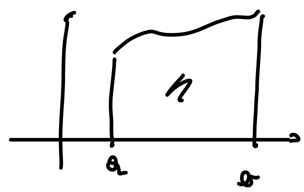
Lezione del 13/12/2012

$f = f(x,y)$ $\quad f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, D regolare , f continua

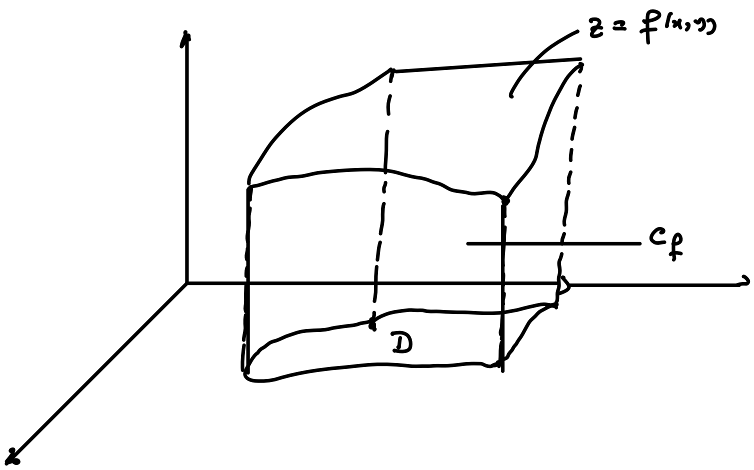
Se $f \geq 0$ $\quad \iint_D f(x,y)\,dxdy$? 3D

$f = f(x)$ $\quad \int_a^b f(x)\,dx \overset{?}{=}$

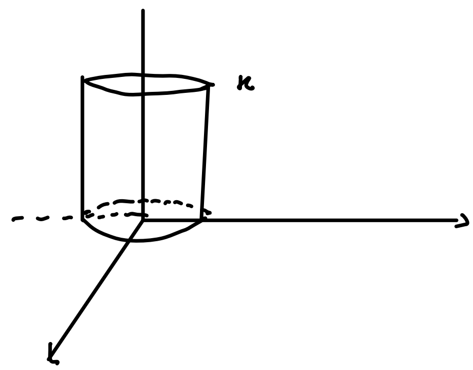


$$\iint_D f(x,y)\,dxdy = \mathrm{VOL}\,(C_f)$$ C_f il cilindroide relativo ad f

$$C_f = \{ (x,y,z) \in \mathbb{R}^3 : \ (x,y) \in D , \ 0 \leq z \leq f(x,y) \}$$



ES. $f(x,y) = K$, $K > 0$, $D = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \}$

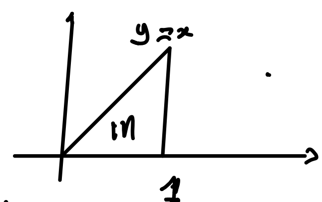


ES. 1 Calcolare il volume del cilindroide relativo alla funzione

$$f(x,y) = x^2 y$$

di base il dominio $D = \{ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \ 0 \leq y \leq x \}$

$$\mathrm{Vol}(C_f) = \iint_D f(x,y)\, dx dy$$



$$= \int_0^1 dx \int_0^x x^2 y \, dy = \int_0^1 x^2 \left(\frac{y^2}{2} \right) \Bigg|_{y=0}^{y=x} dx = \frac{1}{2} \int_0^1 x^4 \, dx$$

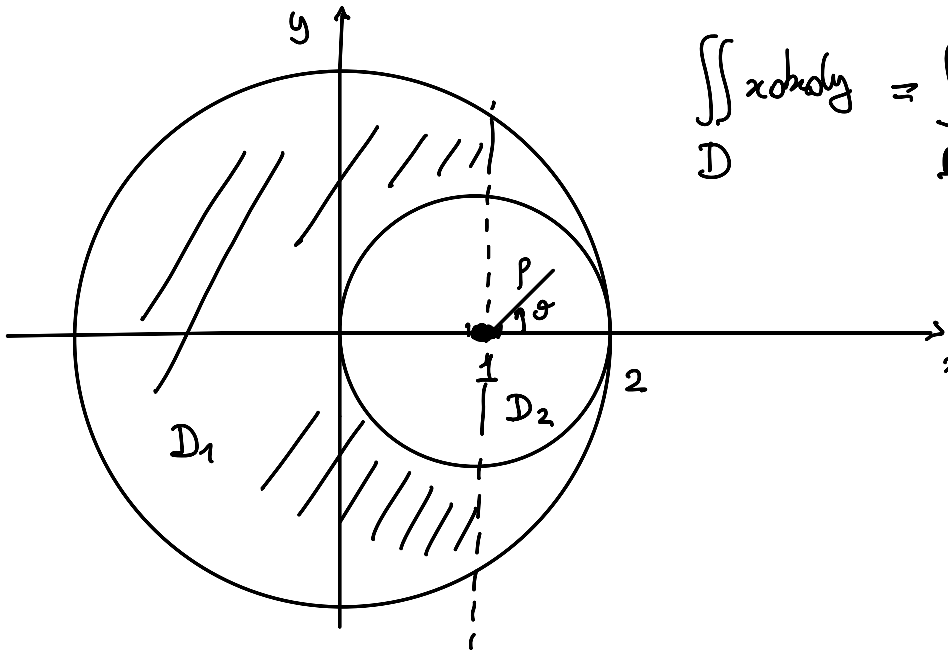
$$= \frac{1}{10}.$$

ES. 1. D sia la regione piana interna al cerchio D_1 di raggio 2 e centro $(0,0)$ ed esterna al cerchio D_2 di raggio 1 e centro $(1,0)$.

Calcolare

$$\iint_D x \, dx dy$$

$$D = D_1 \setminus D_2$$



$$\iint_D x\,dx\,dy = \iint_{D_1} - \; - \iint_{D_2} -$$

$0 \le \rho \le 2$

$0 \le \vartheta \le 2\pi$

$$\iint_{D_1} x\,dx\,dy$$

$$= \iint_T \rho\cos\vartheta \cdot \rho\, d\rho\, d\vartheta = \iint_T \rho^2 \cos\vartheta\, d\rho\, d\vartheta$$

$$= \int_0^2 \rho^2\, d\rho \underbrace{\int_0^{2\pi} \cos\vartheta\, d\vartheta}_{\shortparallel \atop 0} = 0$$

↓

Su D_2 :
$$\begin{cases} x = 1 + \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases}$$

$0 \le \vartheta \le 2\pi$

$0 \le \rho \le 1$

ρ

$$\iint_{D_2} x\,dxdy = \int_0^1 d\rho \int_0^{2\pi} (1+\rho\cos\vartheta)\,\rho\, d\vartheta =$$

$$= \int_0^1 d\rho \int_0^{2\pi} (\rho + \rho^2\cos\vartheta)\, d\vartheta =$$

$$= \int_0^1 \left[2\pi\rho + \rho^2 \underbrace{(\sin\vartheta)_0^{2\pi}}_{=0}\right] d\vartheta = 2\pi\int_0^1 \rho\, d\rho = \pi.$$

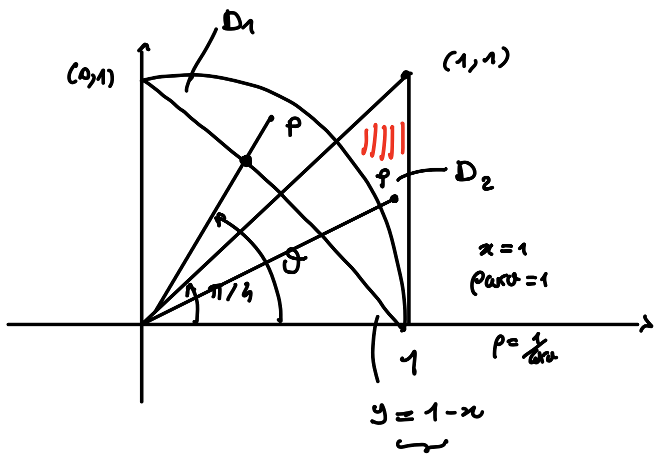
<u>Da svolgere</u>: $\displaystyle\iint_D \frac{xy}{1+(x^2+y^2)^2}\, dxdy$, dove

$$D = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 2,\ x\ge 0,\ y\ge 0\}$$

2.

$$\iint_D \frac{x}{x^2+y^2}\, dxdy$$

$$= \iint_{D_1} - \quad + \iint_{D_2} -$$



Su D_1: $\frac{\pi}{4} \le \vartheta \le \frac{\pi}{2}$; $\quad \underline{\bar\rho} \le \rho \le 1 \qquad \bar\rho\ ?$

$$\frac{1}{\sin\vartheta+\cos\vartheta} \le \rho \le 1 \qquad \rho\sin\vartheta = 1-\rho\cos\vartheta \iff \rho(\sin\vartheta+\cos\vartheta) = 1$$

$$\rho = \frac{1}{\sin\theta + \cos\theta}$$

$$\iint_{D_1} \frac{x}{x^2+y^2}\,dx\,dy = \iint_{T_1} \frac{\cancel{\rho}\cos\theta}{\cancel{\rho^2}}\,\cancel{\rho}\,d\rho\,d\theta = \int_{\pi/4}^{\pi/2} d\theta \int_{\frac{1}{\sin\theta+\cos\theta}}^{1} \cos\theta\,d\rho$$

$$= \int_{\pi/4}^{\pi/2} \cos\theta\left(1 - \frac{1}{\sin\theta+\cos\theta}\right)d\theta = \underbrace{\int_{\pi/4}^{\pi/2} \cos\theta\,d\theta}_{\text{facile}} - \underbrace{\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta}_{?}$$

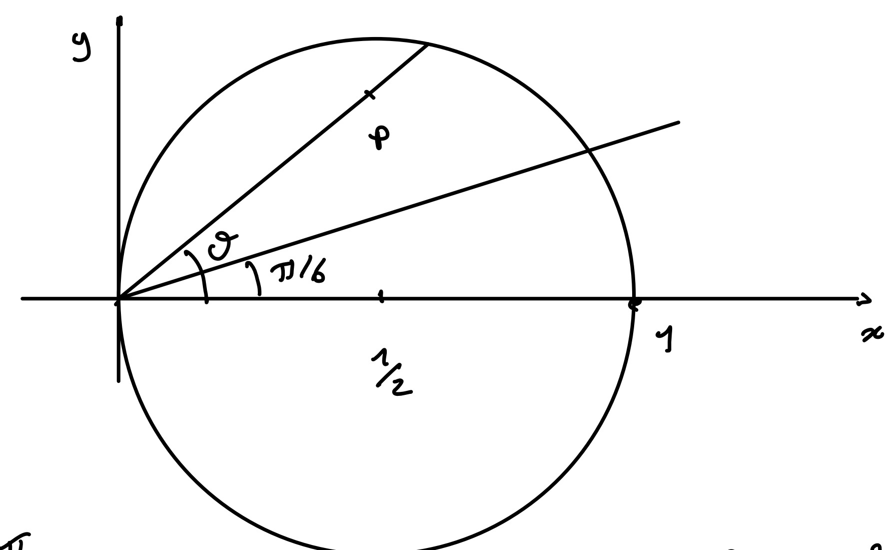
f.le parametriche ...

Su D_2: $0 \le \theta \le \frac{\pi}{4}$; $1 \le \rho \le \frac{1}{\cos\theta}$

$$\iint_{D_2} \frac{x}{x^2+y^2}\,dx\,dy = \iint_{T_2} \cos\theta\,d\rho\,d\theta = \int_0^{\pi/4} d\theta \int_1^{\frac{1}{\cos\theta}} \cos\theta\,d\rho$$

$$= \int_0^{\pi/4} \cos\theta\left(\frac{1}{\cos\theta} - 1\right)d\theta = \frac{\pi}{4} - \left(\sin\theta\right)_0^{\pi/4} = \frac{\pi}{4} - \frac{\sqrt{2}}{2}.$$

3.



$$\iint_D \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$$

$\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$ $0 \le \rho \le \cos\theta$

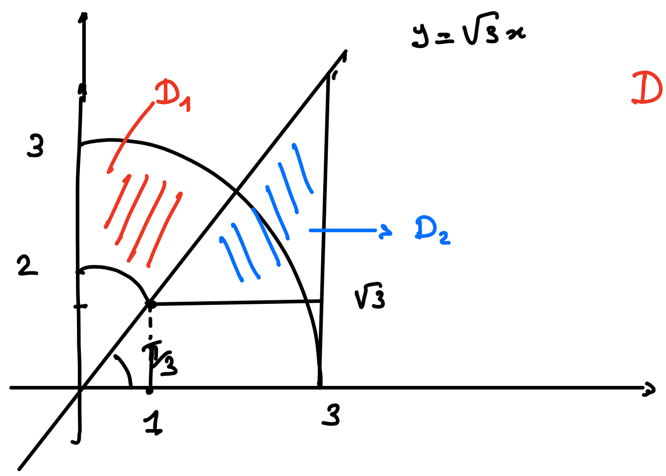
$$= \iint_T \frac{\rho}{\sqrt{1-\rho^2}}\, d\rho\, d\theta = \int_{\pi/6}^{\pi/2} d\theta \int_0^{\cos\theta} \frac{\rho}{\underbrace{\sqrt{1-\rho^2}}}\, d\rho$$

$$= -\frac{1}{\cancel{2}} \int_{\pi/6}^{\pi/2} \frac{\cancel{2}}{3} \Big[(1-\rho^2)^{3/2} \Big]_{\rho=0}^{\rho=\cos\theta} d\theta =$$

$$= -\frac{1}{3} \int_{\pi/6}^{\pi/2} \Big[(1-\cos^2\theta)^{3/2} - 1 \Big] d\theta$$

$$= -\frac{1}{3} \int_{\pi/6}^{\pi/2} \Big[\sin^3\theta - 1 \Big] d\theta$$

9.



$D = D_1 \cup D_2$

$y = x\sqrt{3}$

$= \sqrt{3}x$

$tg\,\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

$$\iint_D \frac{1}{y}\, dx\,dy = \iint_{D_1} - + \iint_{D_2} -$$

Su D_1: $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$; $2 \le \rho \le 3$: $\displaystyle\iint_{D_1} \frac{1}{y}\, dx\,dy = \iint_{T_1} \frac{\cancel{\rho}}{\cancel{\rho}\sin\theta}\, d\rho\, d\theta$

$$= \iint_{T_1} \frac{1}{\sin\vartheta}\, d\rho\, d\vartheta = \int_2^3 d\rho \int_{\pi/3}^{\pi/2} \frac{d\vartheta}{\sin\vartheta} \qquad \int \frac{1}{\sin\vartheta}\, d\vartheta$$

$\sin 2\left(\frac{\vartheta}{2}\right) =$

$$\begin{cases} x^2+y^2 = 4 \\ y = \sqrt{3}x \end{cases} \iff \begin{cases} x^2 + 3x^2 = 4 \\ y = \sqrt{3}x \end{cases} \iff \begin{cases} 4x^2 = 4 \\ y = \sqrt{3}x \end{cases} \iff \begin{cases} x^2 = 1 \\ y = \sqrt{3}x \end{cases}$$

$$\begin{cases} x = 1 \\ y = \sqrt{3} \end{cases} \qquad \begin{cases} x = -1 \\ y = -\sqrt{3} \end{cases}$$

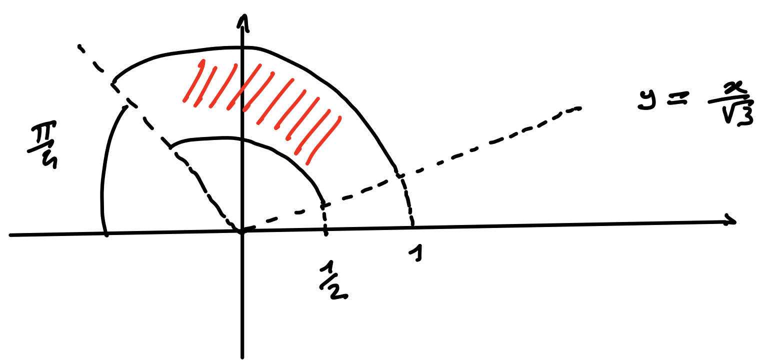
Su D_2: $\quad 1 \le x \le 2, \quad \sqrt{3} \le y \le \sqrt{3}x$

$$\iint_{D_2} \frac{1}{y}\, dx\, dy = \int_1^2 dx \int_{\sqrt{3}}^{\sqrt{3}x} \frac{1}{y}\, dy = \int_1^2 \Big[\log|y|\Big]_{y=\sqrt{3}}^{y=\sqrt{3}x} dx =$$

$$= \int_1^2 \left[\log \sqrt{3}\,|x| - \log\sqrt{3}\right] dx = \int_1^2 \left[\log(\sqrt{3}\,x) - \frac{1}{2}\log 3\right] dx \,..$$

... facile

Da svolgere



$$\iint_D \frac{\arcsin^2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\, dx\, dy \qquad \left[\text{R.: } \frac{7\pi}{72}\left(\pi^2 - \pi\sqrt{3} - 6\right)\right]$$