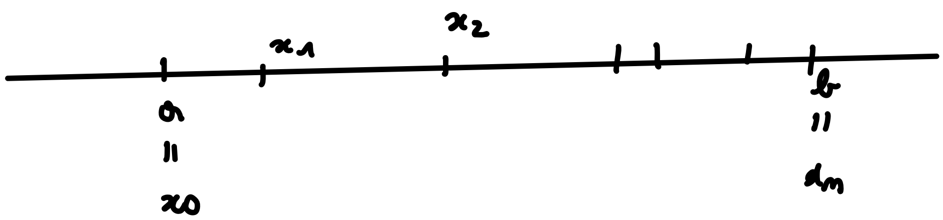
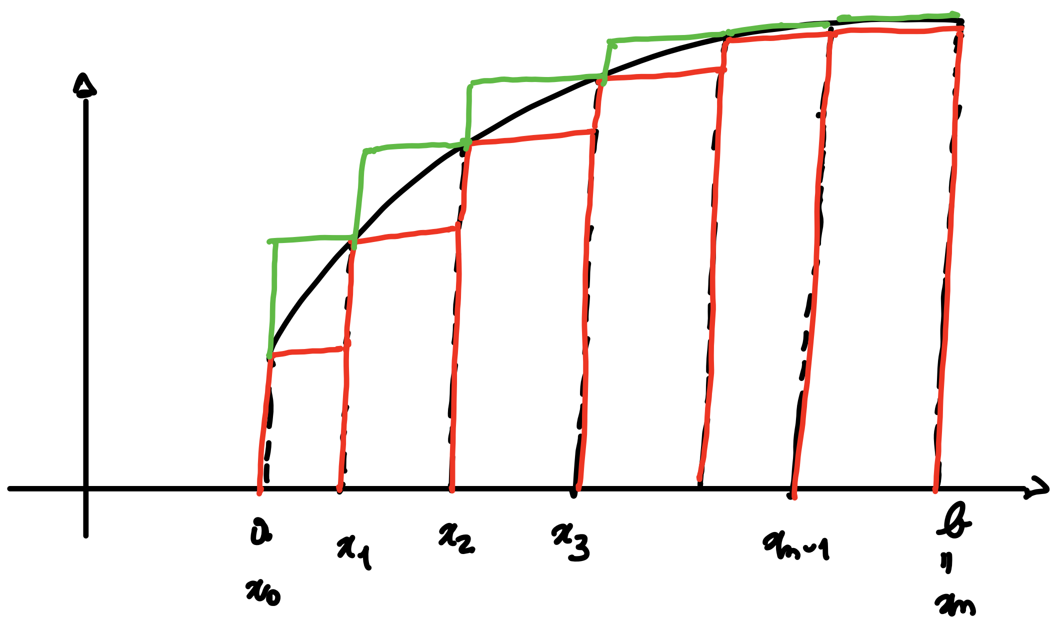
Lezione del 12/12/2022

$f = f(x)$, $f : [a,b] \longrightarrow \mathbb{R}$ limitata

$$a = x_0 < x_1 < \cdots < x_n = b \qquad P$$





$$s(f,P) = \sum_{i=0}^{n-1} \underbrace{\inf_{[x_i, x_{i+1}]} f}_{} \cdot (x_{i+1} - x_i)$$ somma int. inferiore di f

$$S(f,P) = \sum_{i=0}^{n-1} \underbrace{\sup_{[x_i, x_{i+1}]} f}_{} \cdot (x_{i+1} - x_i)$$ somma integrale superiore di f

$$s(f, P_1) \le S(f, P_2)$$

$$\{s(f, P_1)\} \qquad \{S(f, P_2)\}$$

separati

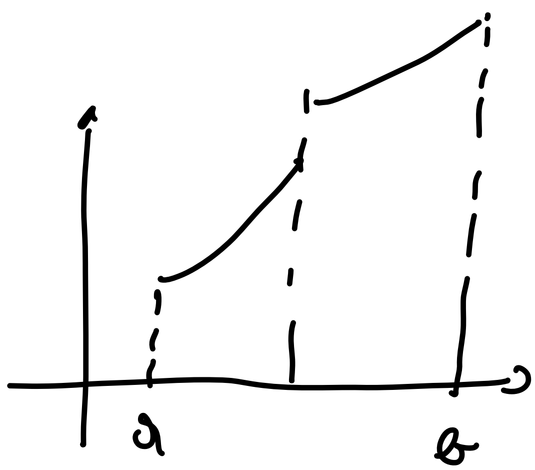
$$\sup_{P} s(f, P) \le \inf_{P} S(f, P)$$

Se $\displaystyle \sup_{P} s(f, P) = \inf_{P} S(f, P)$

$$= \int_a^b f(x)\,dx$$

Se $f \geq 0$,

$\displaystyle \int_a^b f(x)\,dx$ = Area del rettangoloide di f, di base $[a,b]$



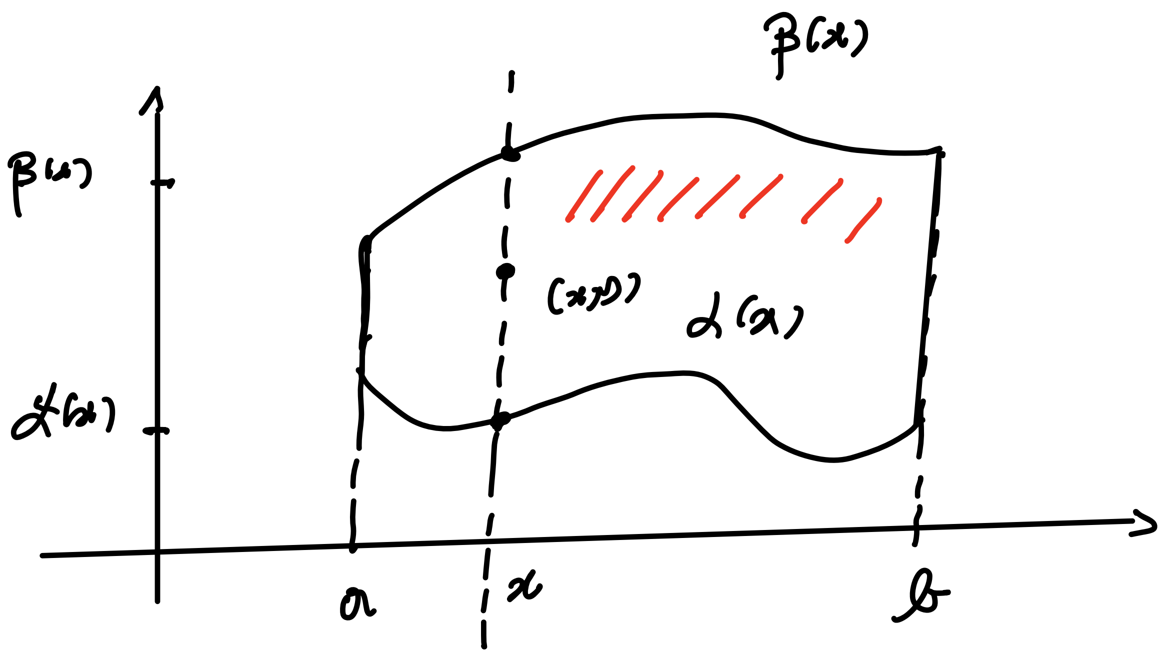
Def. (Dominio normale rispetto ad un asse coordinato)

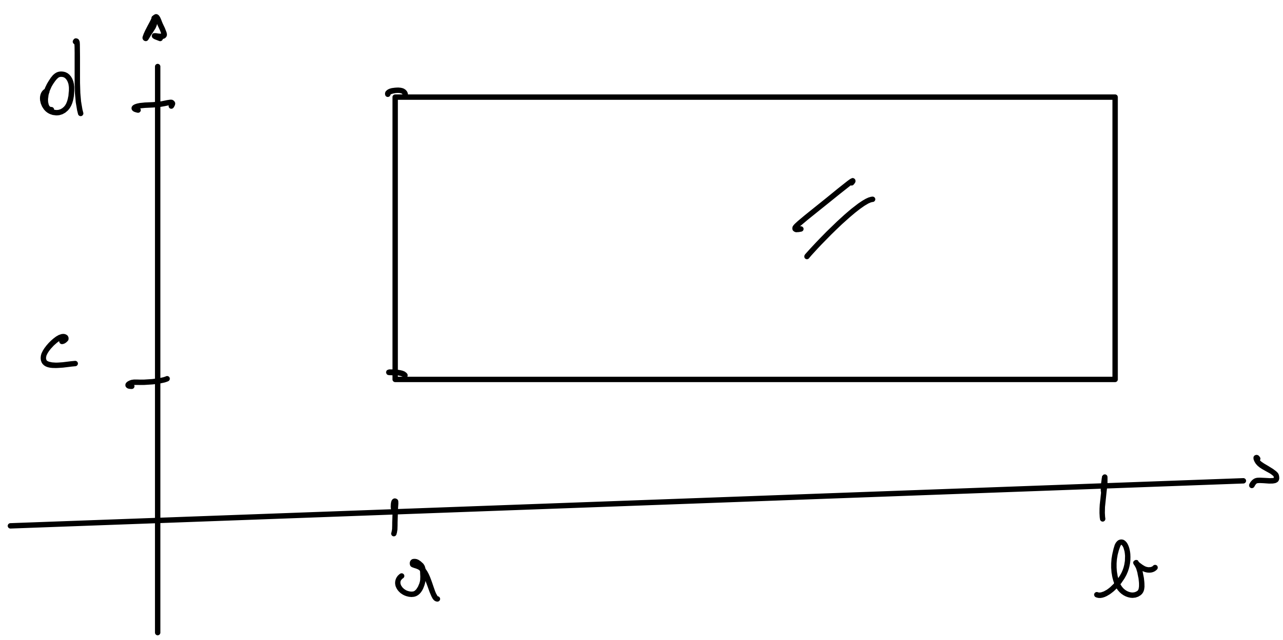
Un dominio normale rispetto all'asse x è un sottoinsieme $D \subseteq \mathbb{R}^2$ definito da

$$D = \left\{ (x,y) \in \mathbb{R}^2 : x \in [a,b], \; \alpha(x) \le y \le \beta(x) \right\}$$

$\alpha(x), \beta(x)$ funzioni continue in $[a,b]$

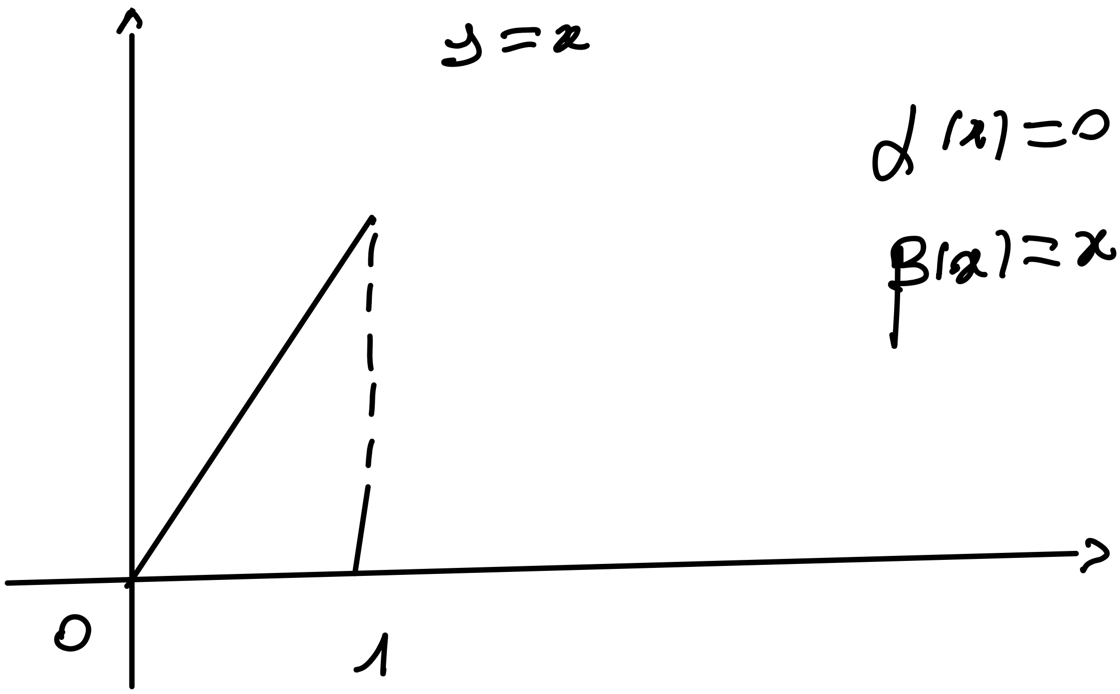
toli che $\qquad \alpha(x) \leq \beta(x), \quad \forall x \in [a,b]$



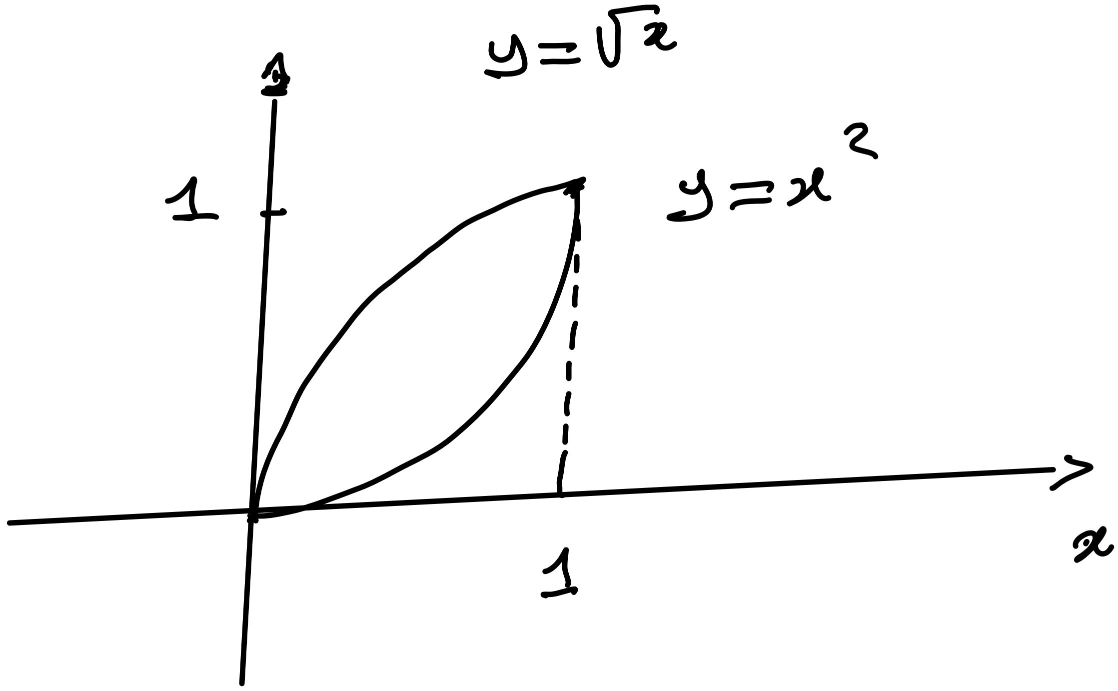


$D = [a,b] \times [c,d]$

$\alpha(x) = c \quad , \quad \beta(x) = d$



$$D = \left\{ (x,y) \in \mathbb{R}^2 : \ x \in [0,1], \ 0 \leq y \leq x \right\}$$



Def. Si dice <u>area</u> di D, la quantità

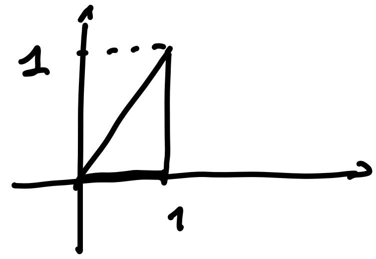
$$\boxed{m(D) = \text{Area}(D) = \int_a^b [\beta(x) - \alpha(x)]\,dx}$$

ES. $D = [a,b] \times [c,d]$

$$m(D) = \int_a^b [c - d]\,dx = (a-b)\cdot(c-d)$$

$$D = \{ (x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1, \quad 0 \le y \le x \}$$

$$m(D) = \int_0^1 [x - 0]\,dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}$$



$$D = \{ (x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1, \quad x^2 \le y \le \sqrt{x} \}$$

$$m(D) = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \frac{2}{3}\left(x\sqrt{x}\right)_0^1 - \frac{1}{3}\left(x^3\right)_0^1$$

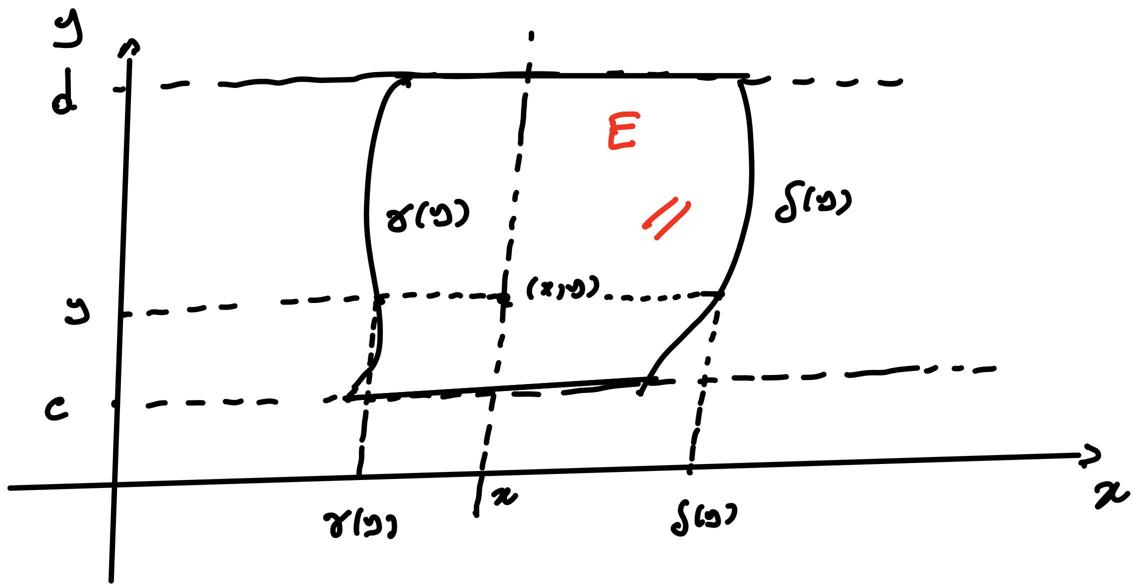
$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Def. $E \subseteq \mathbb{R}^2$ dominio normale rispetto all'asse y se E si scrive nella forma

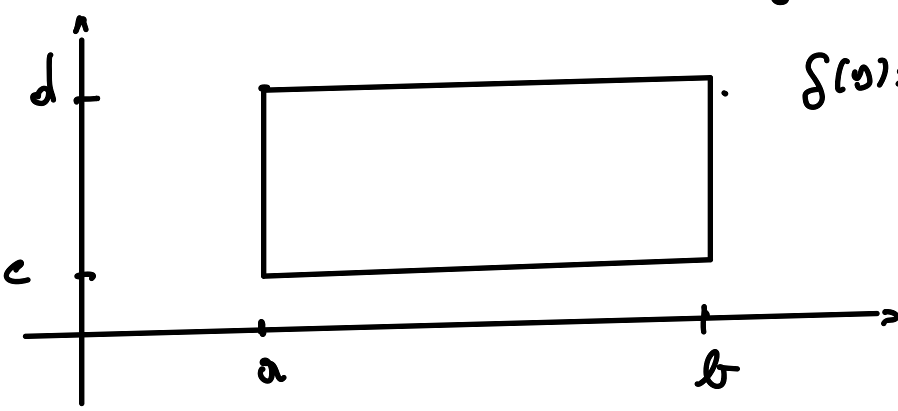
$$E = \{ (x,y) \in \mathbb{R}^2 : \ c \le y \le d, \quad \gamma(y) \le x \le \delta(y) \}$$

$\gamma(y), \delta(y)$ funzioni continue in $[c,d]$

tali che $\gamma(y) \leq \delta(y), \quad \forall y \in [c,d]$

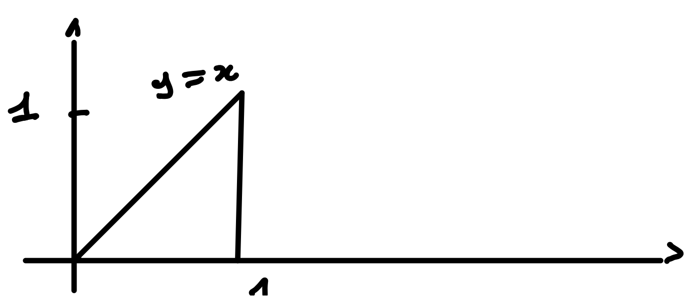


ES.



$\gamma(y) = a$

$\delta(y) = b$



$0 \leq y \leq 1$

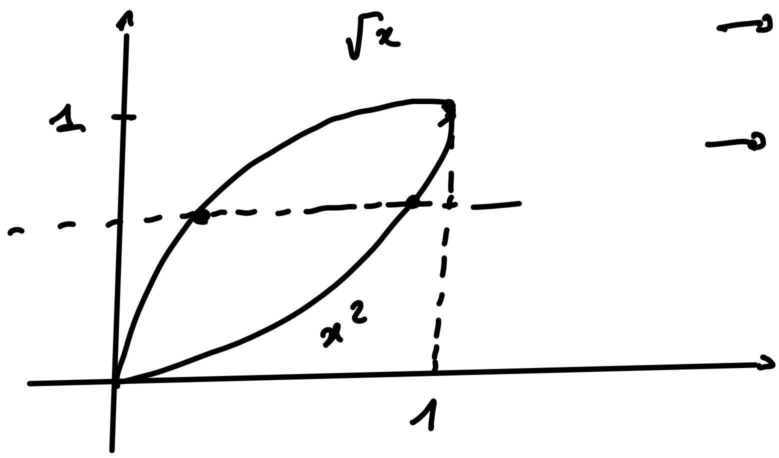
$\gamma(y) = y$

$\delta(y) = 1$

$$0 \leq y \leq 1$$

$$y \leq x \leq 1$$

$$D = \{ (x,y) \in \mathbb{R}^2 : \ 0 \leq y \leq 1, \ y \leq x \leq 1 \}$$



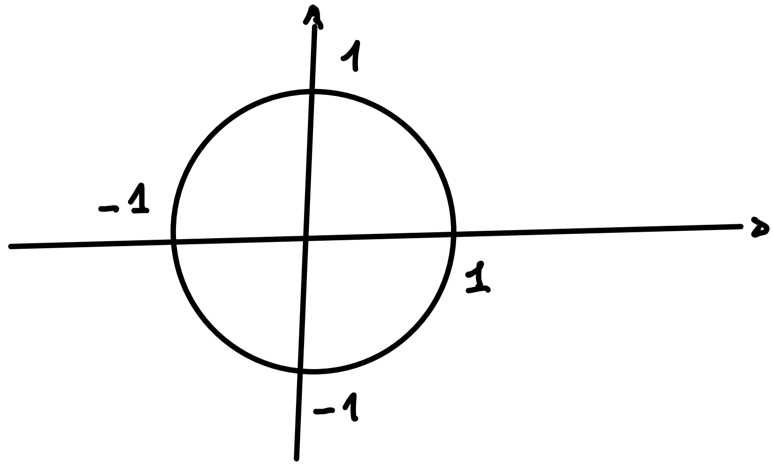
$\to \ y = \sqrt{x}$

$\to \ y = x^2$

$x = y^2$

$x = \sqrt{y}$

$$D = \left\{ \begin{array}{l} 0 \leq y \leq 1 \\ \underbrace{y^2}_{\gamma(y)} \leq x \leq \underbrace{\sqrt{y}}_{\delta(y)} \end{array} \right\}$$



$x^2 + y^2 = 1$

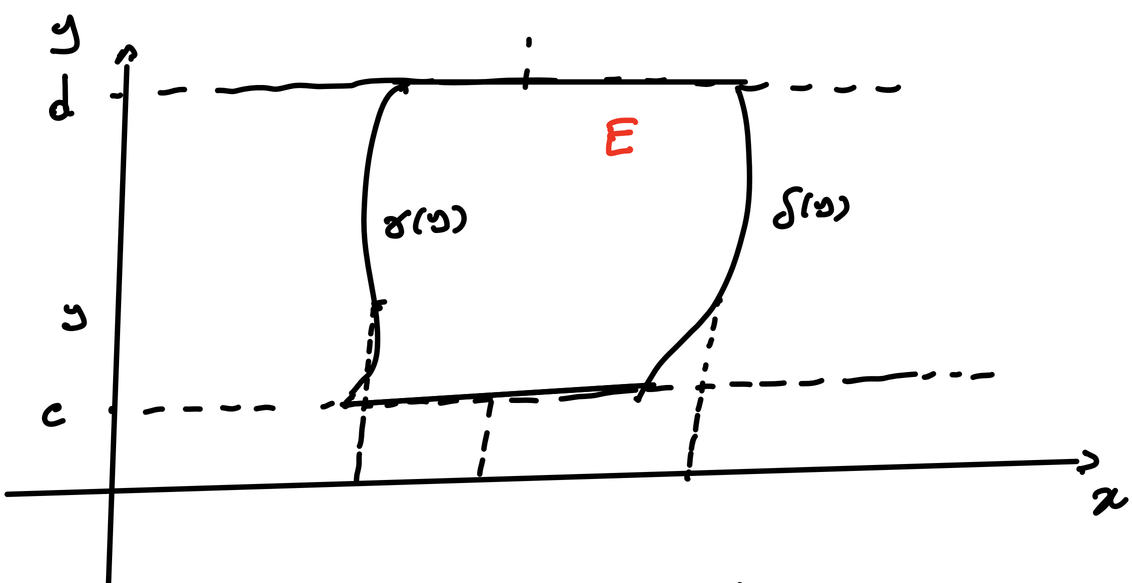
$y = \pm\sqrt{1-x^2}$

$x = \pm\sqrt{1-y^2}$

$$D = \{ -1 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \}$$

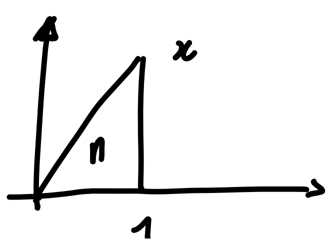
$$= \{ -1 \leq y \leq 1, \quad -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \}$$

Area di un dominio normale rispetto all'asse y



$$\left[m(E) = \text{Area}(D) = \int_c^d [\delta(y) - \gamma(y)]\,dy \right]$$

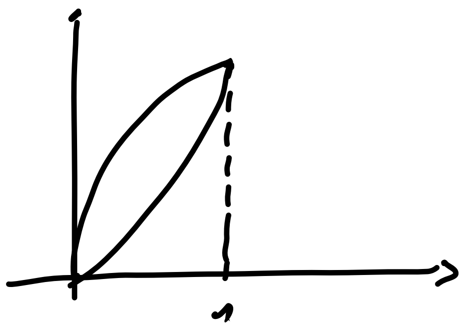
ES.



$m(D) = \frac{1}{2}$

$0 \leq y \leq 1 , \quad y \leq x \leq 1$

$$m(D) = \int_0^1 (1-y)\,dy = 1 - \left(\frac{y^2}{2}\right)\Big|_0^1 = \frac{1}{2}$$



$\gamma(y) = y^2$

$\delta(y) = \sqrt{y}$

$$m(D) = \int_0^1 \left(\sqrt{y} - y^2\right) dy = \frac{1}{3}$$

Area del cerchio di raggio 1 : π

<u>**Def**</u> D normale rispetto all'asse x :

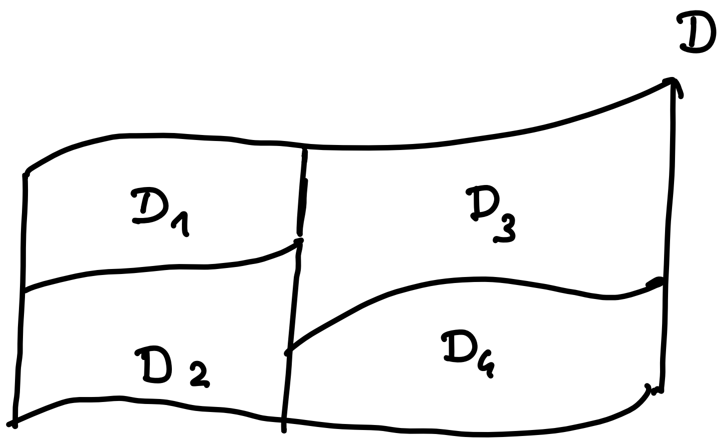
decomposizione di D in domini normali rispetto all'asse x è un insieme del tipo

$\mathcal{P} = \{ D_1, D_2, \dots, D_N \}$ dove

1. $D_i \subseteq D$, D_i normale rispetto ad x

2. $\mathring{D}_i \cap \mathring{D}_j = \emptyset \quad \forall i \neq j$

3. $\displaystyle\bigcup_{i=1}^{N} D_i = D$



$f = f(x,y)$, $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ limitata.

$$s(f, \mathcal{P}) = \sum_{i=1}^{N} \inf_{D_i} f \cdot m(D_i) \,.$$

$$S(f, \mathcal{P}) = \sum_{i=1}^{N} \sup_{D_i} f \cdot m(D_i) \,.$$

. . . .

a) $\{ s(f,P) : P \text{ dec. di } D \}$

<u>separati</u>

$\{ S(f,P) : \text{"} \quad \text{"} \}$

P_1, P_2 di D: $\quad s(f,P_1) \leq S(f,P_2)$

$$\sup_P s(f,P) \leq \inf_P S(f,P)$$

<u>**Def.**</u> Se i due insiemi numerici sono contigui,

$$\sup_P s(f,P) = \inf_P S(f,P),$$

$f(x,y)$ integrabile secondo Riemann su D.

In tal caso,

$$\underbrace{\iint_D f(x,y)\,dx\,dy} = \sup_P s(f,P) = \inf_P S(f,P)$$

integrale doppio di f esteso a D

Prop Se f è continua in D, f è integrabile secondo Riemann in D.

Prop $\iint_D (\alpha f + \beta g)\,dx\,dy = \alpha \iint_D f\,dx\,dy + \beta \iint_D g\,dx\,dy$

$D = D_1 \cup D_2$: $\iint_D f = \iint_{D_1} f + \iint_{D_2} f$

Formule di riduzione

Supponiamo che D normale rispetto ad x

$$D = \{ (x,y) \in \mathbb{R}^2 : x \in [a,b],\ \alpha(x) \le y \le \beta(x) \}.$$

Se $f : D \to \mathbb{R}$ è continua, si ha

$$\iint_D f(x,y)\,dx\,dy = \int_a^b dx \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy .$$

Supponiamo che E normale rispetto ad y

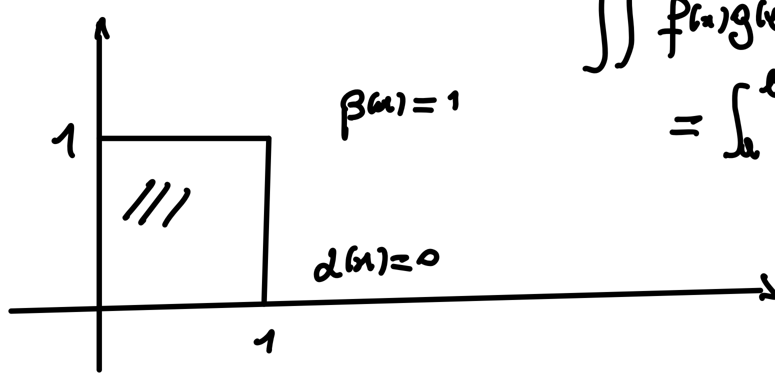
$$E = \{ (x,y) \in \mathbb{R}^2 : y \in [c,d],\ \gamma(y) \le x \le \delta(y) \}.$$

Se $f : E \to \mathbb{R}$ è continua, si ha

$$\iint_E f(x,y)\,dx\,dy = \int_c^d dy \int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx$$

$$\iint_D x \arcsin y \,dx\,dy$$

$D = [0,1] \times [0,1]$



$$\iint f(x)g(y)\,dx\,dy = \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy$$

$$= \int_0^1 dx \int_0^1 x \arcsin y \,dy = \int_0^1 x\,dx \cdot \int_0^1 \arcsin y \,dy$$

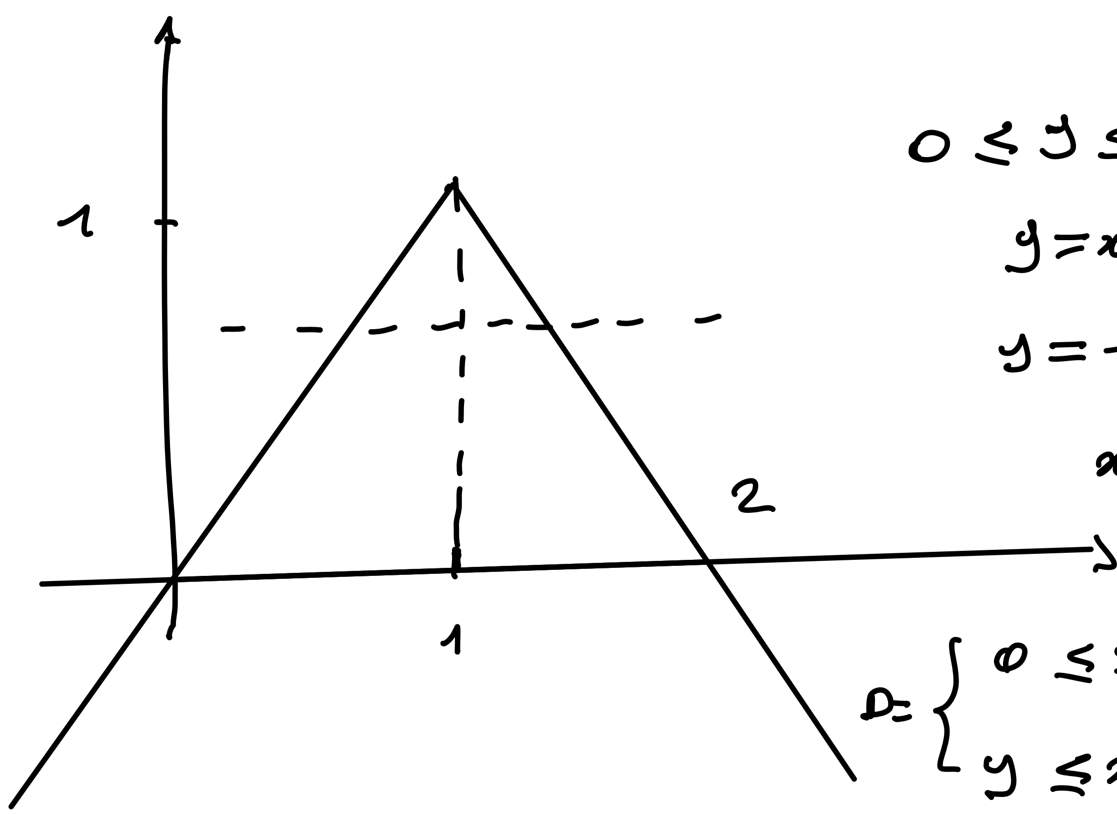
$$= \left(\frac{x^2}{2}\right)_0^1 \cdot \int_0^1 \underbrace{\arcsin y}_{?}\,dy$$

$$= \frac{1}{2} \int_0^1 \text{arcsen}\, y \, dy$$

$$\int \text{arcsen}\, y \, dy = y \,\text{arcsen}\, y - \int \frac{y}{\sqrt{1-y^2}} \, dy = \dots$$

2) $$\iint_D x y^2 \, dx\, dy$$

$$D = \{ (x,y) \in \mathbb{R}^2 : y \le x, \; y \le -x+2, \; y \ge 0 \}$$



$0 \le y \le 1$

$y = x$

$y = -x + 2$

$\downarrow$

$x = 2 - y$

$$D = \begin{cases} 0 \le y \le 1 \\ y \le x \le 2-y \end{cases}$$

$$\iint_D - \quad = \iint_{D_1} - \quad + \iint_{D_2} -$$

$$D_1 = \{ (x,y) : \ 0 \leq x \leq 1, \quad 0 \leq y \leq x \}$$

$$\iint_{D_1} xy^2 \, dx dy = \int_0^1 dx \int_0^x xy^2 \, dy \quad =$$

$$= \int_0^1 x \left(\underbrace{\int_0^x y^2 dy}_{?} \right) dx \quad =$$

$$= \int_0^1 x \left[\frac{y^3}{3} \right]_{y=0}^{y=x} dx = \int_0^1 x \cdot \frac{x^3}{3} = \frac{1}{3} \int_0^1 x^4 dx$$

$$= \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15} .$$

$$D_2 = \{ 1 \leq x \leq 2 , \quad 0 \leq y \leq -x+2 \}$$

$$\iint_{D_2} xy^2 \, dx dy = \int_1^2 dx \int_0^{-x+2} xy^2 \, dy =$$

$$= \int_1^2 x\left(\int_0^{-x+2} y^2\,dy\right)dx =$$

$$= \int_1^2 x\left(\frac{y^3}{3}\right)\Big|_{y=0}^{y=-x+2} dx =$$

$$= \int_1^2 x\,\frac{(2-x)^3}{3}\,dx = \ldots$$

$$D = \begin{cases} 0 \le y \le 1 \\ y \le x \le 2-y \end{cases}$$

$$\iint_D xy^2\,dx\,dy = \int_0^1 dy \int_y^{2-y} x\underline{\underline{y}}^2\,dx$$

$$= \int_0^1 y^2\left(\int_y^{2-y} x\,dx\right)dy$$

$$= \int_0^1 y^2 \left(\frac{x^2}{2} \right)_{x=y}^{x=2-y} dy =$$

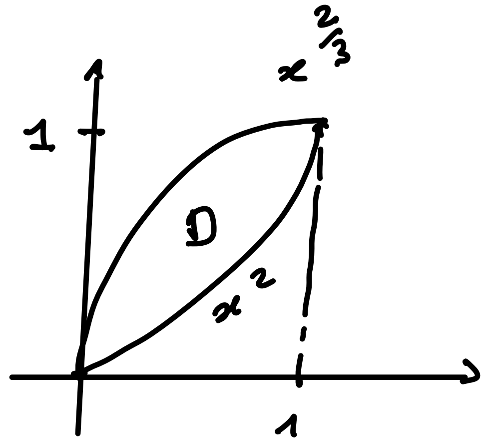
$$= \frac{1}{2} \int_0^1 y^2 \left[(2-y)^2 - y^2 \right] dy =$$

$$= \frac{1}{2} \int_0^1 y^2 \left[4 + \cancel{y^2} - 4y - \cancel{y^2} \right] dy$$

$$= \frac{1}{2} \left[4 \int_0^1 y^2 dy - 4 \int_0^1 y^3 dy \right]$$

$$= 2 \left[\frac{1}{3} - \frac{1}{4} \right] = 2 \, \frac{4-3}{12} = \frac{1}{6}$$

$$\iint_D x e^{y^2}\,dx\,dy$$



$$= \int_0^1 dx \int_{x^2}^{x^{2/3}} x e^{y^2}\,dy = \int_0^1 x\left(\int_{x^2}^{x^{2/3}} e^{y^2}\,dy\right)dx$$

$$D = \quad 0 \leq y \leq 1 \ : \ y^{3/2} \leq x \leq \sqrt{y}$$

$$y = x^{2/3} \iff x = y^{3/2}$$

$$y = x^2 \iff x = \sqrt{y}$$

$$\iint_D x e^{y^2}\,dx\,dy = \int_0^1 e^{y^2}\left(\int_{y\sqrt{y}}^{\sqrt{y}} x\,dx\right)dy$$

$$= \frac{1}{2}\int_0^1 e^{y^2}\left[x^2\right]_{x=y\sqrt{y}}^{\sqrt{y}}\,dy = \frac{1}{2}\int_0^1 e^{y^2}\left[y - y^3\right]dy$$

$$= \frac{1}{2}\left[\int_0^1 \underbrace{y e^{y^2}} dy - \int_0^1 y^3 e^{y^2} dy\right]$$

$$\frac{1}{2}\int y^2 \left(2y e^{y^2} dy\right) = \frac{1}{2}\left[y^2 e^{y^2} - 2\int y e^{y^2} dy\right]$$

int. parti $= \frac{1}{2}\left[y^2 e^{y^2} - e^{y^2}\right]$