad , \forall (x,y) \in E_0 \cap I_\delta(x_0,y_0)$$



Teorema (Moltiplicatori di Lagrange)

$f, g \in C^1(A)$, sia $(x_0,y_0) \in E_0$ regolare per g, ossia

$$\nabla g(x_0,y_0) \neq 0 .$$

Allora, se (x_0,y_0) è un estremo relativo per f, vincolato al vincolo E_0, esiste $\lambda_0 \in \mathbb{R}$ tale che

$$\nabla f(x_0,y_0) = \lambda_0 \nabla g(x_0,y_0)$$

$$\begin{cases} f_x(x_0,y_0) = \lambda_0\, g_x(x_0,y_0) \\ f_y(x_0,y_0) = \lambda_0\, g_y(x_0,y_0) \\ g(x_0,y_0) = 0 \end{cases}$$

λ_0 = moltiplicatore di Lagrange.

vincolati

OSS. Gli estremi relativi di f su E_0 si ottengono determinando i punti critici della

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda\, g(x,y):$$

$$\Longleftrightarrow \quad \begin{cases} f_x - \lambda g_x = 0 \\ f_y - \lambda g_y = 0 \\ g(x,y) = 0 \end{cases} \Longleftrightarrow \underbrace{\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases}}_{\text{incognite } x,y,\lambda}$$

$$\mathcal{L}_\lambda = -g = 0$$

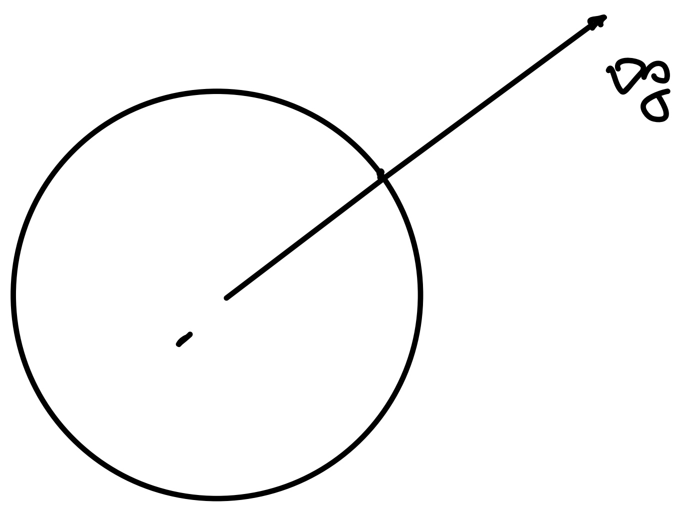
ES. Trovare gli estremi di $f(x,y) = e^{x+y}$

sul vincolo $x^2+y^2 = 1$.

$$\underbrace{g(x,y)}_{E_0} = x^2+y^2-1 \quad , \quad \nabla g = (2x,2y) = (0,0)$$

solo per $(x,y) = (0,0) \notin E_0$

$\Rightarrow$ tutti i punti di E_0 sono regolari



$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

$$= e^{x+y} - \lambda(x^2+y^2-1)$$

$$\begin{cases} e^{x+y} - 2\lambda x = 0 \\ e^{x+y} - 2\lambda y = 0 \\ x^2+y^2 = 1 \end{cases}$$

$$\begin{cases} e^{x+y} = 2\lambda x \\ e^{x+y} = 2\lambda y \\ x^2+y^2 = 1 \end{cases}$$

$$\begin{cases} e^{x+y} = 2\lambda x \\ \cancel{2}\lambda x = \cancel{2}\lambda y \\ x^2+y^2 = 1 \end{cases}$$

$\lambda \neq 0$

$$
\begin{cases}
e^{x+y} = 2\lambda x \\
x = y \\
x^2 + y^2 = 1
\end{cases}
$$

$$\downarrow$$

$$
\begin{cases}
e^{x+y} = 2\lambda x \\
x = y \\
2y^2 = 1 \iff y = \pm \frac{\sqrt{2}}{2}
\end{cases}
$$

$$
\begin{cases}
y = -\frac{\sqrt{2}}{2} \\
x = -\frac{\sqrt{2}}{2} \\
e^{-\sqrt{2}} = \cancel{2}\lambda\left(-\frac{\sqrt{2}}{\cancel{2}}\right) = -\sqrt{2}\lambda
\end{cases}
$$

$$\iff \quad \lambda = -e^{-\sqrt{2}}/\sqrt{2}$$

$$\begin{cases} y = \frac{\sqrt{2}}{2} \\ x = \frac{\sqrt{2}}{2} \\ e^{\sqrt{2}} = \sqrt{2}\lambda \Leftrightarrow \lambda = \frac{e^{\sqrt{2}}}{\sqrt{2}} \end{cases}$$

$$A = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$B = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = e^{-\sqrt{2}}$$

$$\wedge$$

$$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = e^{\sqrt{2}}$$

A = p.to di minimo vincolato

B = p.to di massimo vincolato

2) Determinare il massimo di

$$f = x^2 + y^2$$

soggetta al vincolo

$$E_0 : \quad g(x,y) = e^{x^2} + e^{y^2} - 4 = 0$$

$$\nabla g = \left(2x e^{x^2} , 2y e^{y^2} \right) = (0,0)$$

$$\Leftrightarrow \quad (x,y) = (0,0) \notin E_0$$

$$\mathcal{L}(x,y,\lambda) = f - \lambda g = x^2+y^2 - \lambda\left(e^{x^2}+e^{y^2}-4\right)$$

$$\begin{cases} \cancel{2}x - \cancel{2}x\lambda e^{x^2} = 0 \\ \cancel{2}y - \cancel{2}y\lambda e^{y^2} = 0 \\ e^{x^2}+e^{y^2} = 4 \end{cases} \iff \begin{cases} x(1-\lambda e^{x^2}) = 0 \\ y - \lambda y e^{y^2} = 0 \\ e^{x^2}+e^{y^2} = 4 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y - \lambda y e^{y^2} = 0 \\ e^{y^2} = 4-1 = 3 = e^{\log 3} \end{cases} \quad (S_1)$$

$$\cup \begin{cases} 1 - \lambda e^{x^2} = 0 \\ y - \lambda y e^{y^2} = 0 \\ e^{x^2}+e^{y^2} = 4 \end{cases} \quad (S_2)$$

$$(S_1)\quad \begin{cases} x=0 \\ y=\pm\sqrt{\log 3} \\ y-\lambda y\, e^{y^2}=0 \end{cases}$$

$A=(0,\sqrt{\log 3})$

$B=(0,-\sqrt{\log 3})$

$$\begin{cases} x=0 \\ y=\sqrt{\log 3} \\ \sqrt{\log 3}-3\lambda\sqrt{\log 3}=0 \iff \lambda=\frac{1}{3} \end{cases}$$

$$\begin{cases} x=0 \\ y=-\sqrt{\log 3} \\ -\sqrt{\log 3}+3\sqrt{\log 3}\,\lambda=0 \iff \lambda=\frac{1}{3} \end{cases}$$

$$(S_2)\quad \begin{cases} \lambda e^{x^2}=1 \\ y-\lambda y\, e^{y^2}=0 \\ e^{x^2}+e^{y^2}=4 \end{cases}$$

$\boxed{\underline{y=0}}$

$$\lambda \quad \begin{cases} \lambda e^{x^2} = 1 \\ \lambda e^{y^2} = 1 \\ e^{x^2} + e^{y^2} = 4 \end{cases}$$

$$\begin{cases} \lambda e^{x^2} = 1 \\ \lambda e^{y^2} = 1 \\ 1 + 1 = 4\lambda \iff 4\lambda = 2 \iff \lambda = \frac{1}{2} \end{cases}$$

$$\begin{cases} e^{x^2} = 2 \\ e^{y^2} = 2 \\ \lambda = \frac{1}{2} \end{cases} \iff \begin{cases} x^2 = \log 2 \\ y^2 = \log 2 \\ \lambda = \frac{1}{2} \end{cases}$$

$$\begin{cases} x = -\sqrt{\log 2} \\ y = \sqrt{\log 2} \\ \lambda = 1/2 \end{cases} \cup \begin{cases} x = -\sqrt{\log 2} \\ y = -\sqrt{\log 2} \\ \lambda = 1/2 \end{cases}$$

$$\cup \begin{cases} x = \sqrt{\log 2} \\ y = -\sqrt{\log 2} \\ \lambda = 1/2 \end{cases} \cup \begin{cases} x = \sqrt{\log 2} \\ y = \sqrt{\log 2} \\ \lambda = 1/2 \end{cases}$$

4 punti critici :

$$\left(-\sqrt{\log 2}, \sqrt{\log 2}\right), \left(-\sqrt{\log 2}, -\sqrt{\log 2}\right)$$

$$\left(\sqrt{\log 2}, -\sqrt{\log 2}\right), \left(\sqrt{\log 2}, \sqrt{\log 2}\right)$$

$$(\sqrt{\log 3}, 0), \quad (-\sqrt{\log 3}, 0)$$

$$(0, \sqrt{\log 3}), \quad (0, -\sqrt{\log 3})$$

$$f(\pm\sqrt{\log 3}, 0) = f(0, \pm\sqrt{\log 3})$$

$$= \log 3$$

$$f(\underbrace{\pm\sqrt{\log 2}, \pm\sqrt{\log 2}}) = 2\log 2 = \log 4$$

$$(\pm\sqrt{\log 2}, \pm\sqrt{\log 2})$$ massimo vincolato

ES Determinare gli estremi di

$f = x^2 y$, sul vincolo

$x^2 + y^2 = 1$