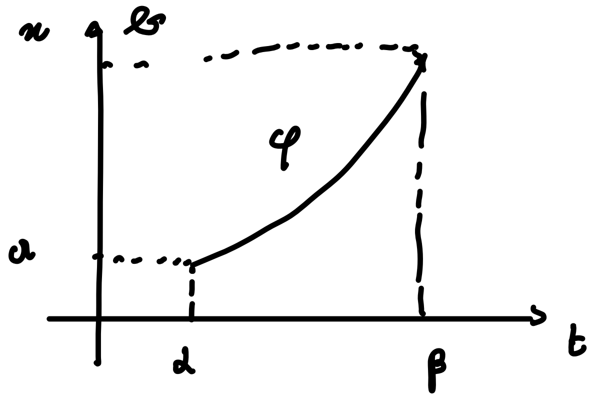
Formula del cambiamento di variabili negli integrali doppi

$f:[a,b] \longrightarrow \mathbb{R}$ continua ; $\quad f = f(x)$

$\varphi = \varphi(t)$, $\quad \varphi : [\alpha, \beta] \longrightarrow [a,b]$

$\qquad t \in \quad \longrightarrow \quad \varphi(t) = x \in [a,b]$

invertibile , $\varphi \in C^1$, $\varphi' \neq 0$

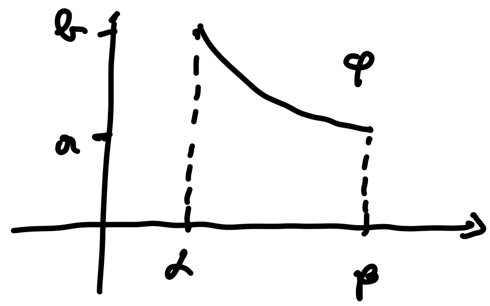


$\varphi' > 0$

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,|\varphi'(t)|\,dt$$

$x = \varphi(t)$: $dx = \varphi'(t)\,dt$; $x = a \longrightarrow t = \alpha$; $x = b \rightarrow t = \beta$

φ è strett. decrescente :



$$\int_a^b f(x)dx = \int_\beta^\alpha f(\varphi(t))\,\varphi'(t)dt$$

$x = a \to t = \beta$

$x = b \to t = \alpha$

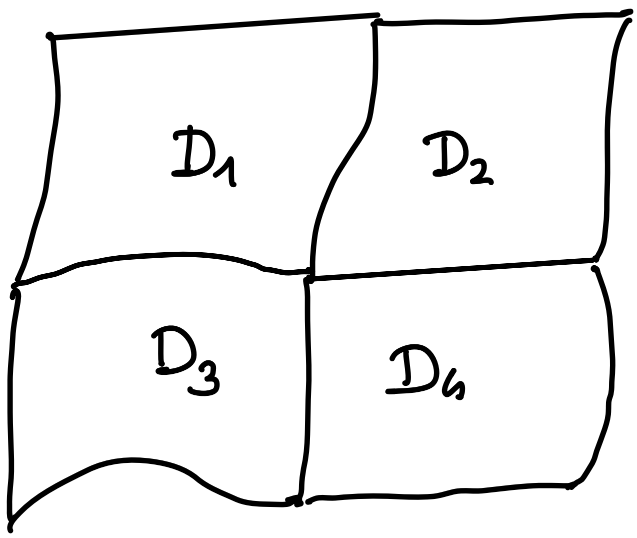
$$= -\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = \int_\alpha^\beta f(\varphi(t))\,|\varphi'(t)|dt$$

$f = f(\underline{\underline{x}}, \underline{\underline{y}})$?

<u>**Def**</u>- <u>Dominio regolare di $\mathbb{R}^2$</u> : un dominio regolare di $\mathbb{R}^2$ è un insieme $D \subseteq \mathbb{R}^2$ che sia unione di un numero finito di domini normali regolari ($D = \{ (x,y) : x \in [a,b];\ \alpha(x) \le y \le \beta(x) \}$

$\alpha, \beta \in C^1([a,b])$)

a due a due privi di punti interni in comune:

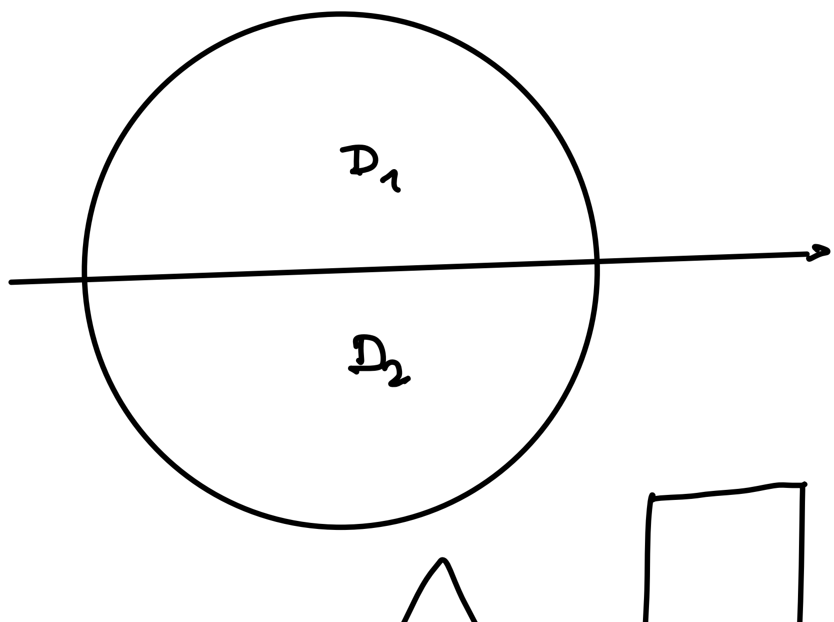


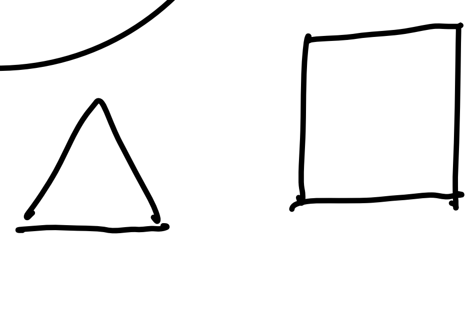
$\mathring{D}_i \cap \mathring{D}_j = \emptyset$

$\forall i \neq j$

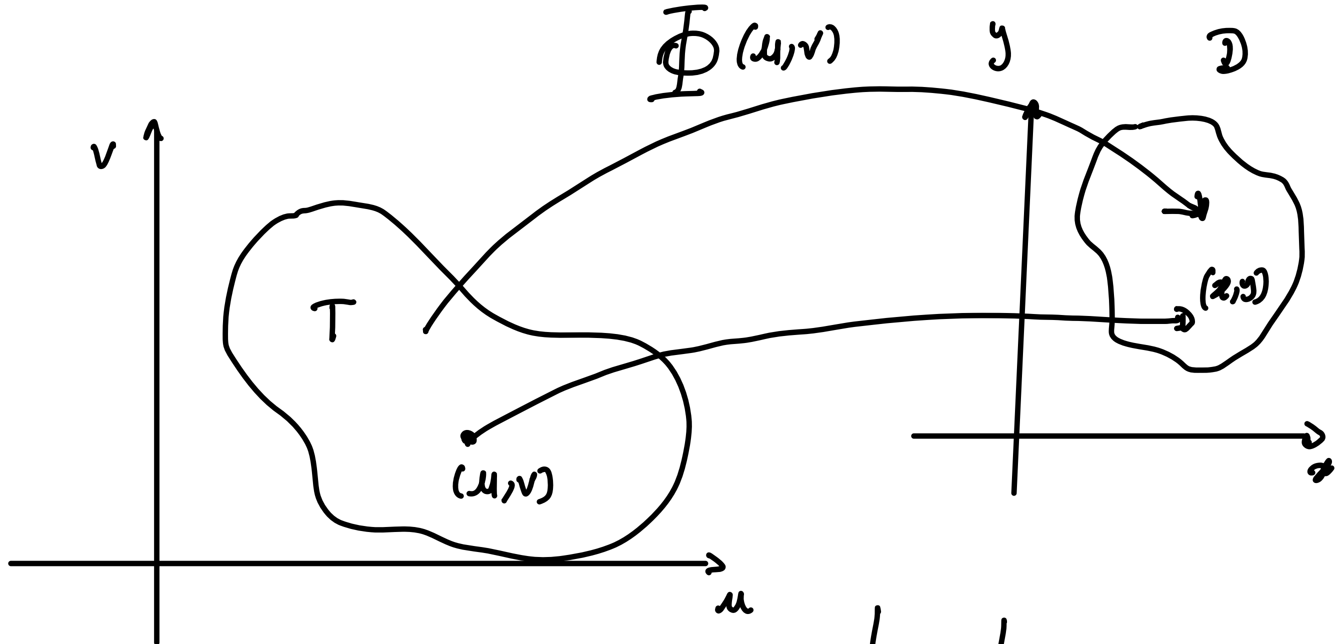
$\bigcup_{i=1}^{4} D_i = D$

∂D





$D \subseteq \mathbb{R}^2_{x,y}$ $\qquad$ $T \subseteq \mathbb{R}^2_{u,v}$



$$\Phi : \underset{\sim}{(u,v) \in T} \longrightarrow \underset{\sim}{(x(u,v), y(u,v)) \in D}$$

x, y f.mi di u,v

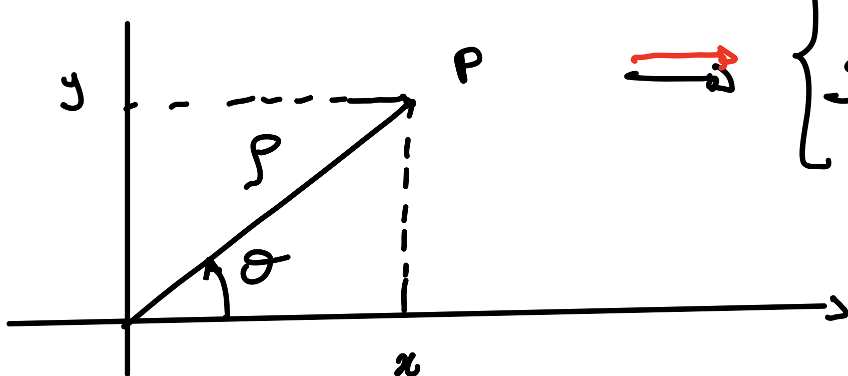
$\Phi \in C^1(T)$

$$\Phi \equiv \begin{cases} x = x(u,v) & \cdot \\ y = y(u,v) & \cdot \end{cases} \in C^1$$

Determinante jacobiano di Φ :

$$\det \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = |\overbrace{J_\Phi}^{\text{matrice jacobiana}}|$$

ES. Trasformazione alle coordinate polari



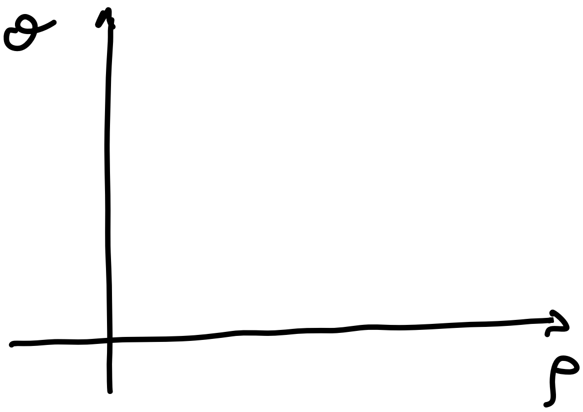
$$\Longrightarrow \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

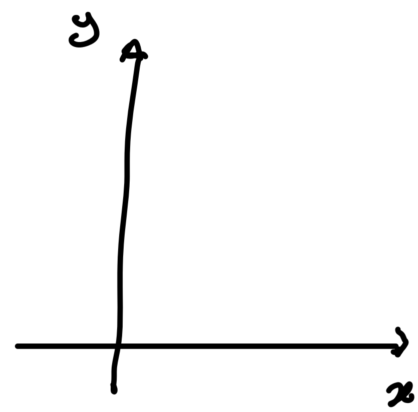
(ρ, θ) coord. polari

$u = \rho, \quad v = \theta$

$$\Phi : (\rho, \theta) \in \mathbb{R}^2 \longrightarrow (\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2$$

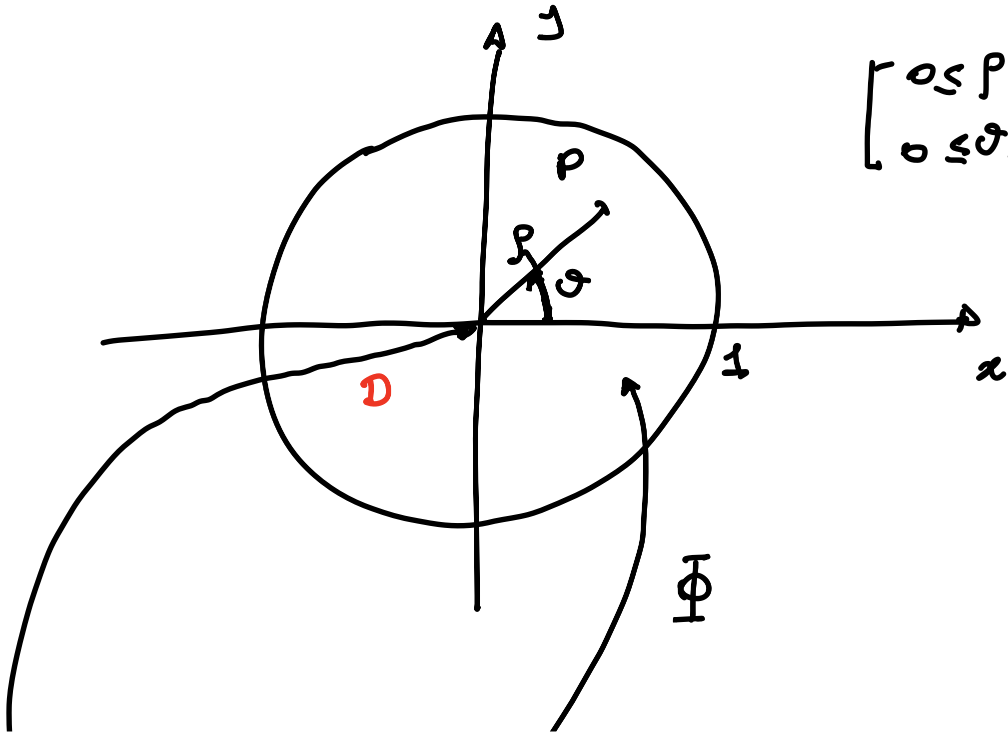
trasf. alle coordinate polari



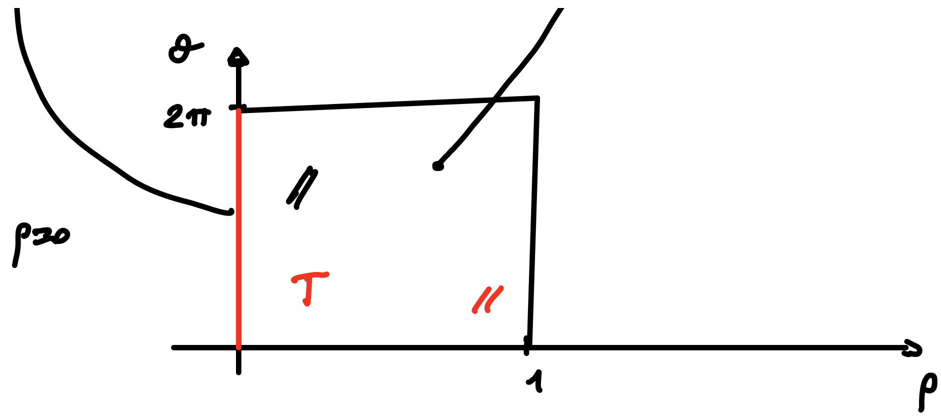


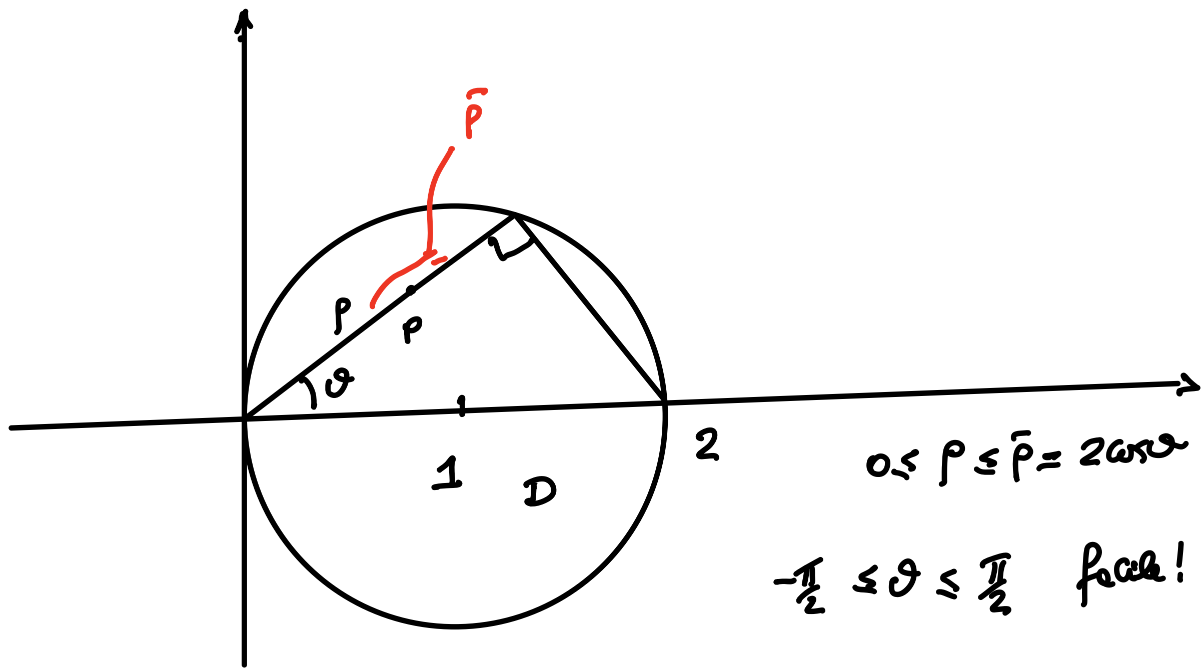
$$\det \frac{\partial(x,y)}{\partial(\rho,\vartheta)} = \begin{vmatrix} x_\rho & x_\vartheta \\ y_\rho & y_\vartheta \end{vmatrix} = \begin{vmatrix} \cos\vartheta & -\rho\sin\vartheta \\ \sin\vartheta & \rho\cos\vartheta \end{vmatrix}$$

$$= \rho \leftarrow$$



$$\begin{bmatrix} 0 \le \rho \le 1 \\ 0 \le \vartheta \le 2\pi \end{bmatrix}$$





$0 \le \rho \le \bar{\rho} = 2\cos\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ facile!

$$(x-1)^2 + y^2 = 1$$

$$T = \left\{ \begin{array}{l} 0 \le \rho \le 2\cos\theta \\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{array} \right\}$$

$$\bar{\rho} = 2\cos\theta$$

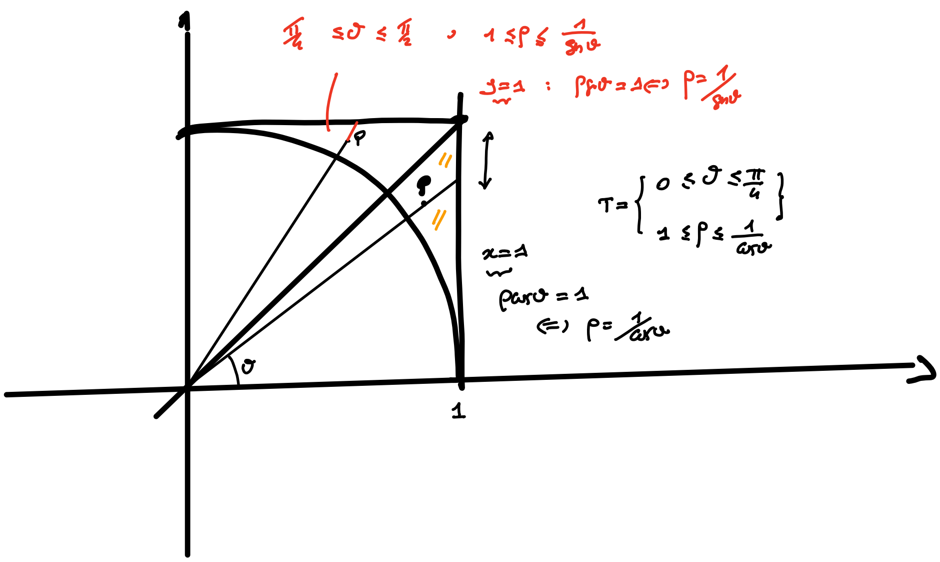
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} :$$

$$(\rho\cos\theta - 1)^2 + \rho^2\sin^2\theta = 1$$

$$\rho^2\cos^2\theta - 2\rho\cos\theta + \cancel{1} + \rho^2\sin^2\theta = \cancel{1}$$

$$\rho^2 = 2\cancel{\rho}\cos\theta$$

$$\underbrace{\rho = 2\cos\theta}_{\text{equazione polare della circonferenza}}$$



<u>Teorema</u> $D \subseteq \mathbb{R}^2_{x,y}$, $T \subseteq \mathbb{R}^2_{u,v}$ regolari.

$\underline{\Phi} : T \longrightarrow D$ di classe C^1, biunivoca e

tale che $\det \dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$, $\forall (u,v) \in T$.

Allora, se $f = f(x,y)$, $f : D \to \mathbb{R}$ è continua, Si ha la formula $\qquad |\varphi'(t)|$

$$\left[\iint_D f(x,y)\,dx\,dy = \iint_T f(x(u,v),y(u,v))\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv\right]$$

OSS. Se consideriamo le coordinate polari:

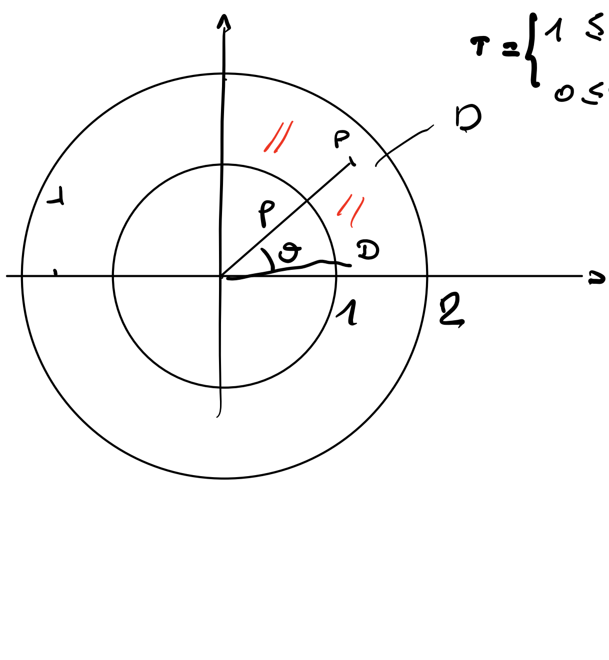
Φ non è lineare!

$|J_\Phi| = \rho = 0$ se $\rho = 0$

1)

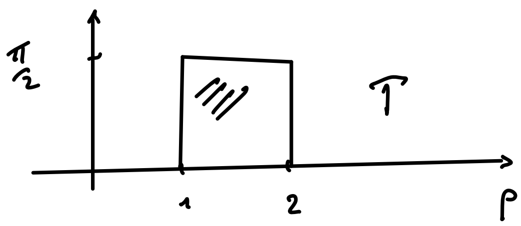
$$\iint_D \arcsin\frac{y}{\sqrt{x^2+y^2}}\,dx\,dy = \qquad \begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases}$$

$$T = \begin{Bmatrix} 1 \le \rho \le 2 \\ 0 \le \vartheta \le \frac{\pi}{2} \end{Bmatrix}$$



$$= \iint_T \left(\arcsin \frac{\rho \sin\theta}{\sqrt{\underbrace{\rho^2\cos^2\theta + \rho^2\sin^2\theta}_{\rho^2}}}\right)\rho \, d\rho \, d\theta =$$

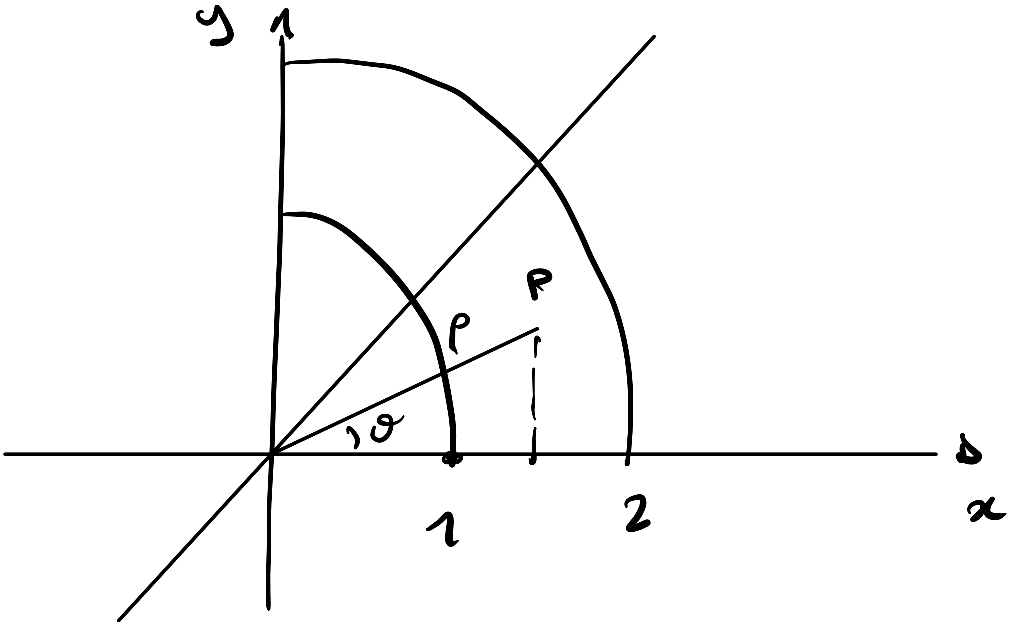
$$= \iint_T \left(\arcsin \frac{\cancel{\rho} \sin\theta}{\cancel{\rho}}\right)\rho \, d\rho \, d\theta = \iint_T \rho\theta \, d\rho \, d\theta = \text{f.la di riduzione}$$



$$= \int_1^2 d\rho \int_0^{\frac{\pi}{2}} \rho\theta \, d\theta = \int_1^2 \rho \, d\rho \int_0^{\frac{\pi}{2}} \theta \, d\theta = \left(\frac{\rho^2}{2}\right)_{\rho=1}^{\rho=2} \left(\frac{\theta^2}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}(4-1)\left(\frac{\pi^2}{4}\right) = \frac{3}{16}\pi^2$$

2) $$\iint_D \frac{y^2}{x^4}\,dx\,dy =$$



3) $$\iint_D (\underbrace{x^2+y^2}) \,dx\,dy = \iint_T \rho^2 \cdot \rho \, d\rho \, d\theta = \iint_T \rho^3 \, d\rho \, d\theta$$

normale rispetto a θ

$\downarrow$

$$T = \left\{ \begin{array}{l} 0 \le \theta \le \frac{\pi}{3} \\ 1 \le \rho \le 2\cos\theta \end{array} \right\}$$

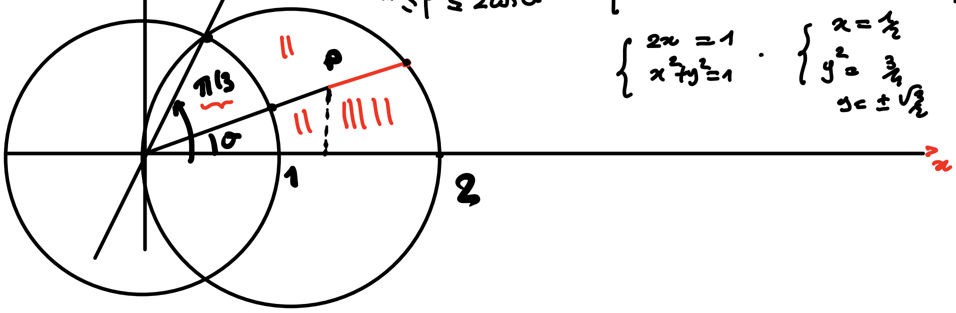
$$\begin{cases} x^2+y^2=1 \\ (x-1)^2+y^2=1 \end{cases}$$

$$\begin{cases} x^2-(x-1)^2=0 \\ x^2+y^2=1 \end{cases}$$

$$\begin{cases} 2x=1 \\ x^2+y^2=1 \end{cases} \quad \cdot \quad \begin{cases} x=\frac{1}{2} \\ y^2=\frac{3}{4} \\ y=\pm\frac{\sqrt{3}}{2} \end{cases}$$

$$\begin{cases} \cos\theta=\frac{1}{2} \\ \sin\theta=\frac{\sqrt{3}}{2} \end{cases}$$

$$\theta=\frac{\pi}{3}$$



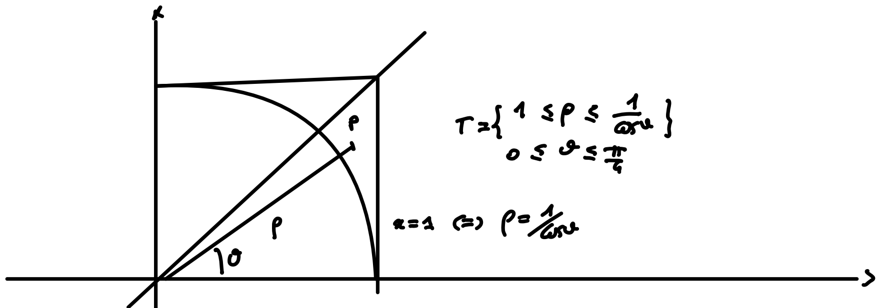
$$\iint_T \rho^3 \, d\rho \, d\theta = \text{formule di riduzione} = \int_0^{\pi/3} d\theta \int_1^{2\cos\theta} \rho^3 \, d\rho =$$

$$= \int_0^{\pi/3} \left(\frac{\rho^4}{4}\right)_1^{2\cos\theta} d\theta = \frac{1}{4} \cdot \int_0^{\pi/3} (16\cos^4\theta - 1) \, d\theta$$

$\int \cos^4\theta = \dots ?$

$\cos^2\theta \cos^2\theta$

$\cos^2\theta (1 - \sin^2\theta) = \dots$



$T = \left\{ 1 \le \rho \le \frac{1}{\cos\theta} ;\ 0 \le \theta \le \frac{\pi}{4} \right\}$

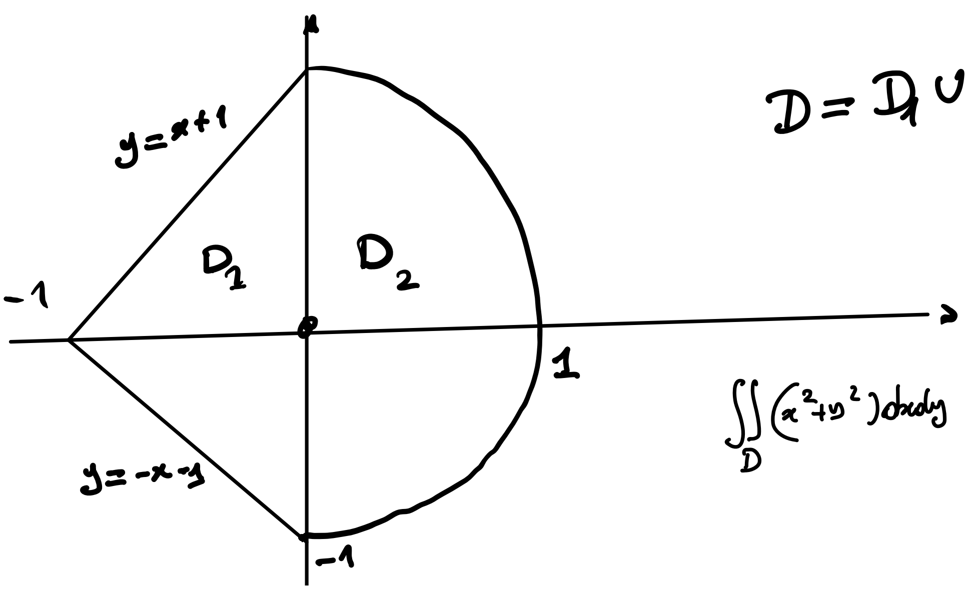
$x = 1 \iff \rho = \frac{1}{\cos\theta}$

$$\iint_D (x^2+y^2) \, dx \, dy = \iint_T \rho^2 \cdot \rho \, d\rho \, d\theta$$

$$= \int_0^{\pi/4} d\theta \int_1^{1/\cos\theta} \rho^3 \, d\rho$$

$$= \frac{1}{4} \int_0^{\pi/4} \left(\frac{1}{\cos^4\theta} - 1 \right) d\theta = \dots$$

4)



$D = D_1 \cup D_2$

$$\iint_D (x^2 + y^2) \, dx \, dy$$

$$= \iint_{D_1} - \; + \iint_{D_2} -$$

$$\iint_{D_1} (x^2+y^2)\,dx\,dy = \int_{-1}^{0} dx \int_{-x-1}^{x+1} (x^2+y^2)\,dy = \int_{-1}^{0} \Big[x^2 \underbrace{(x+1-x+1)}_{\text{polinomio}} + \left(\frac{y^3}{3}\right)\Big|_{y=-x-1}^{y=x+1} \Big]$$

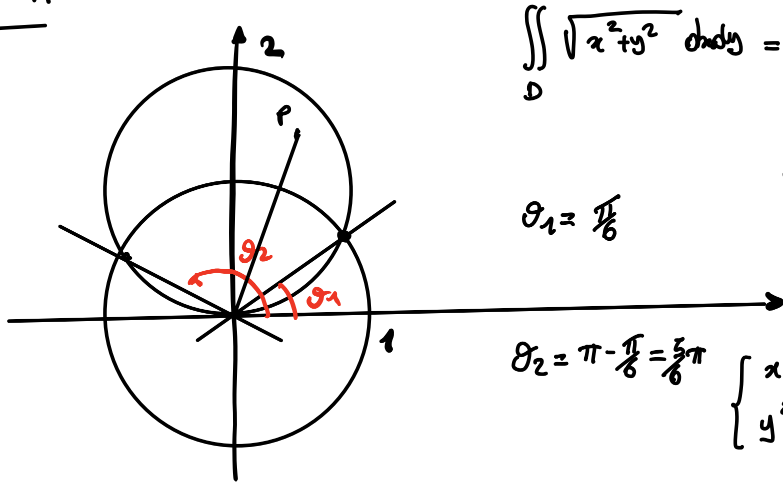
D_1 w asse x

$$\iint_{D_2} (x^2+y^2)\,dx\,dy =$$

$0 \le \rho \le 1 \quad , \quad \vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$= \iint_{T} \rho^3 \, d\rho \, d\vartheta$$

DA SVOLGERE



$$\iint_{D} \sqrt{x^2+y^2}\,dx\,dy = ?$$

$$\begin{cases} x^2+y^2=1 \\ x^2+(y-1)^2=1 \end{cases}$$

$\vartheta_1 = \frac{\pi}{6}$

$\vartheta_2 = \pi - \frac{\pi}{6} = \frac{5}{6}\pi$

$$\begin{cases} x^2+y^2=1 \\ y^2-(y-1)^2=0 \end{cases}$$

$$\begin{cases} x^2+y^2=1 \\ \cancel{y^2}-\cancel{y^2}+2y-1=0 \end{cases}$$

$$\begin{cases} x^2 = 1-\frac{1}{4} = \frac{3}{4} \\ y=\frac{1}{2} \end{cases}$$

$$\frac{\pi}{6} \le \vartheta \le \frac{5}{6}\pi$$

$$1 \le \rho \le$$

$$\begin{cases} \cos\vartheta = \frac{\sqrt{3}}{2} \\ \sin\vartheta = \frac{1}{2} \end{cases} \quad ? \quad \theta = \frac{\pi}{6}$$

$$\begin{cases} x = \pm\frac{\sqrt{3}}{2} \\ y = \frac{1}{2} \end{cases}$$

$$x^2 + (y-1)^2 = 1 \qquad \begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases}$$

$$\rho^2\cos^2\vartheta + \rho^2\sin^2\vartheta - 2\rho\sin\vartheta + \cancel{1} = \cancel{1}$$

$$\rho^{\cancel{2}} = 2\cancel{\rho}\sin\vartheta \iff \boxed{\rho = 2\sin\vartheta}$$

$$T = \left\{ 1 \leq \rho \leq 2\sin\vartheta, \quad \frac{\pi}{6} \leq \vartheta \leq \frac{5}{6}\pi \right\}$$

$$\iint_D \sqrt{\underbrace{x^2+y^2}_{\rho}}\,dxdy = \iint_T \rho\cdot\rho\,d\rho d\vartheta = \int_{\frac{\pi}{6}}^{\frac{5}{6}\pi} d\vartheta \int_1^{2\sin\vartheta} \rho^2 d\rho$$

$$= \int_{\frac{\pi}{6}}^{\frac{5}{6}\pi} d\vartheta \left(\frac{\rho^3}{3}\right)_{\rho=1}^{\rho=2\sin\vartheta} = \frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi} \left(8\sin^3\vartheta - 1\right) d\vartheta$$

$$\int \sin^3\vartheta\, d\vartheta \overset{?}{=} \int \sin^2\vartheta \cdot \sin\vartheta\, d\vartheta = \int (1-\cos^2\vartheta)\sin\vartheta\, d\vartheta =$$

$$= \int \sin\vartheta\, d\vartheta - \int \cos^2\vartheta \sin\vartheta\, d\vartheta$$

$$= -\cos\vartheta + \frac{\cos^3\vartheta}{3} + c$$