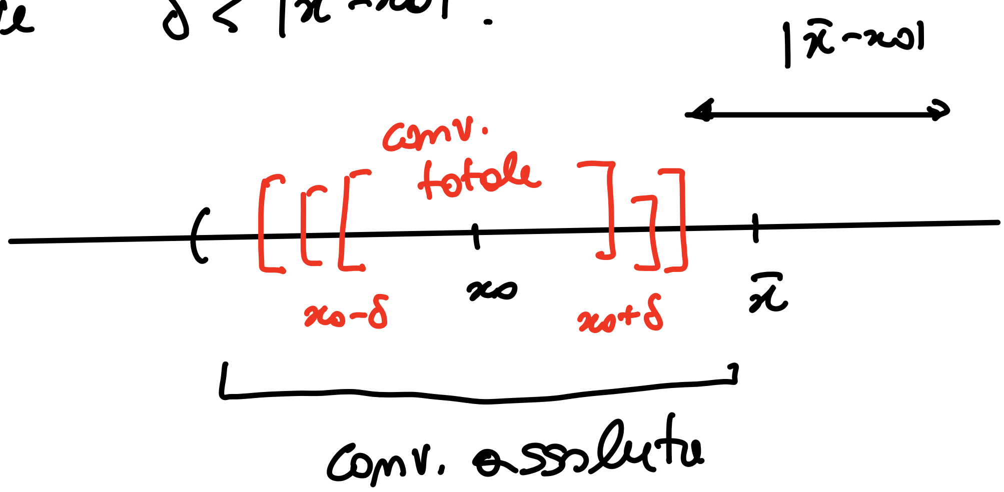
Lezione del 07/10/2022

Lemma $\sum_{n=0}^{\infty} a_n (x-x_0)^n$. Supponiamo che la serie converga per $x=\bar{x} \neq x_0$.
Allora la serie converge assolutamente per ogni x tale che $|x-x_0| < |\bar{x}-x_0|$ e converge totalmente per ogni x tale che

$$|x-x_0| \leq \delta \iff x \in [x_0-\delta, x_0+\delta]$$

dove $\delta < |\bar{x}-x_0|$.



$$x_0 + |\bar{x} - x_0| = \bar{x}$$

$$x_0 - |\bar{x} - x_0| = x_0 - \bar{x} + x_0$$
$$= 2x_0 - \bar{x}$$

Sia x tale che:

Dim. $|x - x_0| < |\bar{x} - x_0|$

$$\sum_{n=0}^{\infty} |a_n| \, |x - x_0|^n < \infty \quad \text{TESI}$$

Poiché per ipotesi, la serie converge per $x = \bar{x}$, $\bar{x} \neq x_0$, abbiamo

$$\lim_{n \to \infty} |a_n| \, |\bar{x} - x_0|^n = 0$$

$\Rightarrow \exists M > 0$ tale che

$$|a_n| \, |\bar{x} - x_0|^n \leq M, \quad \forall n \in \mathbb{N}$$

$$|a_n| \, |x - x_0|^n = |a_n| \, |\bar{x} - x_0|^n \left(\frac{|x - x_0|}{|\bar{x} - x_0|} \right)^n$$

$$\leq M \underbrace{\left(\frac{|x-x_0|}{|\bar{x}-x_0|}\right)}_{h_x}^{n} = M h_x^{n}$$

$h_x < 1$

$$\Rightarrow \sum h_x^{n} < \infty \text{ converge}$$

$$\Rightarrow \text{criterio del confronto} \Rightarrow \sum_{n=0}^{\infty} |a_n| |x-x_0|^n < \infty$$

Convergenza totale : scegliamo δ tale che $0 < \delta < |\bar{x}-x_0|$, vogliamo mostrare che la serie converge totalmente in

$$[x_0-\delta, x_0+\delta]$$

Sia $x \in [x_0-\delta, x_0+\delta] \iff |x-x_0| \leq \delta$

$$|a_n||x-x_0|^n = |a_n|\,|\bar{x}-x_0|^n \left(\frac{|x-x_0|}{|\bar{x}-x_0|}\right)^n$$

$$\le M \left(\underbrace{\frac{|x-x_0|}{|\bar{x}-x_0|}}\right)^n$$

$$\le M \left(\underbrace{\frac{\delta}{|\bar{x}-x_0|}}\right)^n$$

(perché $\delta < |\bar{x}-x_0|$)

$$h = \frac{\delta}{|\bar{x}-x_0|} < 1 .$$ Quindi $\Longrightarrow$

$$|a_n||x-x_0|^n \le M\, h^n ,$$ dove

$$\sum_{n=0}^{\infty} h^{n} < +\infty \quad , \forall x \in [x_0-\delta, x_0+\delta]$$

$\Rightarrow$ la serie conv. totalmente

in $[x_0-\delta, x_0+\delta]$.

Teorema (Raggio di convergenza)

Per la serie $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

vale solo una delle seguenti

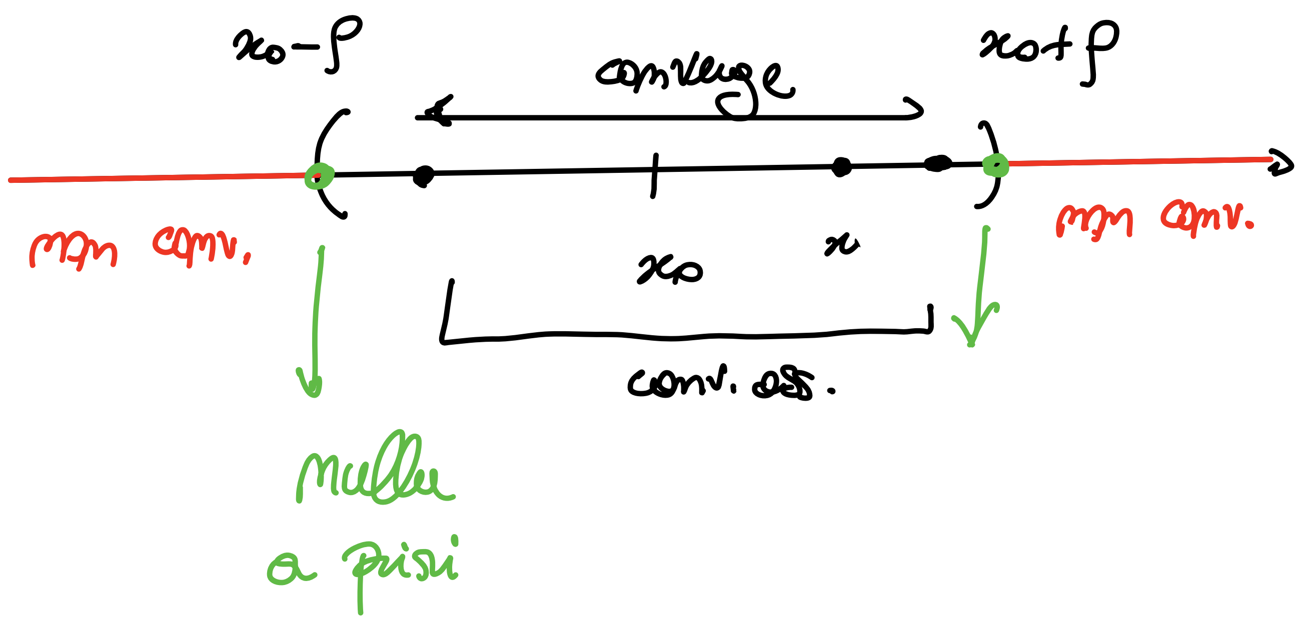
proprietà:

1) la serie converge solo per $x=x_0$.

2) la serie converge $\forall x \in \mathbb{R}$.

3) esiste una costante $\rho > 0$ tale che la serie converge per $x \in \,]x_0 - \rho, x_0 + \rho[$ $\quad (|x - x_0| < \rho)$

<u>non converge</u> quando $|x - x_0| > \rho$



Nei casi (1) e (2): $\rho = 0$

$= +\infty$

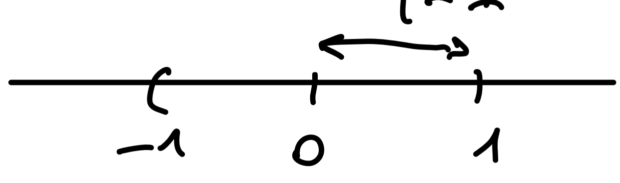
$\rho \in [0, +\infty]$

raggio di convergenza della serie.

Se $\rho > 0$, $]x_0 - \rho, x_0 + \rho[$ si chiama

intervallo di convergenza

ES $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \forall x \in]-1,1[$$

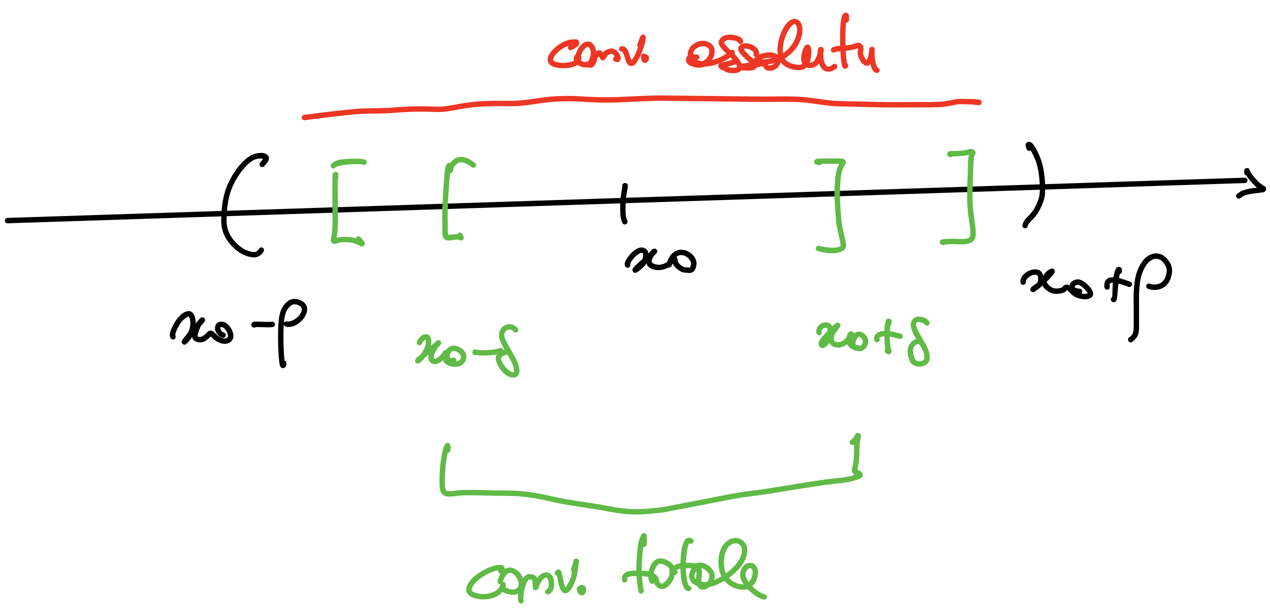


OSS: Se $\rho > 0$, dal Teorema di convergenza assoluta e totale: la serie converge assolutamente in

$$]x_0 - \rho, x_0 + \rho[$$

e converge totalmente in ogni intervallo

$$[x_0 - \delta, x_0 + \delta] \quad \text{dove } 0 < \delta < \rho$$



Agli estremi $x = x_0 \pm \rho$

nulla si può dire !!

Criterio della radice

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad \{a_n\} \subseteq \mathbb{R}$$

Supponiamo che $\exists \lim_{n\to\infty} \sqrt[n]{|a_n|} = \ell$

$\ell \in [0, +\infty]$.

Allora

$$\rho = 1/\ell = \begin{cases} +\infty & \text{se } \ell = 0 \\ 0 & \text{se } \ell = +\infty \\ \in]0, +\infty[& \text{se } \ell \in]0, +\infty[\end{cases}$$

$$\sum_{n=0}^{\infty} 1 \cdot x^n \qquad a_n = 1$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$$

$$\rho = 1$$

$$-(-2)$$

$$\sum_{n=0}^{\infty} \underbrace{(-1)^n 2^n}_{a_n} (x+2)^n$$

$$x_0 = -2$$

$$\lim_{n\to\infty} \sqrt[n]{|(-1)^n 2^n|} = \lim_{n\to\infty} \sqrt[n]{2^n} = 2$$

$\rho = 1/2$ raggio di conv.

$$]x_0-\rho, x_0+\rho[\; = \;]-2-\tfrac{1}{2}, -2+\tfrac{1}{2}[$$

$$= \left]-\frac{5}{2}, -\frac{3}{2}\right[$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \qquad \underline{\underline{\forall x \in \mathbb{R}}}$$

$\rho = +\infty$

$$a_n = \frac{1}{n!}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{\underbrace{\sqrt[n]{n!}}_{\downarrow\, +\infty}} = 0$$

$\rho = +\infty$

$$\sum_{n=0}^{\infty} n!\, x^n \qquad \rho = 0$$

$x_0 = 0$ converge $\underline{\text{solo}}$ in $\underline{x_0 = 0}$

Criterio del rapporto

Supponiamo che $\exists \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \ell$

$\ell \in [0, +\infty]$. Allora

$$\rho = \frac{1}{\ell}$$

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \rho$$

$$\lim_{n\to\infty} \left| \frac{(-1)^n \, 2^n}{(-1)^{n+1} \, 2^{n+1}} \right| = \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{2^n} \cdot 2}$$

$$= \frac{1}{2} \qquad \rho = 1/2$$

ES. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{3^n+1} =$

$$= \sum_{n=1}^{\infty} \frac{\left(2\left(x-1/2\right)\right)^n}{3^n+1} =$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{3^n+1} \left(x - 1/2\right)^n$$

$x_0 = 1/2$

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{\cancel{2^n}}{3^n+1} \cdot \frac{3^{n+1}+1}{\cancel{2^n}\cdot 2}$$

$$\cancel{\frac{1}{a_{n+1}}}$$

$$3^{n+1} = 3^n \cdot 3$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{\overbrace{3^{n+1}+1}^{\sim 3^{n+1}}}{\underbrace{3^n+1}_{\sim 3^n}}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{\cancel{3^n}\cdot 3}{\cancel{3^n}} = \frac{3}{2}$$

$$]x_0-\rho, x_0+\rho[\; = \;]\frac{1}{2}-\frac{3}{2}, \frac{1}{2}+\frac{3}{2}[$$

$$=]-1, 2[$$

int. convergenza

$x = -1$:

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n+1} \left(-1 - \tfrac{1}{2}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{3^n+1} \left(-\frac{3}{2}\right)^n$$

$$\left((-1)\tfrac{3}{2}\right)^n$$

$$= (-1)^n \left(\frac{3}{2}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{\cancel{2^n}}{3^n+1} \cdot \frac{3^n}{\cancel{2^n}}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{3^n+1}$$

$$\lim_{n\to\infty} \left|\frac{3^n}{3^n+1}\right| = 1$$

$\Rightarrow$ la serie non può convergere

$x = 2$

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}\left(2-\frac{1}{2}\right)^n =$$

$$= \sum_{n=1}^{\infty} \left(\frac{3^n}{3^n+1}\right) = +\infty$$

$$1$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)^{2n}}{(4n-1)^{2n}} (x-1)^n$$

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{\log \log n} (2x+1)^n$$

$$\sum_{n=0}^{\infty} 2^n (\log x)^{2n}$$

<u>serie di funzioni</u>

$a_n (x-x_0)^n$

$x>0$

$$y = (\log x)^2$$

$$\sum_{n=0}^{\infty} 2^n y^n \qquad \sqrt[n]{2^n} = 2 \to 2$$

$$\rho = \frac{1}{2}$$

$$\downarrow$$

converge (assolutamente) in

$$y \in \left]-\frac{1}{2}, \frac{1}{2}\right[$$

$$\iff \log^2 x \in \left]-\frac{1}{2}, \frac{1}{2}\right[$$

$$\iff -\frac{1}{2} < \log^2 x < \frac{1}{2}$$

$$\iff \log^2 x < \frac{1}{2}$$

$$t^2 < \frac{1}{2} \iff -\frac{\sqrt{2}}{2} < \underset{\log x}{\underset{\parallel}{t}} < \frac{\sqrt{2}}{2}$$

$$\iff -\frac{\sqrt{2}}{2} < \log x < \frac{\sqrt{2}}{2}$$

$$\iff e^{-\frac{\sqrt{2}}{2}} < x < e^{\frac{\sqrt{2}}{2}}$$

La serie in x converge nell'intervallo

$$\left] e^{-\frac{\sqrt{2}}{2}}, e^{\frac{\sqrt{2}}{2}} \right[.$$

$$\sum_{n=0}^{\infty} 2^n y^n$$ converge totalmente

$y \in [-\delta, \delta]$, $0 < \delta < \frac{1}{2}$

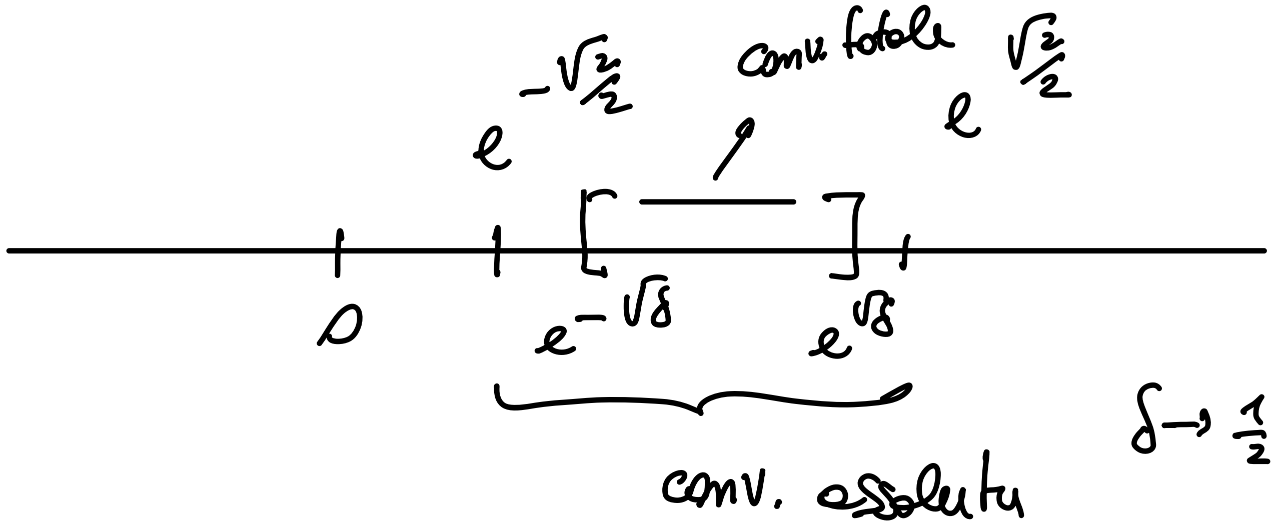
$\Leftrightarrow \log^2 x \leq \delta$

$\Leftrightarrow -\sqrt{\delta} \leq \log x \leq \sqrt{\delta}$

$\Leftrightarrow e^{-\sqrt{\delta}} \leq x \leq e^{\sqrt{\delta}}$

$[e^{-\sqrt{\delta}}, e^{\sqrt{\delta}}]$

conv. totale



Agli estremi?