Campi Elettromagnetici

Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

a.a. 2019-2020 - Laurea “Triennale” – Secondo semestre - Secondo anno

Università degli Studi di Napoli “Parthenope”

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Color legend

New formulas, important considerations,
important formulas, important concepts

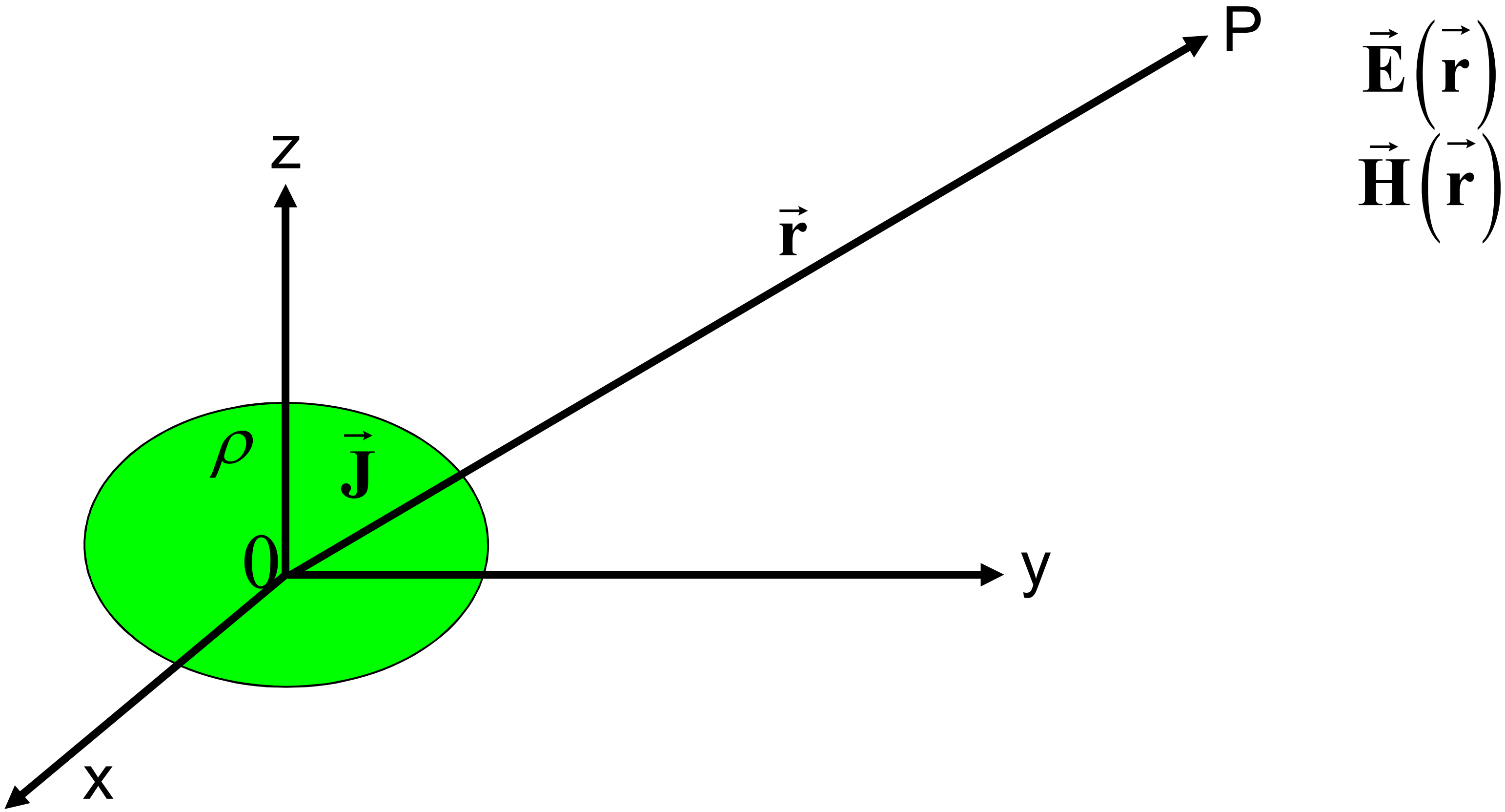
Very important for the discussion

Memo

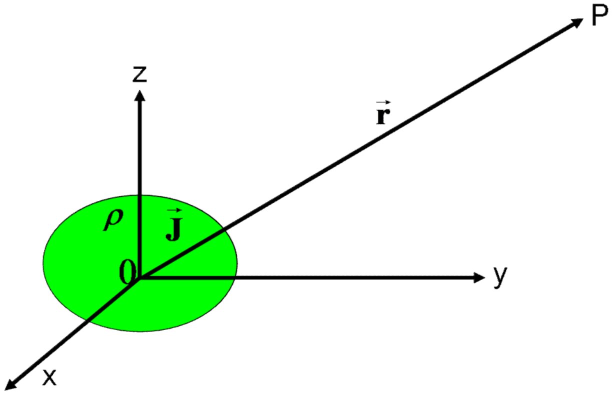
Mathematical tools to be exploited

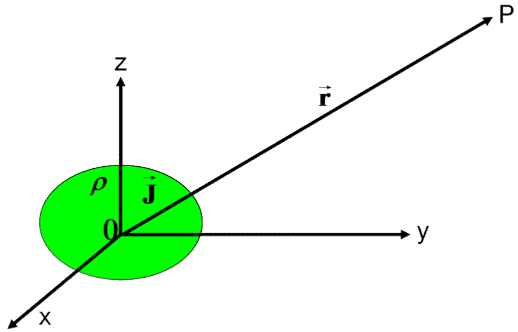
Mathematics

Radiation problem

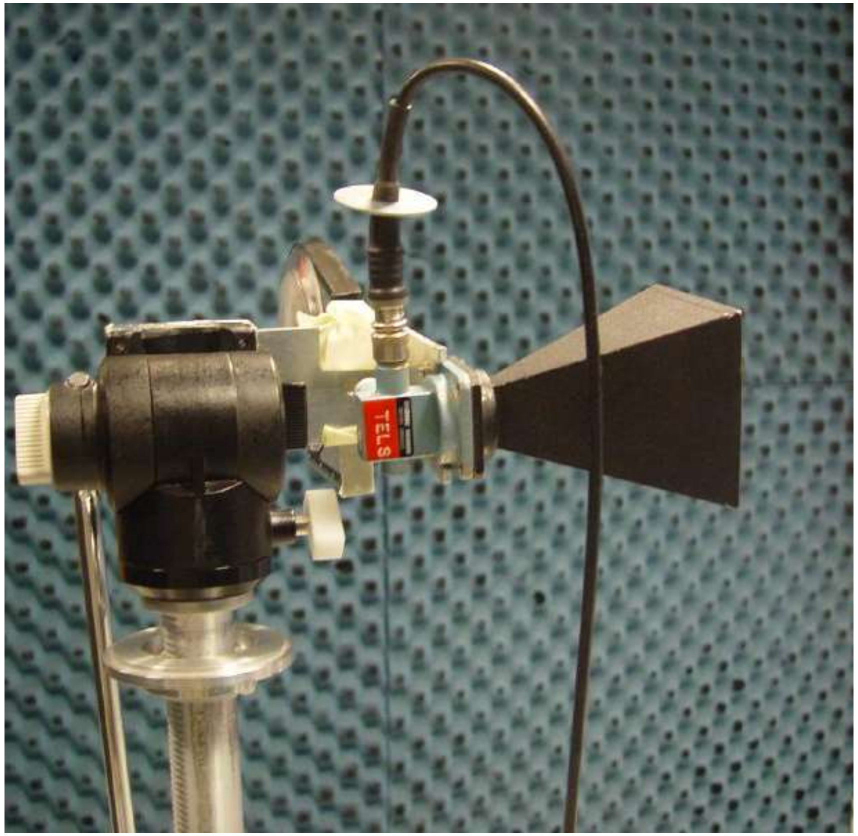


An antenna (or aerial) is an electrical device which converts electric power into radio waves, and vice versa.

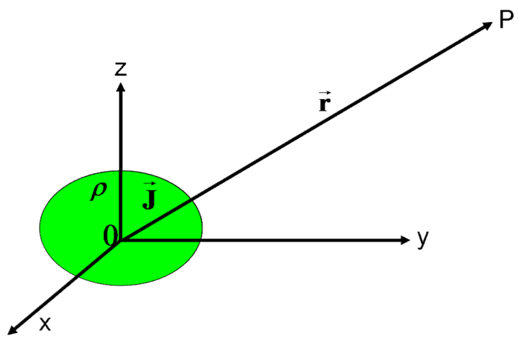




Horn antenna

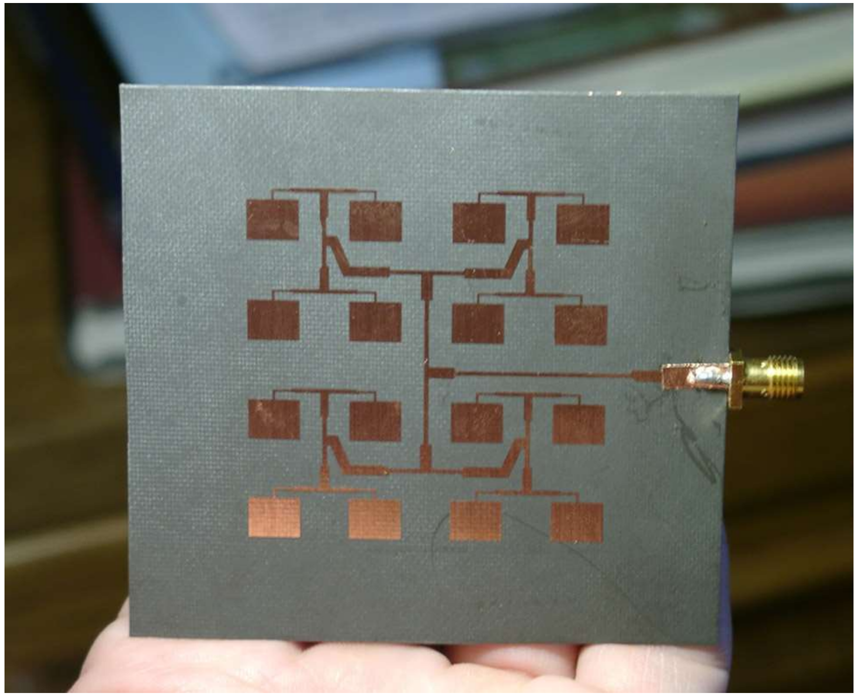


Dipole antenna



Helix or helical antenna

Microstrip antenna



Radiation problem

- We approach the radiation problem of an antenna assuming that the sources are known and the media in which propagation occurs is simple.
- Mathematically we exploit the potentials solution.

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Medium: linear, isotropic, homogeneous (time-invariant and space-invariant), non-dispersive in space and time

Phasor domain

$$
\begin{cases}
\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\
\nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\
\nabla \cdot \varepsilon \boldsymbol{E} = \rho \\
\nabla \cdot \mu \boldsymbol{H} = 0
\end{cases}
$$

Symbols and notations

$\vec{\mathbf{r}}$ $\mathbf{r}$

$\vec{\mathbf{E}}$ $\mathbf{E}$

vectors

$\vec{\mathbf{E}}(\vec{\mathbf{r}})$ $\vec{\mathbf{E}}(\mathbf{r})$ $\mathbf{E}(\mathbf{r})$ Vector fields

$\phi(\vec{\mathbf{r}})$ $\phi(\mathbf{r})$ Scalar fields

Radiation problem

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Phasor domain

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Radiation problem & potentials

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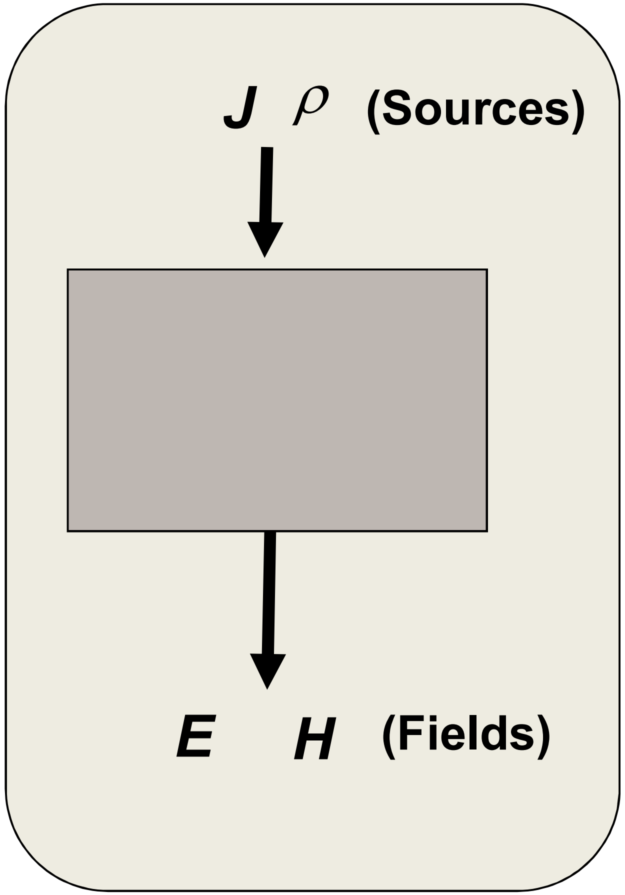

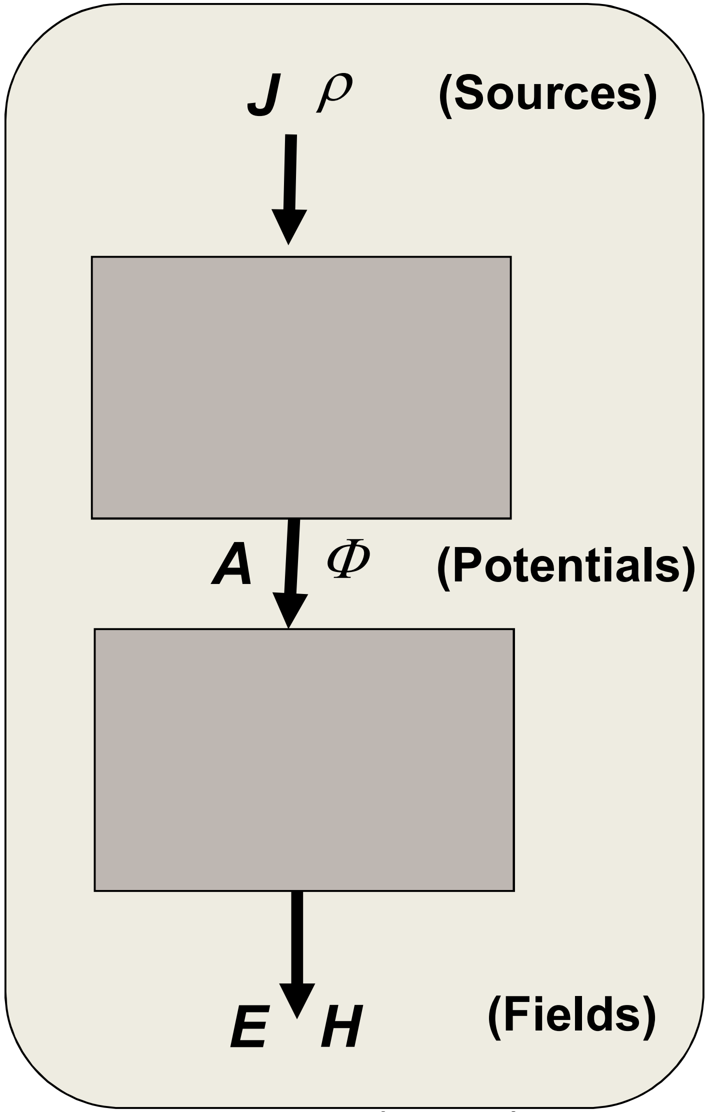

… mathematical tools that we will exploit today…

$$\nabla \times \mathbf{C} = \mathbf{0} \quad \Rightarrow \quad \exists\, \Phi \quad : \quad \mathbf{C} = \nabla \Phi$$

$$\nabla \cdot \mathbf{C} = 0 \quad \Rightarrow \quad \exists\, \mathbf{A} \quad : \quad \mathbf{C} = \nabla \times \mathbf{A}$$

Radiation problem & potentials

$$\begin{cases} \boxed{\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \boxed{\nabla \cdot \mu \boldsymbol{H} = 0} \end{cases}$$

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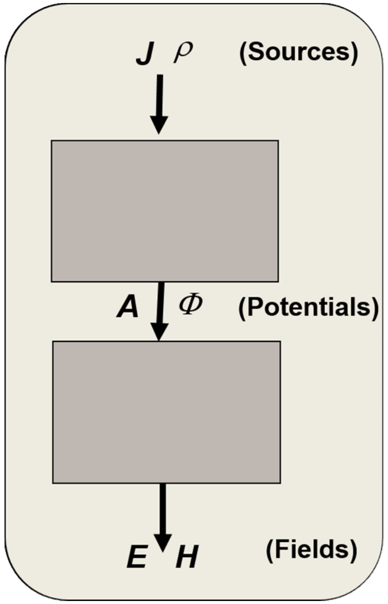
$$\cdot\, \mu \mathbf{H} = 0 \quad \Rightarrow \boxed{\mu \mathbf{H} = \nabla \times \mathbf{A}}$$

$$\times \mathbf{E} + j\omega \boxed{\mu \mathbf{H}} = 0 \quad \Rightarrow \nabla \times \mathbf{E} + j\omega \nabla \times \mathbf{A} = 0 \quad \Rightarrow \nabla \times [\mathbf{E} + j\omega \mathbf{A}] = 0 \quad \Rightarrow [\mathbf{E} + j\omega \mathbf{A}] = -\nabla \Phi$$

Radiation problem & potentials

$$\begin{cases} \boxed{\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \boxed{\nabla \cdot \mu \boldsymbol{H} = 0} \end{cases}$$



$$\cdot\, \mu \mathbf{H} = 0 \quad \Rightarrow \boxed{\mu \mathbf{H} = \nabla \times \mathbf{A}}$$

$$\times \mathbf{E} + j\omega\mu \mathbf{H} = 0 \quad \Rightarrow \nabla \times \mathbf{E} + j\omega \nabla \times \mathbf{A} = 0 \quad \Rightarrow \nabla \times \left[\mathbf{E} + j\omega \mathbf{A}\right] = 0 \quad \Rightarrow \boxed{\left[\mathbf{E} + j\omega \mathbf{A}\right] = -\nabla \Phi}$$

Radiation problem & potentials

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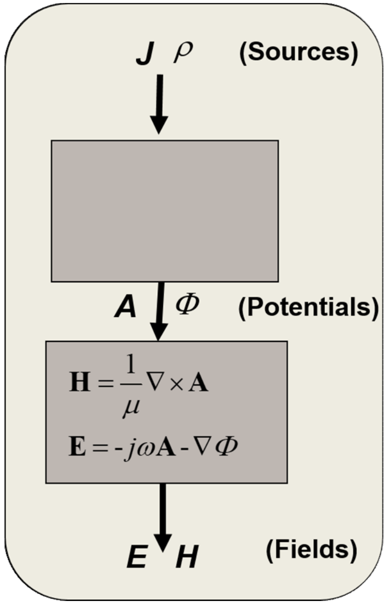

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Radiation problem & potentials

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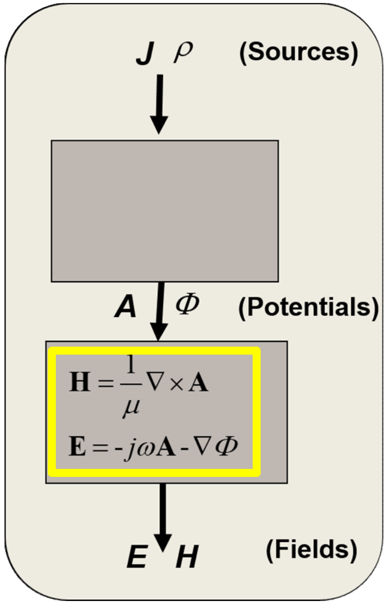


$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\Phi\right.$$

$$\omega^2 \mu\varepsilon = k^2$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E} + \mathbf{J} \Rightarrow \nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{A}\right) = j\omega\varepsilon\left(-j\omega \mathbf{A} - \nabla\Phi\right) + \mathbf{J} \Rightarrow \nabla \times \left(\nabla \times \mathbf{A}\right) = j\omega\mu\varepsilon\left(-j\omega \mathbf{A} - \nabla\Phi\right) + \mu \mathbf{J}$$

$$\Downarrow$$

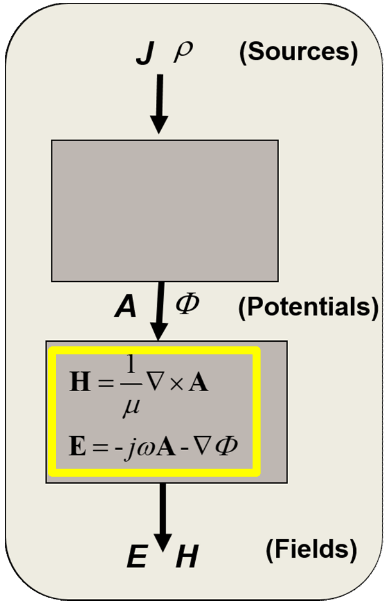
$$\nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} = \omega^2 \mu\varepsilon \mathbf{A} - j\omega\mu\varepsilon\nabla\Phi + \mu \mathbf{J}$$

$$\Rightarrow \nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\nabla\Phi \Rightarrow \nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\Phi\right)$$

$$\nabla \times \left(\nabla \times \mathbf{A}\right) = \nabla\nabla \cdot \mathbf{A} - \nabla$$

Radiation problem & potentials

$$\begin{cases} \nabla\times\boldsymbol{E}=-j\omega\mu\boldsymbol{H} \\ \nabla\times\boldsymbol{H}=j\omega\varepsilon\boldsymbol{E}+\boldsymbol{J} \\ \nabla\cdot\varepsilon\boldsymbol{E}=\rho \\ \nabla\cdot\mu\boldsymbol{H}=0 \end{cases}$$



$$\nabla^2\mathbf{A}+k^2\mathbf{A}=-\mu\mathbf{J}+\nabla\left(\nabla\cdot\mathbf{A}+j\omega\mu\varepsilon\Phi\right.$$

$$\nabla^2\Phi+k^2\Phi=-\frac{\rho}{\varepsilon}-j\omega\left(\nabla\cdot\mathbf{A}+j\omega\mu\varepsilon\Phi\right.$$

$$\omega^2\mu\varepsilon=k^2$$

$$\cdot\mathbf{E}=\frac{\rho}{\varepsilon} \Rightarrow \nabla\cdot\left(-j\omega\mathbf{A}-\nabla\Phi\right)=\frac{\rho}{\varepsilon} \Rightarrow -\nabla\cdot\left(\nabla\Phi\right)-j\omega\nabla\cdot\mathbf{A}=\frac{\rho}{\varepsilon} \Rightarrow -\nabla^2\Phi-j\omega\nabla\cdot\mathbf{A}=\frac{\rho}{\varepsilon}$$

$$\Downarrow$$

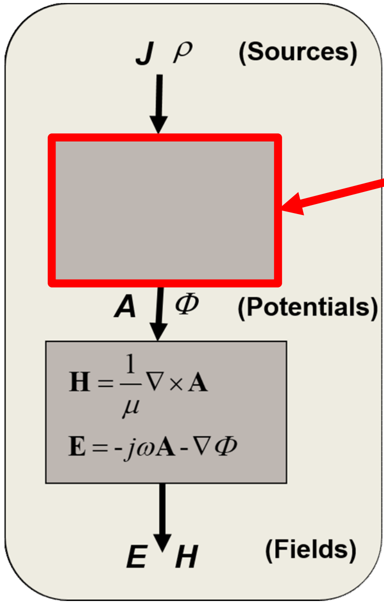
$$\nabla^2\Phi \quad +k^2\Phi \quad =-\frac{\rho}{\varepsilon} \quad -j\omega\nabla\cdot\mathbf{A}-jj\omega\omega\mu\varepsilon\Phi$$

$$+\omega^2\mu\varepsilon\Phi \qquad\qquad +\omega^2\mu\varepsilon\Phi$$

$$\nabla\cdot\left(\nabla\Phi\right)=\nabla^2\Phi$$

Radiation problem & potentials

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$



$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\Phi\right.$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\Phi\right.$$

$$\omega^2 \mu\varepsilon = k^2$$

$$\cdot\mathbf{E} = \frac{\rho}{\varepsilon} \Rightarrow \nabla \cdot \left(-j\omega\mathbf{A} - \nabla\Phi\right) = \frac{\rho}{\varepsilon} \Rightarrow -\nabla \cdot \left(\nabla\Phi\right) - j\omega\nabla \cdot \mathbf{A} = \frac{\rho}{\varepsilon} \Rightarrow -\nabla^2\Phi - j\omega\nabla \cdot \mathbf{A} = \frac{\rho}{\varepsilon}$$

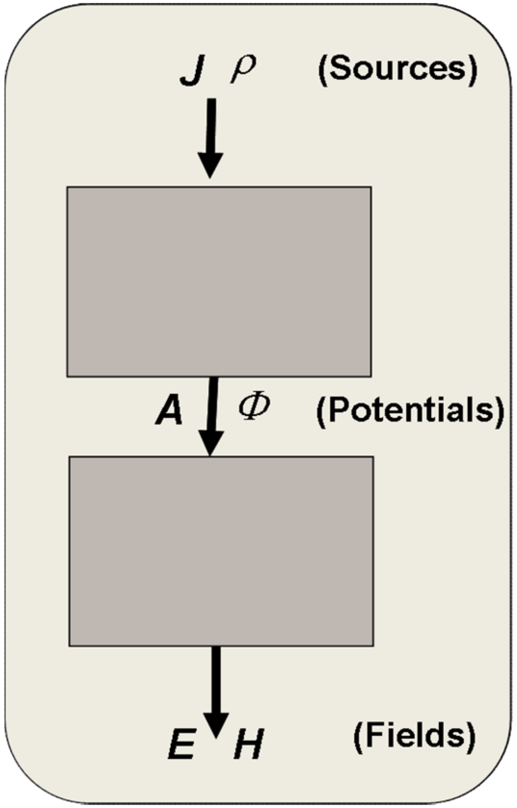
$$\Downarrow$$

$$\nabla^2\Phi \quad +k^2\Phi \quad = -\frac{\rho}{\varepsilon} \quad - j\omega\nabla \cdot \mathbf{A} - jj\omega\omega\mu\varepsilon\Phi$$

$$+\omega^2\mu\varepsilon\Phi \qquad\qquad +\omega^2\mu\varepsilon\Phi$$

$$\nabla \cdot \left(\nabla\Phi\right) = \nabla^2\Phi$$

Potentials



$\mathbf{J} \quad \rho$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi)$$

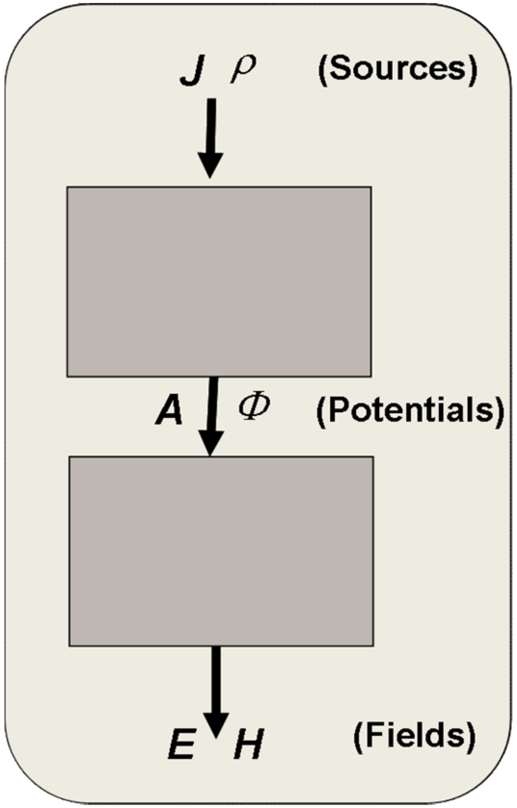
$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi\right)$$

$\mathbf{A} \quad \Phi$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

$\mathbf{E} \quad \mathbf{H}$

Potentials



$$J \quad \rho$$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi\right)$$

$$A \quad \Phi$$

Mathematical tools

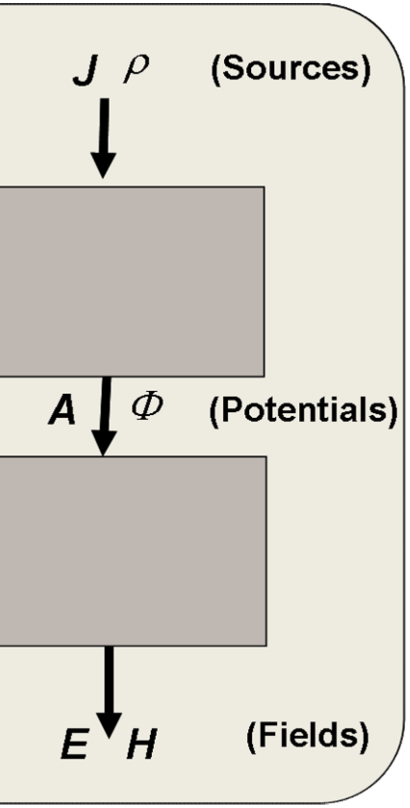
$$\mathbf{C} = \mathrm{C_x}\left(x,y,z\right)\hat{i}_x + \mathrm{C_y}\left(x,y,z\right)\hat{i}_y + \mathrm{C_z}\left(x,y,z\right)\hat{i}_z$$

$$\Phi = \Phi\left(x,y,z\right)$$

$$\nabla^2\Phi = \nabla\cdot\left(\nabla\Phi\right) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\mathbf{C} = \nabla^2\mathrm{C}_x\,\hat{i}_x + \nabla^2\mathrm{C}_y\,\hat{i}_y + \nabla^2\mathrm{C}_z\,\hat{i}_z$$

Potentials



$\mathbf{J} \quad \rho$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi\right)$$

$\mathbf{A} \quad \Phi$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = \cdots \Rightarrow \begin{cases} \nabla^2 A_x + k^2 A_x = \\ \nabla^2 A_y + k^2 A_y = \\ \nabla^2 A_z + k^2 A_z = \end{cases}$$

$$\nabla^2 \Phi + k^2 \Phi = \cdots$$

$$\nabla^2 \mathbf{C} = \nabla^2 C_x \hat{i}_x + \nabla^2 C_y \hat{i}_y + \nabla^2 C_z \hat{i}_z$$

Mathematical tools

$$\mathbf{C} = \mathrm{C_x}(x,y,z)\hat{i}_x + \mathrm{C_y}(x,y,z)\hat{i}_y + \mathrm{C_z}(x,y,z)\hat{i}_z$$

$$\mathbf{A} = \mathrm{A_x}(x,y,z)\hat{i}_x + \mathrm{A_y}(x,y,z)\hat{i}_y + \mathrm{A_z}(x,y,z)\hat{i}_z$$

$$\Phi = \Phi(x,y,z)$$

I) $\nabla \cdot \mathbf{C} = 0 \quad \Rightarrow \quad \exists\, \mathbf{A} \quad : \quad \mathbf{C} = \nabla \times \mathbf{A}$

II) $\nabla \times \mathbf{C} = \mathbf{0} \quad \Rightarrow \quad \exists\, \Phi \quad : \quad \mathbf{C} = \nabla \Phi$

Potentials & uniqueness

$$\text{I)} \qquad \nabla \cdot \mathbf{C} = 0 \quad \Rightarrow \qquad \exists\, \mathbf{A} \quad : \quad \mathbf{C} = \nabla \times \mathbf{A}$$

Let us suppose that a vector $\mathbf{A}_0$ exists such that $\nabla \times \mathbf{A}_0 = \mathbf{0}$

$$\nabla \times (\mathbf{A} + \mathbf{A}_0) = \nabla \times \mathbf{A} + \nabla \times \mathbf{A}_0 = \nabla \times \mathbf{A}$$

$$\text{I)} \quad \Rightarrow \quad \mathbf{C} = \nabla \times (\mathbf{A} + \mathbf{A}_0)$$

where $\nabla \times \mathbf{A}_0 = \mathbf{0}$

A is defined but for a vector $\mathbf{A}_o$ that is curl free.

Potentials & uniqueness

$$\text{II)} \qquad \nabla \times \mathbf{C} = \mathbf{0} \quad \Rightarrow \qquad \exists\, \Phi \quad : \quad \mathbf{C} = \nabla \Phi$$

Let us suppose that a scalar Φ_0 exists such that $\nabla \Phi_0 = \mathbf{0}$

$$\nabla\left(\Phi + \Phi_0\right) = \nabla \Phi + \nabla \Phi_0 = \nabla \Phi$$

$$\text{II)} \quad \Rightarrow \quad \mathbf{C} = \nabla\left(\Phi + \Phi_0\right)$$

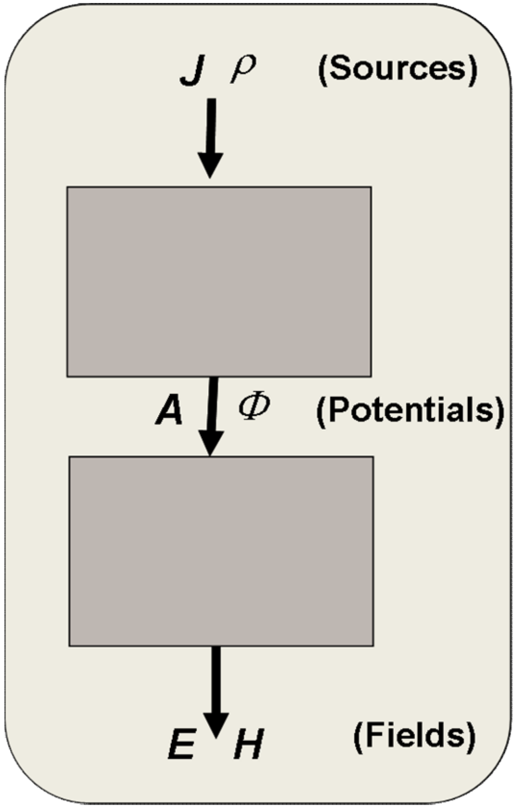
where $\nabla \Phi_0 = \mathbf{0}$

Φ is defined but for a scalar Φ_o that is gradient free.

Potentials

Amongst the infinite couples of potentials, is it possible to find a couple such that

$$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0 \quad ?$$



$\boldsymbol{J}$ ρ

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

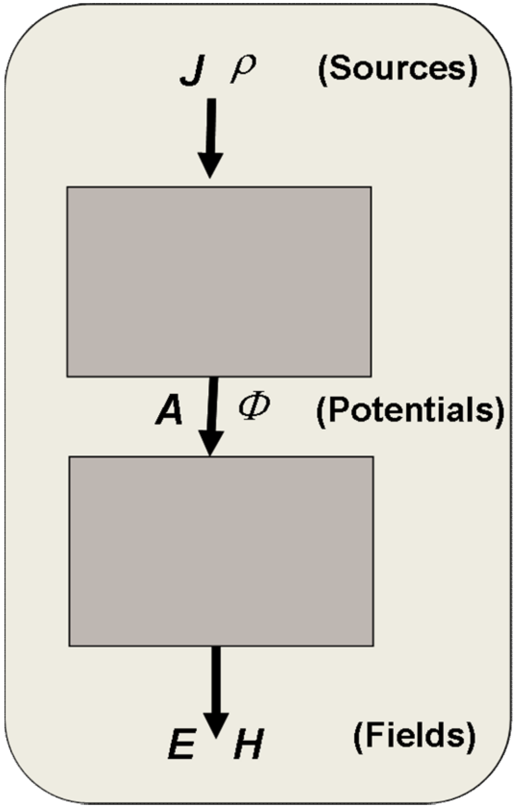
$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \boldsymbol{A} + \omega\varepsilon\mu\Phi\right)$$

$\boldsymbol{A}$ Φ

Potentials

$$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0$$

Lorentz gauge



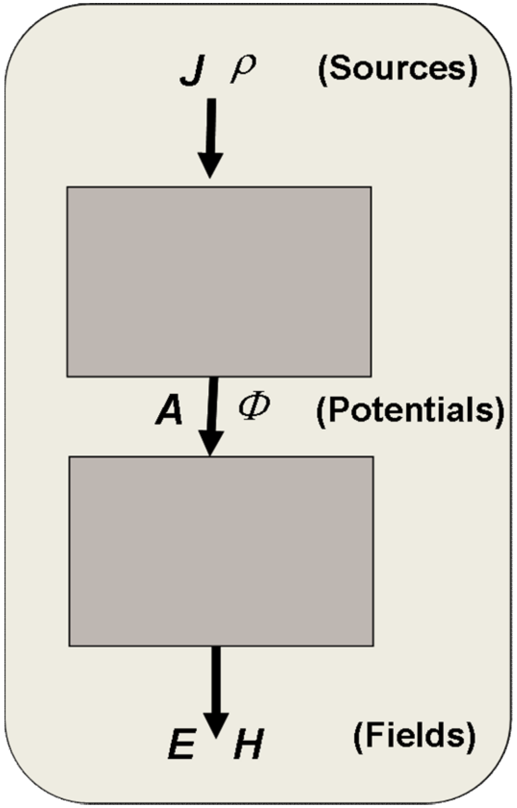
$\boldsymbol{J}$ ρ

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla\left(\cancel{\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi}\right)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\cancel{\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi}\right)$$

$\boldsymbol{A}$ Φ

Potentials



$\boldsymbol{J}$ ρ

$$\begin{cases} \nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} & \textbf{Vector Helmholtz equation} \\ \nabla^2 \Phi + k^2 \Phi = -\dfrac{\rho}{\varepsilon} & \textbf{Scalar Helmholtz equation} \end{cases}$$

$\boldsymbol{A}$ Φ

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega \mathbf{A} - \nabla \Phi$$

$\boldsymbol{E}$ $\boldsymbol{H}$

Potentials

$$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0$$

Lorentz gauge

J ρ

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

$$\cancel{\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon}}$$ **Scalar Helmholtz equation**

A Φ

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

E *H*

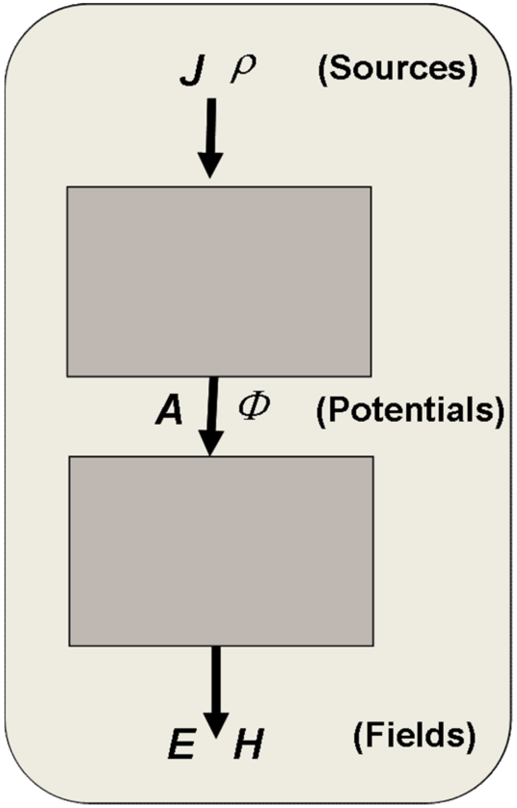
$$\nabla\left(\frac{\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}\right)$$

Note that once ***A*** is calculated by solving the (vector) Helmholtz equation involving ***A*** and ***J***, subsequent calculation of Φ can be straightforwardly achieved by means of the *Lorentz gauge*

$$\Phi = -\frac{\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}$$

thus rendering unnecessary the solution of the (scalar) Helmholtz equation relevant to Φ

Potentials



$\boldsymbol{J}$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

Vector Helmholtz equation

$\boldsymbol{A}$

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla \nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$$

$\boldsymbol{E}$ $\boldsymbol{H}$

Potentials

$\boldsymbol{J}$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

$\boldsymbol{A}$

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

Potentials

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

Let us address the solution of the following scalar Helmholtz equation

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

Potentials

$$\boxed{\nabla^2 A_x + k^2 A_x = -\mu J_x}$$

$$-\mu J_x(\vec{\mathbf{r}}) \longrightarrow \boxed{\quad g(\vec{\mathbf{r}}) \quad} \longrightarrow A_x(\vec{\mathbf{r}})$$

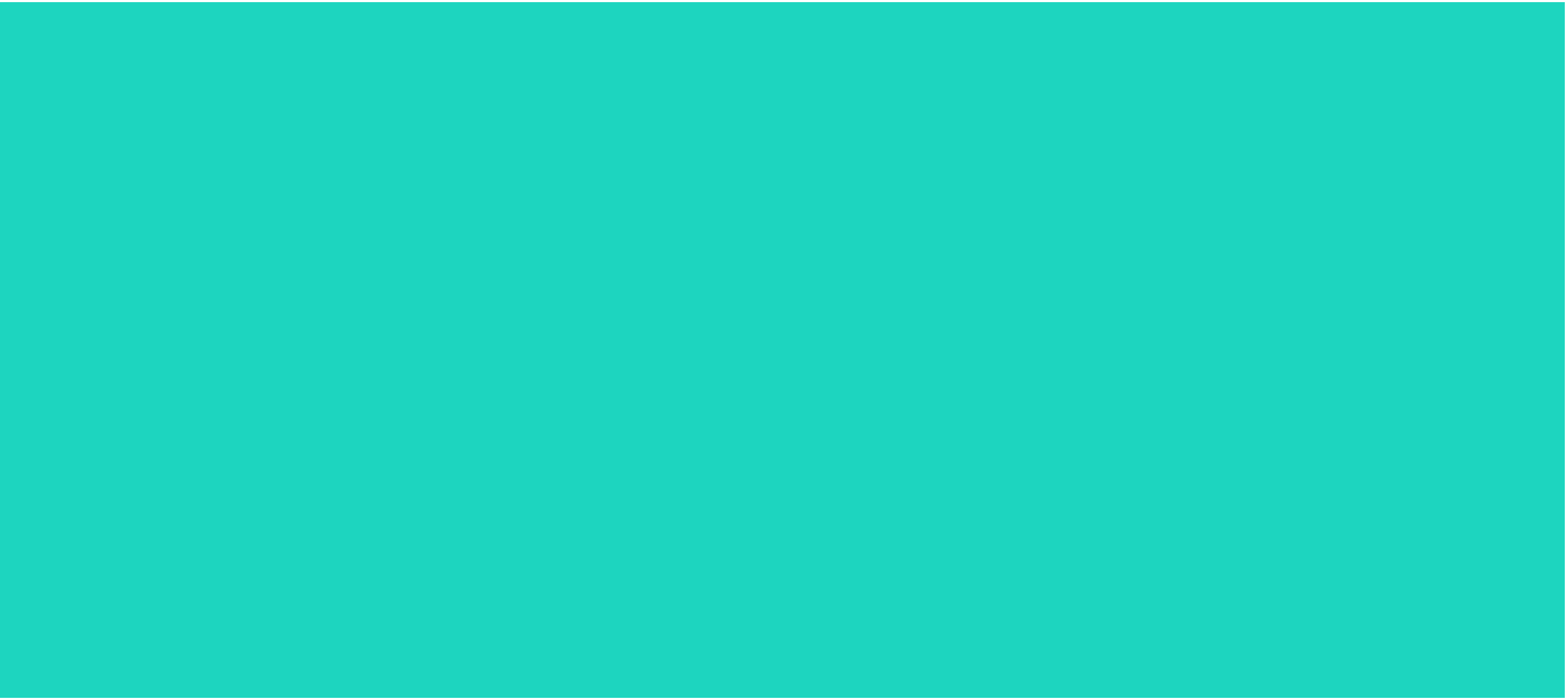
$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{\qquad\qquad} \longrightarrow g(\vec{\mathbf{r}})$$

$$\delta(\vec{\mathbf{r}}-\vec{\mathbf{r}}') \longrightarrow \boxed{\qquad\qquad} \longrightarrow g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')$$

$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\, d\vec{\mathbf{r}}'$$

Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$



Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{\qquad\qquad} \longrightarrow g(\vec{\mathbf{r}})$$

$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$

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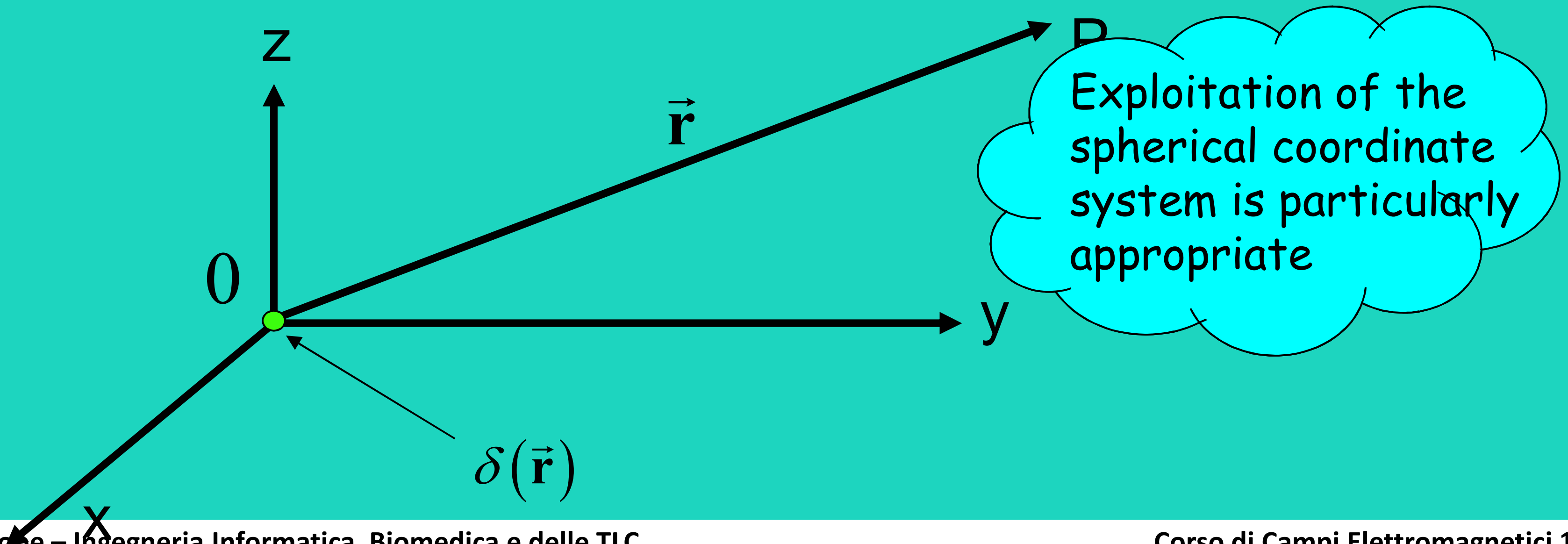
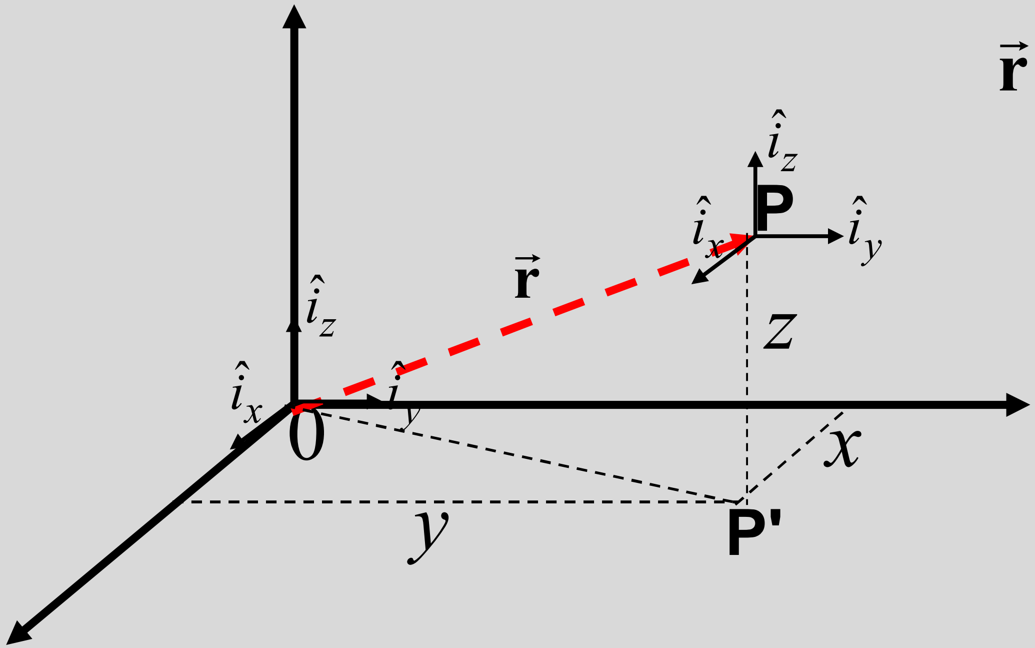


Reference systems: Cartesian

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E_x}(\vec{\mathbf{r}})\hat{i}_x + \mathrm{E_y}(\vec{\mathbf{r}})\hat{i}_y + \mathrm{E_z}(\vec{\mathbf{r}})\hat{i}_z$$

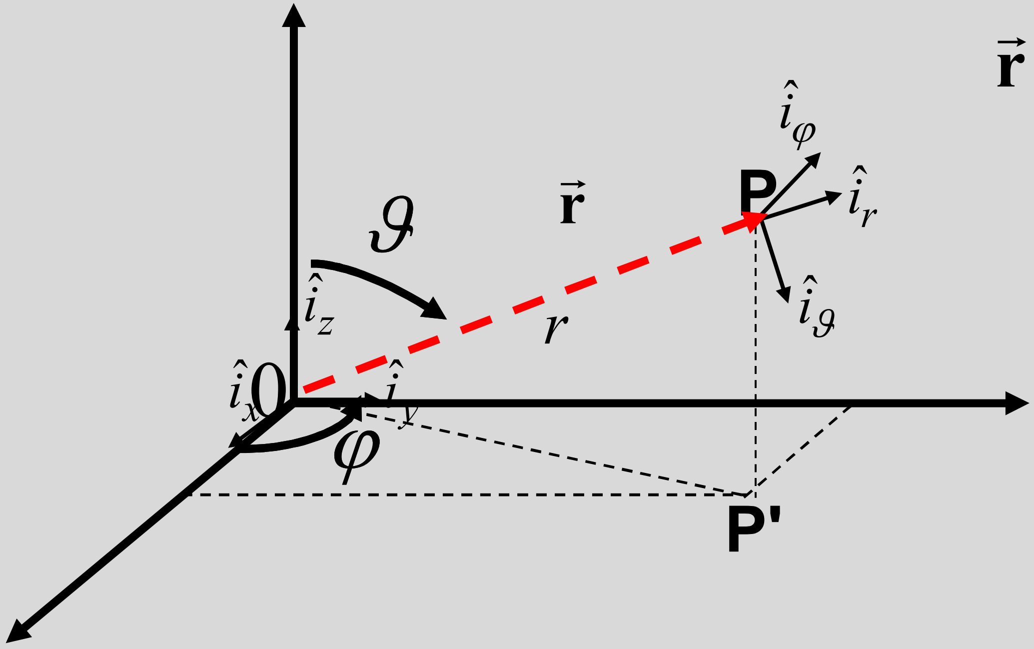
$$\vec{\mathbf{r}} = (x, y, z)$$



Reference systems: Spherical

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_r(\vec{\mathbf{r}})\hat{i}_r + \mathrm{E}_\varphi(\vec{\mathbf{r}})\hat{i}_\varphi + \mathrm{E}_\vartheta(\vec{\mathbf{r}})\hat{i}_\vartheta$$

$$\vec{\mathbf{r}} = (r, \vartheta, \varphi)$$



Potentials

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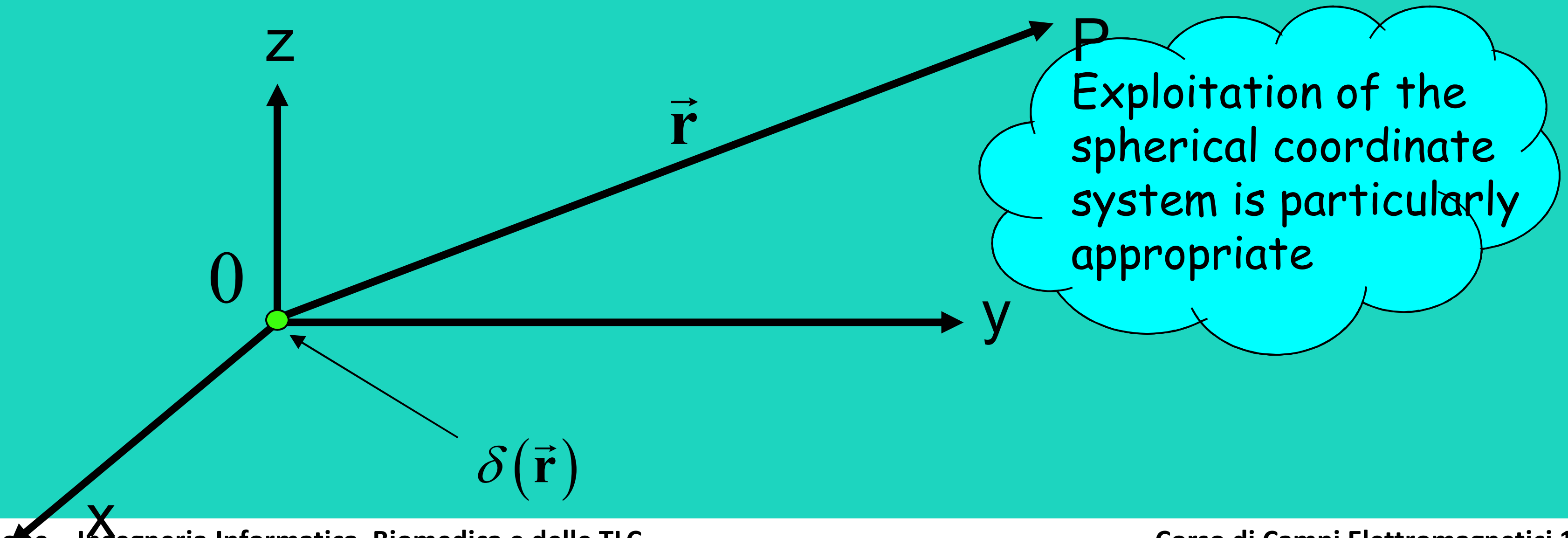
$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow g(\vec{\mathbf{r}})$$

$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$



Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{\qquad\qquad} \longrightarrow g(\vec{\mathbf{r}})$$

$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$

where, in principle, $g(\vec{\mathbf{r}}) = g(r, \vartheta, \varphi)$

However, due to symmetry considerations, the function $A_x(r,\vartheta,\varphi)$ turns out to be independent of ϑ and φ, that is,

$$g(\vec{\mathbf{r}}) = g(r)$$

Accordingly, in the whole three dimensional space the solution of the Helmholtz equation is:

$$g(r) = -\frac{1}{4\pi}\frac{e^{-jkr}}{r}$$

Potentials

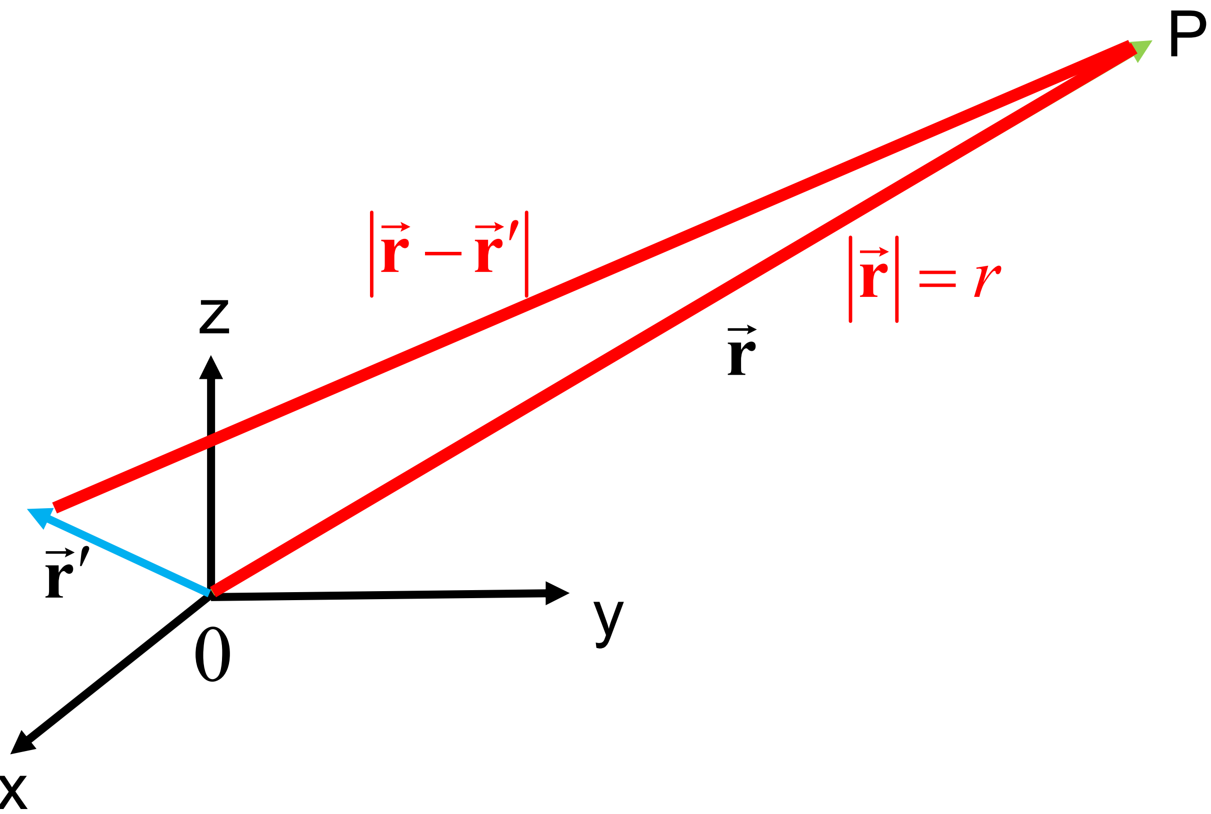
$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$-\mu J_x(\vec{\mathbf{r}}) \longrightarrow \boxed{g(\vec{\mathbf{r}})} \longrightarrow A_x(\vec{\mathbf{r}})$$

$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{\phantom{g(\vec{\mathbf{r}})}} \longrightarrow g(\vec{\mathbf{r}}) = -\frac{1}{4\pi}\frac{e^{-jkr}}{r} = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}|}}{|\vec{\mathbf{r}}|}$$

$$\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}') \longrightarrow \boxed{\phantom{g(\vec{\mathbf{r}})}} \longrightarrow$$

Potentials



$$\delta(\vec{\mathbf{r}}) \rightarrow -\frac{1}{4\pi}\frac{e^{-jkr}}{r} = -\frac{1}{4\pi}\frac{e^{-jk\left|\vec{\mathbf{r}}\right|}}{\left|\vec{\mathbf{r}}\right|}$$

$$\delta(\vec{\mathbf{r}}-\vec{\mathbf{r}}') \rightarrow -\frac{1}{4\pi}\frac{e^{-jk\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|}}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|}$$

Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$-\mu J_x(\vec{\mathbf{r}}) \longrightarrow \boxed{g(\vec{\mathbf{r}})} \longrightarrow A_x(\vec{\mathbf{r}})$$

$$\delta(\vec{\mathbf{r}}) \longrightarrow \boxed{} \longrightarrow g(\vec{\mathbf{r}}) = -\frac{1}{4\pi}\frac{e^{-jkr}}{r} = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}|}}{|\vec{\mathbf{r}}|}$$

$$\delta(\vec{\mathbf{r}}-\vec{\mathbf{r}}') \longrightarrow \boxed{} \longrightarrow = -\frac{1}{4\pi}\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')$$

$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\, d\vec{\mathbf{r}}' = \frac{\mu}{4\pi}\int J_x(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$

Potentials

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

c

$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\, d\,\vec{\mathbf{r}}' \;= \frac{\mu}{4\pi} \int J_x(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|} d\,\vec{\mathbf{r}}'$$

Potentials

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} \quad \Longrightarrow \quad \mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} \int \mathbf{J}(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\,\vec{\mathbf{r}}'$$

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases} \quad \Longrightarrow \quad \begin{aligned} A_x(\vec{\mathbf{r}}) &= \frac{\mu}{4\pi} \int J_x(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\,\vec{\mathbf{r}}' \\ A_y(\vec{\mathbf{r}}) &= \frac{\mu}{4\pi} \int J_y(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\,\vec{\mathbf{r}}' \\ A_z(\vec{\mathbf{r}}) &= \frac{\mu}{4\pi} \int J_z(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\,\vec{\mathbf{r}}' \end{aligned}$$

$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\, d\,\vec{\mathbf{r}}' = \frac{\mu}{4\pi} \int J_x(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\,\vec{\mathbf{r}}'$$

Potentials

$\downarrow$ $\boldsymbol{J}(\boldsymbol{r})$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int \mathbf{J}(\mathbf{r}')\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu}$$

$\downarrow$ $\boldsymbol{E}(\boldsymbol{r})$

$\boldsymbol{H}(\boldsymbol{r})$

Potentials

$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi}\int \mathbf{J}(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\,\vec{\mathbf{r}}'$$