Lezione del 12/10/2022

Condizione sufficiente per la sviluppabilità in serie di Taylor.

$f(x)$, $f \in C^\infty(]a,b[)$, $x_0 \in]a,b[$.

Supponiamo che in un intorno I di x_0,

$\exists\, M, h > 0$ tali che $|f^{(n)}(x)| \leq M h^n$

$\forall x \in I, \ \forall n \in \mathbb{N}$

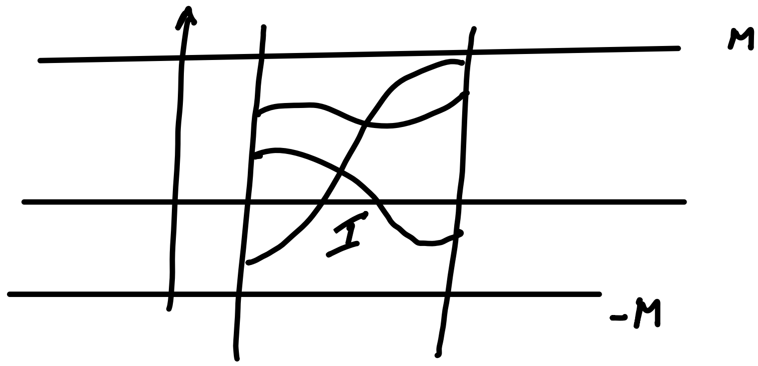
Allora f è sviluppabile in serie di Taylor in x_0

e $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$\forall x \in I$.

Se $h = 1$, $\underbrace{|f^{(n)}(x)| \leq M}_{\text{equilimitatezza}}$ $\forall n$

vale il teorema.



Sviluppi notevoli

$f(x) = e^x \quad , x_0 = 0 \qquad I = [-\delta, \delta]$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$f'(x) = f''(x) =$

$= \dots = f^{(n)}(x) = e^x$

$$|f^{(n)}(x)| = e^x \leq e^{\delta} = M$$

$x \in [-\delta, \delta]$

$x \leq \delta$

$\Rightarrow$ $f(x) = e^x$ è sviluppabile in serie di MacLaurin.

$$f(0) = f'(0) = \dots = f^{(n)}(0) = e^0 = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x \in [-\delta, \delta]$$

$\forall x \in \mathbb{R}$ perché $\delta > 0$ è <u>arbitrario</u>

<u>Funzioni trigonometriche</u>

$f(x) = \cos x$ $\quad$ $f(0) = 1$

$f'(x) = -\sin x$ $\quad$ $f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{(IV)}(x) = +\cos x \qquad f^{(IV)}(0) = 1$

$\dots$

$$|f^{(n)}(x)| \le 1 = M$$

$$\forall x \in [-\delta, \delta],\ \delta > 0$$

$\Rightarrow f(x) = \cos x$ è sviluppabile in serie di MacLaurin e si ha

$$\cos x = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$+ \cdots + \frac{f^{(m)}(0)}{m!} x^m + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

$$+ \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + \cdots$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} \qquad \underline{\forall x \in \mathbb{R}}$$

$$\forall x \in [-\delta, \delta], \ \forall \delta > 0$$

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \;, \qquad f''(0) = 0$$

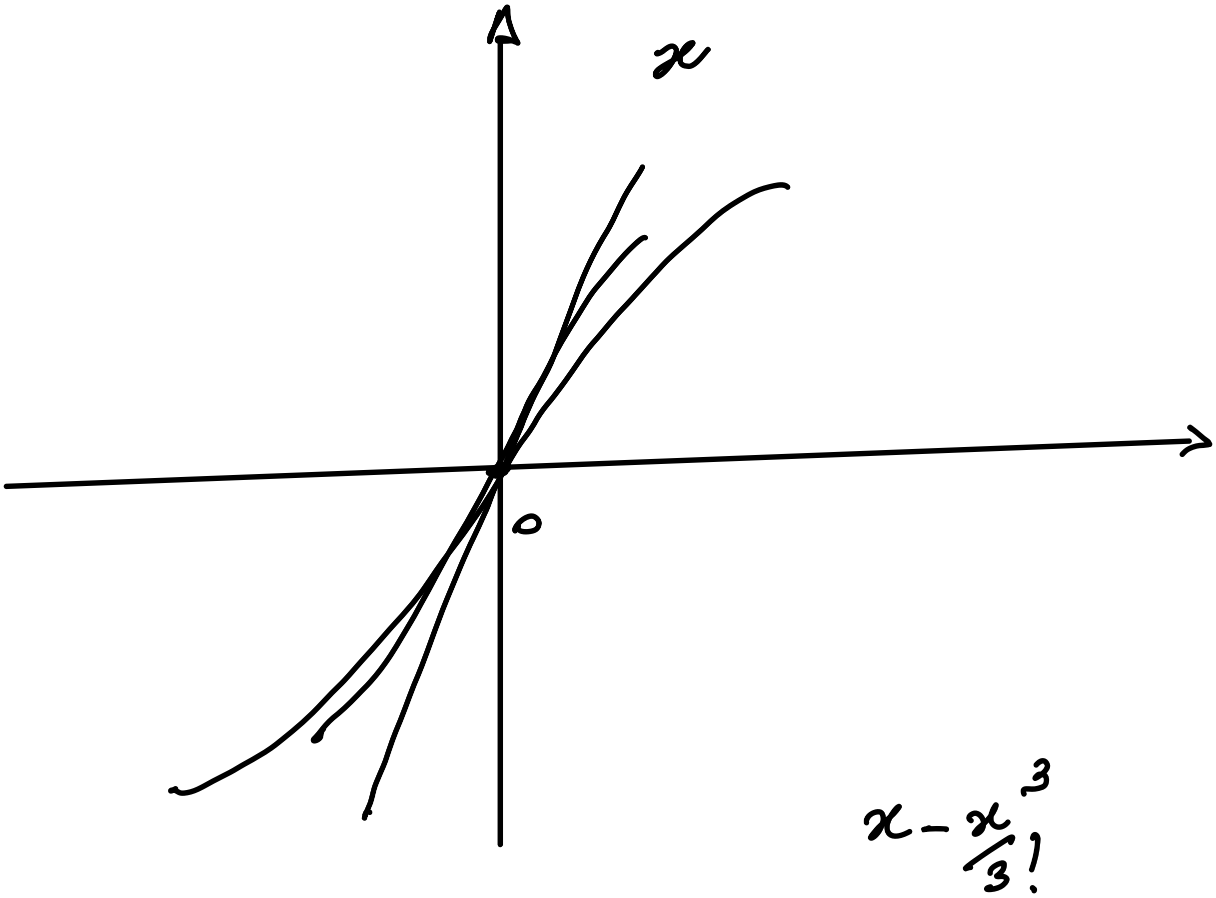
$$f'''(x) = -\cos x \;, \qquad f'''(0) = -1$$

$$\vdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$+ \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \cdots$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} \quad \forall x \in \mathbb{R}$$



Serie geometrica

$$\sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$$

$$\forall x \in]-1,1[$$

$$x \longleftrightarrow -x$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$
$$+ \dots + (-1)^n x^n + \dots$$

$\int_0^x$ integrazione termine a termine

$$\int_0^x \frac{1}{1+t}\,dt \overset{?}{=} \Big[\log|1+t|\Big]_{t=0}^{t=x}$$

$$= \log(1+x)$$

$$-1 < x < 1$$

$$1+x > 0$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$+ \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\underline{\underline{-1 < x < 1}}$$

$$x = 1 \; : \; \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

serie armonica a segni alterni

$$\Rightarrow \log 2 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

$$\forall x \in]-1,1[$$

$$x \longleftrightarrow -x^2$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots +$$

$$+ \cdots + (-1)^n x^{2n} + \cdots$$

$$\int_0^x \frac{1}{1+t^2}\,dt = \Big[\operatorname{arctg} t\Big]_{t=0}^{t=x}$$

$$= \operatorname{arctg} x$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + C$$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$+ (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\forall x \in]-1, 1[$$

$x = 1$: $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ converge pu Leibniz

$$\underset{\substack{\parallel \\ \frac{\pi}{4}}}{\operatorname{arctg} 1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

$x = -1$? $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1}$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$ converge

$$\underbrace{\operatorname{arctg}(-1)}_{\substack{\parallel \\ -\frac{\pi}{2}}} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$\longrightarrow$ banale.