$$\text{P. Cauchy}\begin{cases} y'' = f(x, y, y') \\ y(x_0) = y_0 \\ y'(x_0) = y'_0 \end{cases}$$

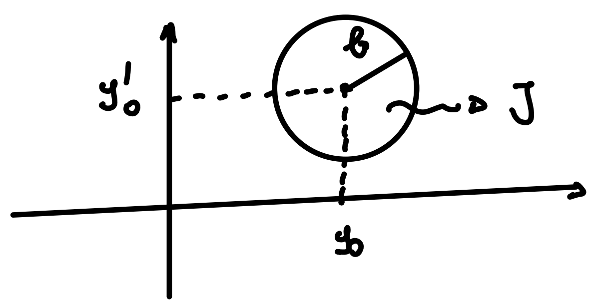
Eq. II° ordine

$x_0, y_0, y'_0 \in \mathbb{R}$

$I = [x_0 - a, x_0 + a]$

$(y_0, y'_0) \in \mathbb{R}^2$ $\qquad a, b > 0$

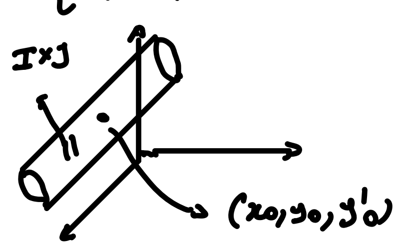
$J = \{ (z, t) \in \mathbb{R}^2 : \| (z,t) - (y_0, y'_0) \| \leq b \}$



$f(x, y, z)$

1) f continua in $I \times J$

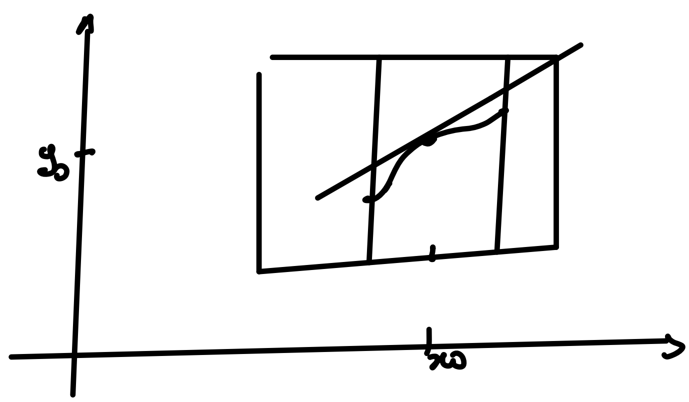
$I \times J$? $= \{ (x, y, z) : x \in I ; (y, z) \in J \}$



2) f LIPSCHITZIANA RISPETTO A (y,z), uniformemente rispetto a x:

$$\exists L>0: \ |f(x,y_1,z_1) - f(x,y_2,z_2)| \leq$$
$$\leq L \ \|(y_1,z_1) - (y_2,z_2)\|$$
$$\forall x \in I, \quad \forall (y_1,z_1), (y_2,z_2) \in J$$

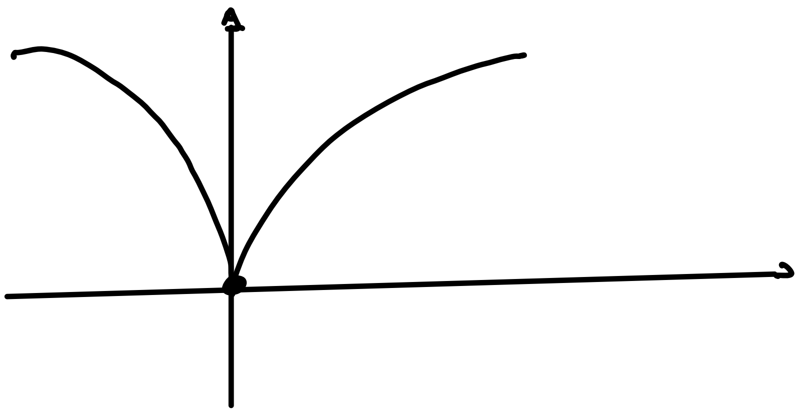
TESI $\Rightarrow$ $\exists\, \delta>0$, $\delta<a$, $\exists!\ y:[x_0-\delta, x_0+\delta] \to \mathbb{R}$ derivabile 2 volte in $[x_0-\delta, x_0+\delta]$ che risolve in tale intervallo il p. di Cauchy.



ES.

$$\begin{cases} y' = 2\sqrt{|y|} \\ y(0) = 0 \end{cases}$$

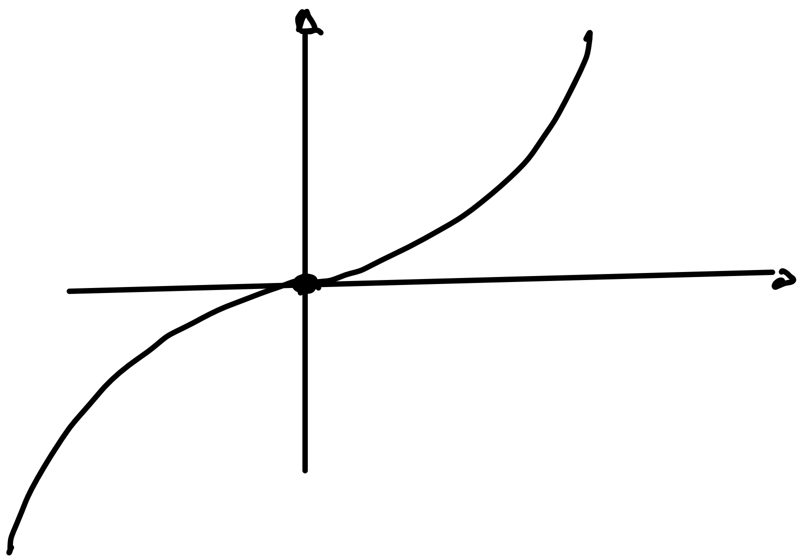
$f(x,y) = 2\sqrt{|y|}$ non è Lipschitz (PROVARE!)



$y \equiv 0$ è soluzione !

$$y(x) = x|x| = \begin{cases} x^2 & \text{se } x \geq 0 \\ -x^2 & \text{se } x \leq 0 \end{cases}$$

$y(0) = 0$



$x>0$: $\quad y' = 2x$

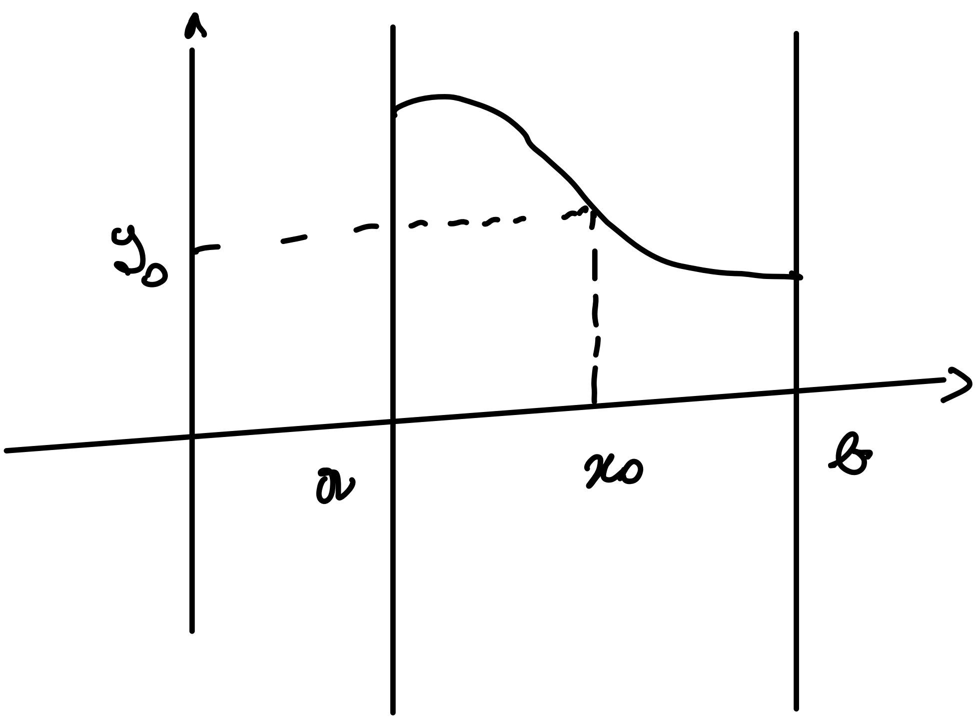
$2\sqrt{|y|} = 2\sqrt{x^2} = 2|x| = 2x$

$x<0$: $\quad y' = -2x$

$2\sqrt{|y|} = 2\sqrt{x^2} = 2|x| = -2x$

$y'(0) = 0 = 2\sqrt{|y(0)|}$

Teorema di esistenza e unicità globale (o in grande) di Cauchy



$f: [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$

1) f continua in $[a,b] \times \mathbb{R}$;

2) f Lipschitz rispetto ad $y \in \mathbb{R}$, uniformemente rispetto ad $x \in [a,b]$:

$\exists L > 0$:

$$|f(x,y_1) - f(x,y_2)| \leq L\,|y_1 - y_2|$$

$$\forall x \in [a,b], \quad \forall y_1, y_2 \in \mathbb{R}.$$

Allora: per ogni $x_0 \in [a,b]$ ed $y_0 \in \mathbb{R}$ $\exists !\ y = y(x)$

$y : [a,b] \to \mathbb{R}$ che risolve in tutto $[a,b]$ il problema

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

$$\begin{cases} y' = \dfrac{1+x^2}{1+y^2} = f(x,y) \\ y(1) = 1 \end{cases}$$

$y(x) = x \quad : \quad y'(x) = 1$

$$\frac{1+x^2}{1+y^2} = \frac{1+x^2}{1+x^2} = 1$$

Equazioni a variabili separabili

$$y' = f(x)\, g(y)$$

$$y' = y \qquad\qquad y' = y^2$$

$$y' = x \operatorname{arctg} y$$

$$\frac{dy}{dx} = f(x)\, g(y) \quad ; \quad g(y) \neq 0$$

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

$$G(y) = F(x) + c$$

<u>ES.</u> $\quad y' = y^2 \qquad \dfrac{dy}{dx} = y^2$

$$\int \frac{dy}{y^2} = \int dx \quad : \quad -\frac{1}{y} = x + c$$

$\forall c \in \mathbb{R}$

$y \neq 0$

$$\frac{1}{y} = c - x \quad : \quad \boxed{y = \frac{1}{c-x}} \quad x \neq c$$

$y = 0$ è soluzione dell'equazione!

$$\rightarrow \begin{cases} y = \dfrac{1}{c-x} \neq 0 \quad \forall c \in \mathbb{R}, \\ \underbrace{y = 0}_{\text{integrale singolare!}} \end{cases}$$

$$\begin{cases} y' = y^2 \\ y(0) = 0 \end{cases} \qquad 0 \neq y(0) = \frac{1}{c} \quad \underline{\text{ASSURDO}}$$

$y(x) = 0$ è <u>soluzione</u>

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases} \qquad 1 = y(0) = \frac{1}{c}$$

$$\Leftrightarrow \quad \underline{c = 1}$$

$$y(x) = \frac{1}{1-x}$$

$$y' = 1 + y^2 \qquad = f(x)\, g(y)$$

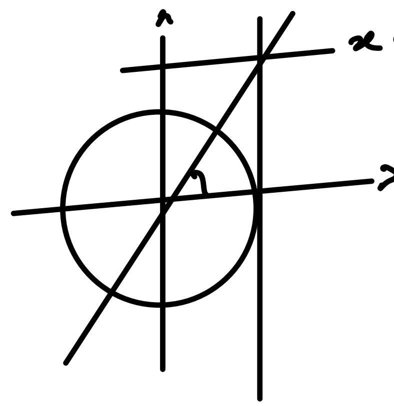
$$f(x) \equiv 1, \quad g(y) = 1 + y^2$$

$$\underbrace{\int \frac{dy}{1+y^2}} = \int dx$$

$$\left[\operatorname{arctg} y = x + c \right] \qquad c \in \mathbb{R}$$

$$y = tg(x+c) \quad , c \in \mathbb{R}$$

$$y' = \cos^2 y \qquad \frac{dy}{dx} = \cos^2 y$$



$$\cos y \neq 0$$

$$\int \underbrace{\frac{dy}{\cos^2 y}}_{?} = \int dx$$

$$tg\, y = x + c \qquad c \in \mathbb{R}$$

(1) $y = \text{arctg}(x+c) + k\pi \qquad k \in \mathbb{Z}, \; c \in \mathbb{R}$

$\cos y = 0 \iff$ (2) $y = \frac{\pi}{2} + k\pi$ soluzioni!

$$y' = 0 = \cos^2 y$$

integrali singolari