# Lezione del 10/10/2022

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad \rho > 0 \text{ raggio di convergenza}$$

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad \forall x \in \,]x_0-\rho, x_0+\rho[$$

$$\int_{x_0}^{x} dt$$

$$= a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \dots + a_n (x-x_0)^n + \dots$$

Derivando termine a termine, otteniamo

$$a_1 + 2a_2 (x-x_0) + 3a_3 (x-x_0)^2 + \dots + n a_n (x-x_0)^{n-1} + \dots$$

$$= \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} \qquad \underline{\text{serie derivata}}$$

Integrando termine a termine in $(x_0, x)$ $\quad (x > x_0)$

$$a_1 (x-x_0)$$

$$\int_{x_0}^{x} a_1 (t-x_0)\, dt = a_1 \left( \frac{(t-x_0)^2}{2} \right)_{t=x_0}^{t=x}$$

$$= \frac{a_1}{2} (x-x_0)^2$$

$$a_0 (x-x_0) + \frac{a_1}{2} (x-x_0)^2 + \frac{a_2}{3} (x-x_0)^3$$

$$+ \dots + \frac{a_n}{n+1}(x-x_0)^{n+1} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}$$ <u>serie integrale</u>

<u>Teorema</u> (di derivazione ed integrazione termine a termine)

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad \forall x \in \,]x_0-\rho, x_0+\rho[\,.$$

Allora la serie derivata e la serie integrale hanno lo stesso raggio di convergenza $\rho$, ed inoltre

- $$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}, \quad \forall x \in \,]x_0-\rho, x_0+\rho[$$

$$\int_{x_0}^{x} f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}, \quad \text{''} \quad \text{''} .$$

- $$\left( \sum_{n=0}^{\infty} a_n (x-x_0)^n \right)' = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$\int_{x_0}^{x} \qquad = \sum_{n \geq 0}^{\infty} \frac{a_n}{n+1} (x-x_0)^{n+1}$$

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n (x-x_0)^{n-1} =$$

$$= a_1 + 2a_2 (x-x_0) + 3a_3 (x-x_0)^2 + \dots$$

$$+ \dots + n\, a_n (x-x_0)^{n-1} + \dots$$

$$2a_2 \qquad + 3 \cdot 2\, a_3 (x-x_0)$$

$$+ \dots + n(n-1)\, a_n (x-x_0)^{n-3} + \dots$$

$$f''(x) = (f')' = \sum_{n=2}^{\infty} n(n-1)\, a_n (x-x_0)^{n-2}$$

$$\forall x \in ]x_0-\rho, x_0+\rho[$$

Applicando il teorema di derivazione termine a termine infinite volte, abbiamo che

$$f \in C^{\infty} \quad e$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdot\ldots\cdot(n-(k-1))\, a_n (x-x_0)^{n-k}$$

$$= \underbrace{k\cdot(k-1)(k-2)\cdots 3\cdot 2\cdot 1\; a_k}_{k!\, a_k} + \ldots$$

$$k!\, a_k + \ldots$$

$$f^{(k)}(x_0) = k!\, a_k$$

$$a_k = \frac{f^{(k)}(x_0)}{k!}$$

Quindi

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n =$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(x_0)}{n!} (x-x_0)^n}_{\odot}$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

$$+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

La $\odot$ serie di Taylor di $f(x)$, di punto iniziale $x_0$.

**Def** $f \in C^{\infty}(]a,b[)$, $x_0 \in ]a,b[$

se vale l'uguaglianza

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$\forall x \in \,]x_0-\delta, x_0+\delta[\, \subseteq \,]a,b[$$

Si dice che $f$ è sviluppabile in serie di Taylor, di punto iniziale $x_0$.

Problema È vero che ogni funzione è sviluppabile in serie di Taylor?

Risposta: NO

Esempio

$$f(x) = \begin{cases} e^{-1/x^2} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

$\underline{x \neq 0}$ :

$$f'(x) = -e^{-1/x^2} D\left(\frac{1}{x^2}\right) =$$

$$= +2e^{-1/x^2} \cdot \frac{1}{x^3}$$

$\underline{x = 0}$ ?

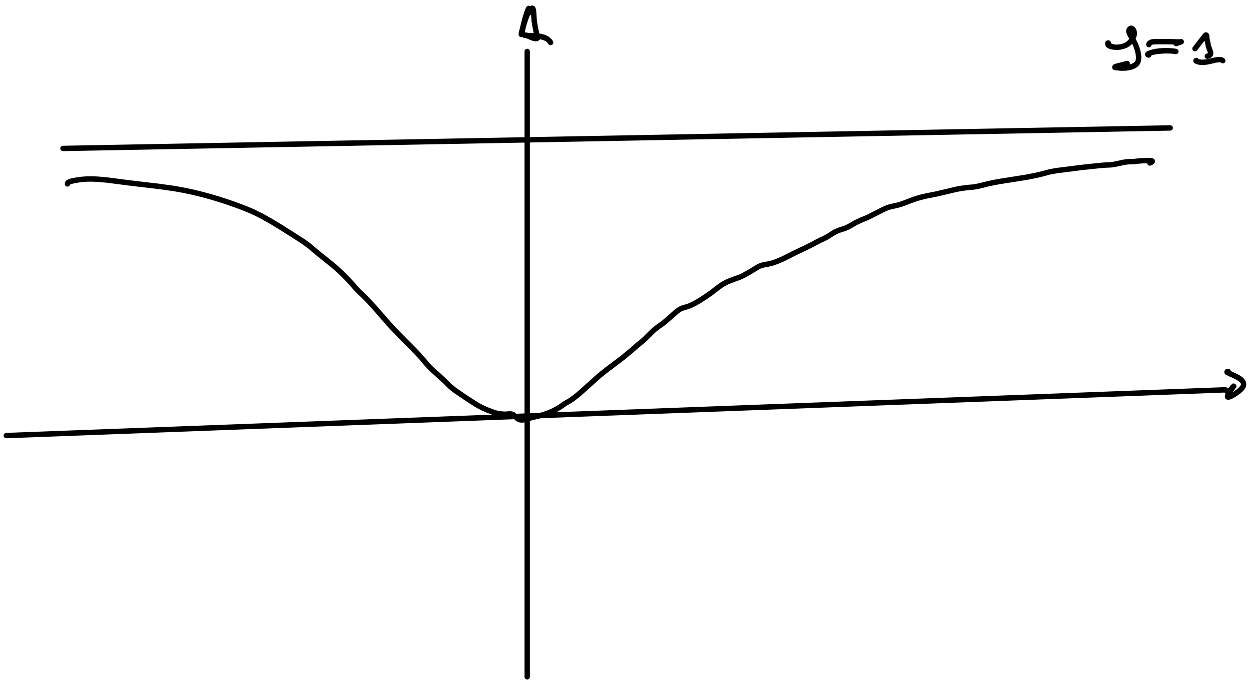
$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} =$$

$$= \lim_{x\to 0} \frac{e^{-1/x^2}}{x} \quad \left[\frac{0}{0}\right]$$

$$= 0 \qquad f'(0) = 0$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} e^{-1/x^2} = 1$$

$$e^0 = 1$$




$f(0)=0$
$f'(0)=0$

$f''(0)=0$

$f^{(n)}(0)=0 \quad \forall n \in \mathbb{N}$

$f \in C^{\infty}(\mathbb{R})$

$$x_0=0 \ : \ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \overbrace{f(0)}^{=0} + \overbrace{f'(0)}^{=0}\, x + \overbrace{\frac{f''(0)}{2}}^{=0}\, x^2$$

$$+ \dots + \underbrace{\frac{f^{(n)}(0)}{n!}}_{=0}\, x^n + \dots$$

$$= 0 \neq f(x)$$

$x_0 = 0$ serie di <u>MacLaurin</u>.

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Determinare l'insieme di

convergenza delle serie

$$\sum_{n=1}^{\infty} \frac{(1-x^2)^n}{n} \qquad y = 1-x^2$$

$$\sum_{n=1}^{\infty} \frac{y^n}{n} \quad (\bullet) \qquad a_n = \frac{1}{n}$$

$$\rho = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1/n}{1/(n+1)}$$

$$= \lim_{n\to\infty} \frac{n+1}{n} = 1$$

La $(\bullet)$ converge (assolutamente)

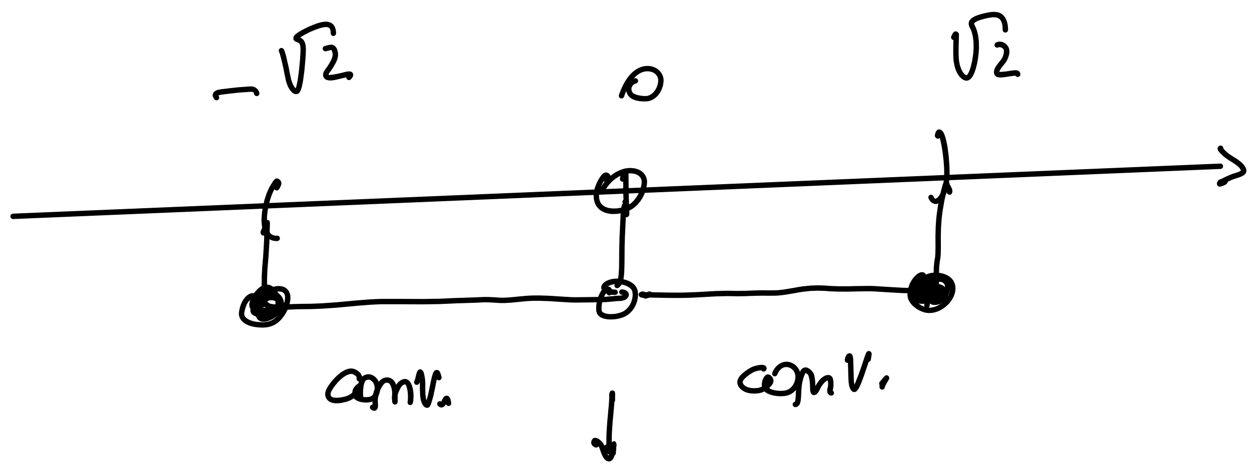
quando $-1 < y < 1 \iff$ la serie iniziale converge

$$-1 < 1 - x^2 < 1$$

$$\iff \quad -1 < x^2 - 1 < 1$$

$$\iff \quad 0 < x^2 < 2$$

$$\iff \quad x \neq 0 \, , \; -\sqrt{2} < x < \sqrt{2}$$



non conv.

(diverge)

$\circ) \sum_{n=0}^{\infty} \frac{y^n}{n}$ $\quad y \in (-1, 1)$

Agli estremi? $\quad y = 1 \longrightarrow$ diverge

$y = -1 \quad : \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converge

$y = -1 \iff 1 - x^2 = -1$

$\iff x^2 = 2 \iff x = \pm\sqrt{2}$

converge

La convergenza è totale quando

$$-\delta \leq y \leq \delta \, , \quad 0<\delta<1$$

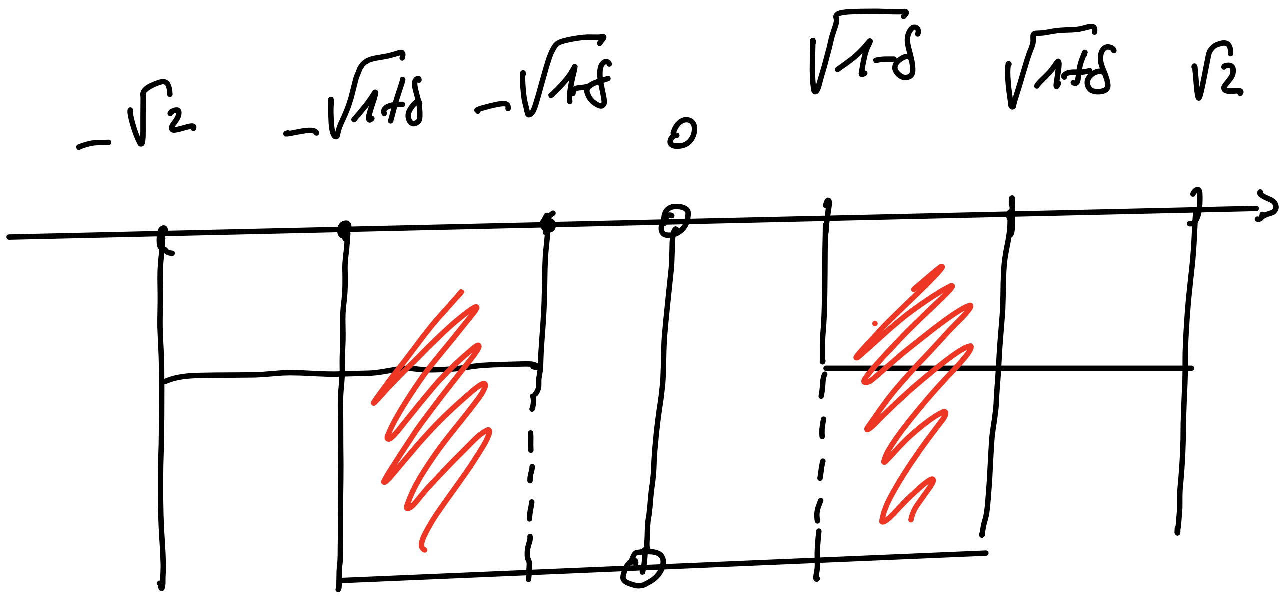
$$\iff \quad -\delta \leq 1-x^2 \leq \delta$$

$$\iff \quad -\delta \leq x^2-1 \leq \delta$$

$$\iff \quad 1-\delta \leq x^2 \leq 1+\delta$$

$$\iff \quad \begin{cases} x^2 \geq 1-\delta \\ x^2 \leq 1+\delta \end{cases}$$

$$\Longleftrightarrow \begin{cases} x \leq -\sqrt{1-\delta}\,, \quad x \geq \sqrt{1-\delta} \\ -\sqrt{1+\delta} \leq x \leq \sqrt{1+\delta} \end{cases}$$




Convergenza totale in

$$\left[-\sqrt{1+\delta}, -\sqrt{1-\delta}\right] \cup \left[\sqrt{1-\delta}, \sqrt{1+\delta}\right]$$

$$0 < \delta < 1$$

$$\sum_{n=0}^{\infty} \left( \frac{x-1}{x} \right)^n \qquad x \neq 0$$

$$\sum_{n=0}^{\infty} \frac{n\,5^n}{2^n+3^n} \Big( \underbrace{4-|x+2|}_{y} \Big)^n$$

$$\sum_{n=0}^{\infty} \frac{\sqrt{n+2}}{n!\,(n+1)} \Big( \underbrace{\arcsin x}_{y} \Big)^n \qquad -1 \leq x \leq 1$$

$$\sum_{n=0}^{\infty} \frac{\sqrt{n+2}}{n!\,(n+1)}\, y^n \qquad \text{converge}$$

$y_0 = 0$ per ogni $y \in \mathbb{R}$

$$\rho = \lim_{n\to\infty} \underbrace{\frac{\sqrt{n+2}}{\cancel{n!}\,\cancel{(n+1)}} \cdot \frac{\cancel{(n+1)!}\,(n+2)}{\sqrt{n+3}}}_{\downarrow\; 1}$$

$$= +\infty$$

$\Leftrightarrow$ la serie in $x$ converge $\forall x \in [-1,1]$

<u>totalmente</u> in $[-1,1]$

<u>Convergenza totale</u>: la serie in $y$ converge totalmente in ogni intervallo $[-\delta, \delta]$ $\forall \delta > 0$

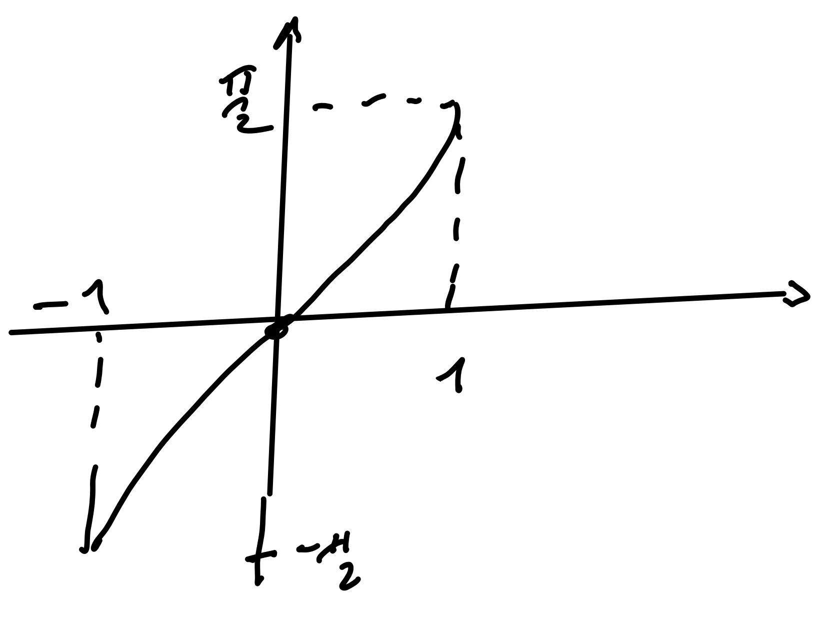
$\Leftrightarrow$ la serie in $x$ converge totalmente

$$\arcsin x \in [-\delta, \delta]$$

$$\Leftarrow \quad |\arcsin x| \leq \delta$$

$\delta = \frac{\pi}{2}$ : $\qquad |\arcsin x| \leq \frac{\pi}{2}$ ?

$$\forall x \in [-1, 1]$$



$$\sum_{n=1}^{\infty} \frac{e^{nx}}{n+\log n} \qquad y = e^{x}$$

$$\sum_{n=1}^{\infty} \frac{y^{n}}{n+\log n} \qquad a_n = \frac{1}{n+\log n}$$

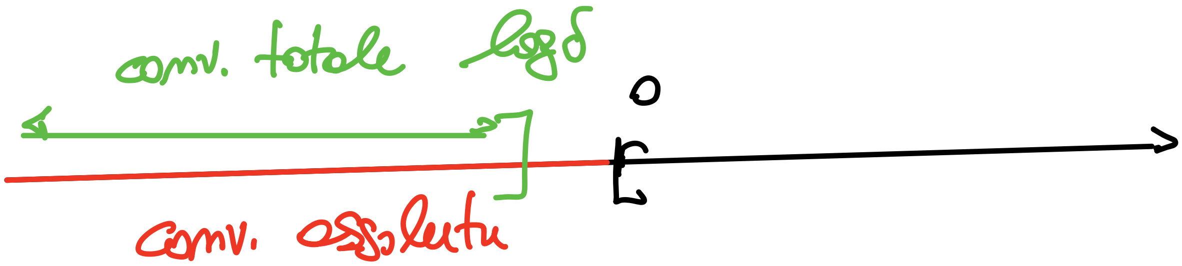
$$\rho = \lim_{n\to\infty} \frac{1}{n+\log n} \cdot (n+1)+\log(n+1)$$

$$= \lim_{n\to\infty} \frac{\overbrace{(n+1)+\log(n+1)}^{\sim (n+1)}}{\underbrace{n+\log n}_{\sim n}} = 1$$

Conv. assoluta quando $-1<y<1$

$$\iff -1 < e^{x} < 1 \iff e^{x} < 1$$

$$\iff \quad x<0 \quad \iff \quad x\in \,]-\infty,0[$$




La serie in $y$ converge totalmente

$$[-\delta,\delta], \qquad 0<\delta<1$$

$$\iff \qquad -\delta \le y \le \delta \qquad , \quad 0<\delta<1$$

$$\iff \qquad e^{x} \le \delta$$

$$\iff \qquad x \le \log\delta$$

$$]\underbrace{-\infty, \log\delta}\,], \quad \forall\, \delta\in\,]0,1[$$

conv. totale

Agli estremi?

$y = -1$ $\underline{\underline{No}}$

$y = 1 \iff e^x = 1 \iff x = 0$

$$\sum \frac{1}{n + \log n} \sim \sum \frac{1}{n} = +\infty$$

$$\sum_{n=0}^{\infty} \frac{[n^2(x^2-1)]^n}{(n+1)^{2n}}$$