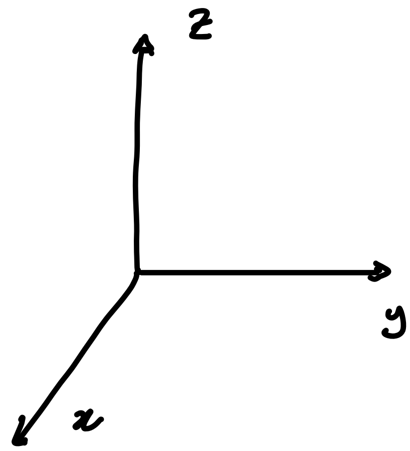
Integrazione tripla

$f(x,y,z)$ $\qquad E \subseteq \mathbb{R}^3$



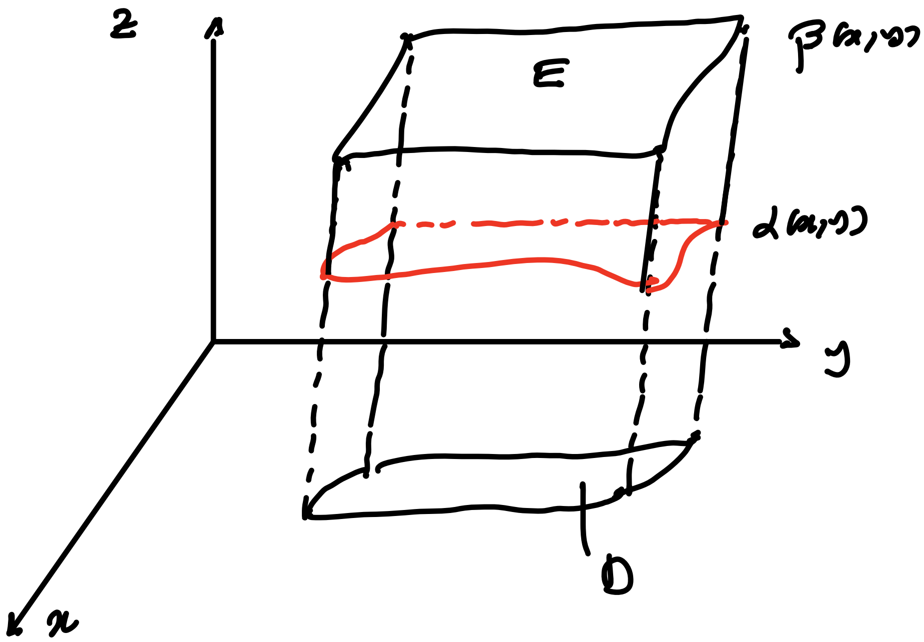
$xy, \; yz, \; xz$

Def. $E \subseteq \mathbb{R}^3$ dominio normale rispetto al piano xy se si può scrivere nella seguente forma:

$$E = \{ (x,y,z) \in \mathbb{R}^3 : \; (x,y) \in D, \; \alpha(x,y) \leq z \leq \beta(x,y) \}$$

$D \subseteq \mathbb{R}^2$ dominio regolare di $\mathbb{R}^2$; $\alpha(x,y)$, $\beta(x,y)$ funzioni continue su D e tali che

$$\alpha(x,y) \leq \beta(x,y), \qquad \forall (x,y) \in D$$



$$Vol(E) = m(E) = \iint_D \left[\beta(x,y) - \alpha(x,y)\right] dx\,dy$$

Dominio normale rispetto a yz:

$$E = \{ (x,y,z) \in \mathbb{R}^3 : (y,z) \in D, \; \alpha(y,z) \le x \le \beta(y,z) \}$$

$$Vol(E) = \iint_D \left[\beta(y,z) - \alpha(y,z) \right] dy\,dz$$

Dominio normale rispetto a xz:

$$E = \{ (x,y,z) \in \mathbb{R}^3 : (x,z) \in \tilde{D},$$

$$\alpha(x,z) \leq y \leq \beta(x,z) \}$$

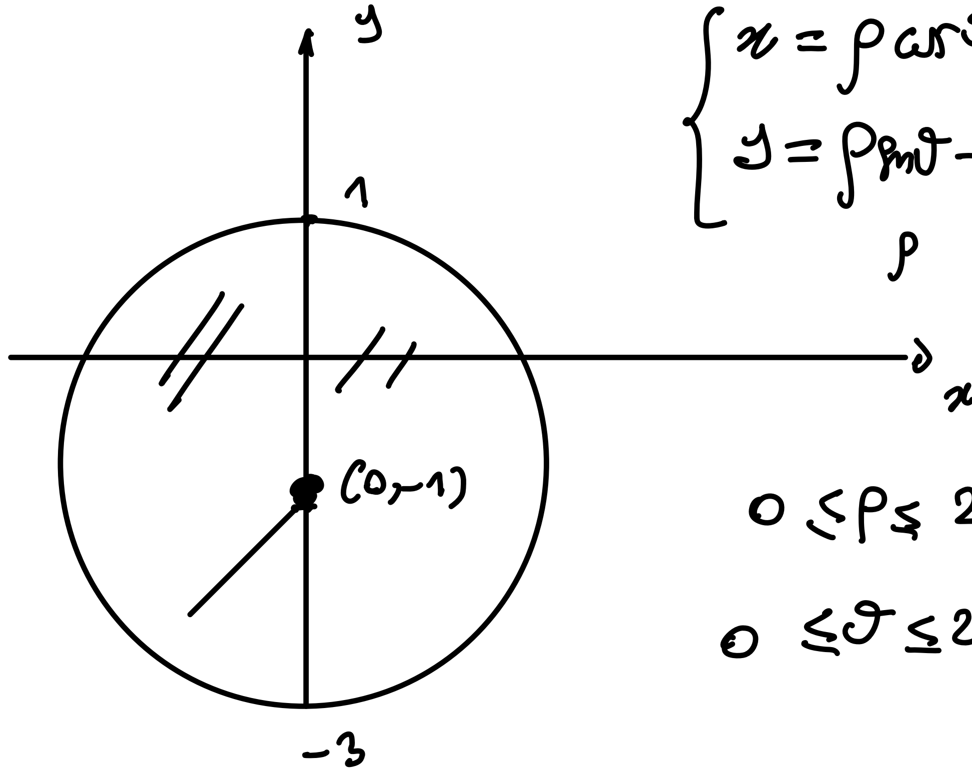
ES.

$$E = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ e}$$

$$\underbrace{x^2+y^2}_{\alpha} \leq z \leq \underbrace{3-2y}_{\beta} \} \quad xy$$

D = cerchio centrato in $(0,-1)$, di raggio 2

$$\mathrm{VOL}(E) = \iint_D \left(3 - 2y - x^2 - y^2\right) dx\,dy =$$



$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta - 1 \end{cases}$$

ρ

$0 \le \rho \le 2$

$0 \le \vartheta \le 2\pi$

$$= \iint_T \Big(3 - 2(\rho\sin\vartheta - 1) - \rho^2\cos^2\vartheta - (\rho\sin\vartheta - 1)^2 \Big)\rho\, d\rho\, d\vartheta$$

$$= \iint_T \Big[3 - \cancel{2\rho\sin\vartheta} + 2 - \rho^2\cos^2\vartheta - \rho^2\sin^2\vartheta + \cancel{2\rho\sin\vartheta} - 1 \Big]$$

$$= \iint_T \big[4 - \rho^2 \big]\, d\rho\, d\vartheta = \underline{\underline{\text{facile}}}$$

$$= \int_0^2 d\rho \int_0^{2\pi} (4-\rho^2)\, d\theta$$

Def. $E \subseteq \mathbb{R}^3$ dominio normale rispetto al piano xy.

Partizione o decomposizione di E in domini normali rispetto a xy

$\mathcal{P} = \{ E_1, E_2, \dots, E_N \}$ t.c.

a) E_i domini normali rispetto a xy;

b) $\mathring{E}_i \cap \mathring{E}_j = \emptyset \quad \forall i \neq j$

c) $\bigcup_{i=1}^{N} E_i = E$

$f = f(x,y,z)$, $\quad f: E \to \mathbb{R}$

limitata.

$\mathcal{P} = \{E_1, \dots, E_N\}$ dec. di E.

$\cup, \mathbb{R}^3$

$$s(f, \mathcal{P}) = \sum_{i=1}^{N} \inf_{E_i} f \cdot m(E_i)$$

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} \sup_{E_i} f \cdot m(E_i)$$

$\forall\, \mathcal{P}_1, \mathcal{P}_2$ decomposizioni di E

$$s(f, P_1) \le S(f, P_2)$$

$$\{ s(f,P) : P \}$$

$$\{ S(f,P) : P \}$$

separati

<u>Def</u> Se sono contigui, si dice che f è integrabile secondo Riemann su E e scriviamo

$$\underbrace{\iiint_E f(x,y,z)\,dx\,dy\,dz}_{\text{integrale triplo di } f \text{ su } E} := \sup_P s(f,P) = \inf_P S(f,P)$$

integrale triplo di f su E.

Prop. Se f è continua su E, f è integrabile.

Formule di riduzione (per gli integrali tripli)

$$E = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in D , \\ \alpha(x,y) \leq z \leq \beta(x,y) \} .$$

Allora, se $f : E \longrightarrow \mathbb{R}$ continua, si ha:

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iint_D dx\,dy \int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz$$

Se E normale rispetto a yz :

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iint_D dy\,dz \int_{\alpha(y,z)}^{\beta(y,z)} f(x,y,z)\,dx$$

Se E normale rispetto a xz:

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iint_D dx\,dz \int_{\alpha(x,z)}^{\beta(x,z)} f(x,y,z)\,dy$$

Es. $\iiint_E e^{x^2}\,dx\,dy\,dz$

$$E = \{ (x,y,z) \in \mathbb{R}^3 : \ 0 \le x \le 1,\ 0 \le y \le x,\ -1 \le z \le 4 \}$$

normale rispetto a (x,z)

$$(x,z) \in D = [0,1] \times [-1,4]$$

$$\iiint_E e^{x^2} dx\,dy\,dz = \iint_D dx\,dz \int_0^x e^{x^2} dy =$$

$$= \iint_D e^{x^2} \cdot x \, dx\,dz =$$

$$= \int_0^1 dx \int_{-1}^4 x e^{x^2} dz =$$

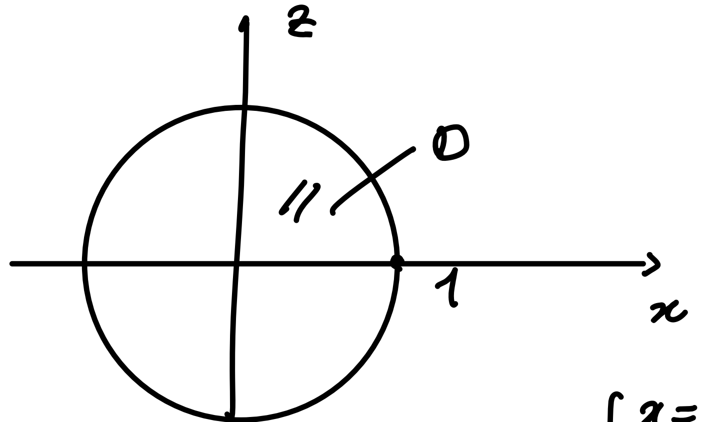
$$= \frac{5}{2} \int_0^1 2x e^{x^2} dx = \frac{5}{2} \left(e^{x^2} \right)_0^1$$

$$= \frac{5}{2} (e - 1)$$

2)

$$\iiint_E x\,dx\,dy\,dz \quad , \quad E=\{(x,y,z)\in\mathbb{R}^3 : z^2+x^2\le 1, \ 0\le y\le 1-x-z\}$$

$$D=\{(x,z)\in\mathbb{R}^3 : x^2+z^2\le 1\}$$



$0\le\rho\le 1$

$0\le\vartheta\le 2\pi$

$$\begin{cases} x=\rho\cos\vartheta \\ z=\rho\sin\vartheta \end{cases}$$

$$= \iint_D dx\,dz \int_0^{1-x-z} x\,dy =$$

$$= \iint_D x\,(1-x-z)\,dx\,dz =$$

$$= \iint_T \rho\cos\vartheta\,(1-\rho\cos\vartheta-\rho\sin\vartheta)\cdot\rho\, d\rho\, d\vartheta$$

$$= \iint_T \rho\,(\rho\cos\vartheta-\rho^2\cos^2\vartheta-\rho^2\sin\vartheta\cos\vartheta)\, d\rho\, d\vartheta$$

$$= \int_0^1 d\rho \int_0^{2\pi} [\rho^2\cos\vartheta-\rho^3\cos^2\vartheta-\rho^3\sin\vartheta\cos\vartheta]\, d\vartheta$$

$$= -\int_0^1 \rho^3 d\rho \int_0^{2\pi} \cos^2\vartheta\, d\vartheta \quad = \dots$$
