Campi Elettromagnetici

Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

a.a. 2019-2020 - Laurea “Triennale” – Secondo semestre - Secondo anno

Università degli Studi di Napoli “Parthenope”

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THEOREMS

Poynting

Time domain – Phasor domain

Uniqueness (Interior problem – Exterior problem)

Time domain – Phasor domain

Equivalence

Phasor domain

Image Theory

Reciprocity

Phasor domain

Color legend

New formulas, important considerations,
important formulas, important concepts

Very important for the discussion

Memo

Mathematical tools to be exploited

Mathematics

Reciprocity theorem

$\vec{\mathbf{E}}_1, \vec{\mathbf{H}}_1$

Consider a source distribution $\vec{\mathbf{J}}_{01}$ with its associated electromagnetic field $\left(\vec{\mathbf{E}}_1, \vec{\mathbf{H}}_1\right)$

Reciprocity theorem

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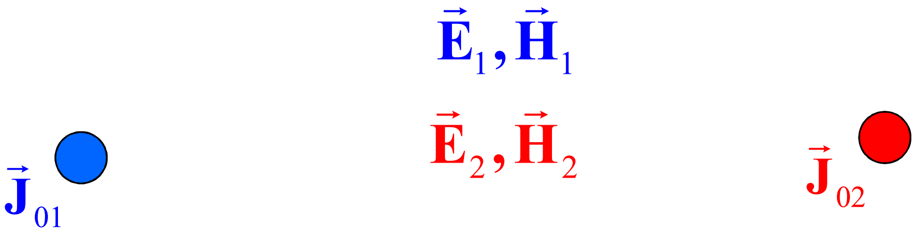
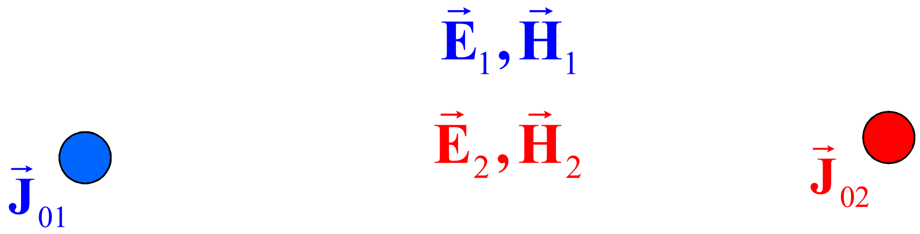
Reciprocity theorem



Consider a source distribution $\vec{\mathbf{J}}_{01}$ with its associated electromagnetic field $\left(\vec{\mathbf{E}}_1, \vec{\mathbf{H}}_1\right)$

Consider a source distribution $\vec{\mathbf{J}}_{02}$ with its associated electromagnetic field $\left(\vec{\mathbf{E}}_2, \vec{\mathbf{H}}_2\right)$

Reciprocity theorem



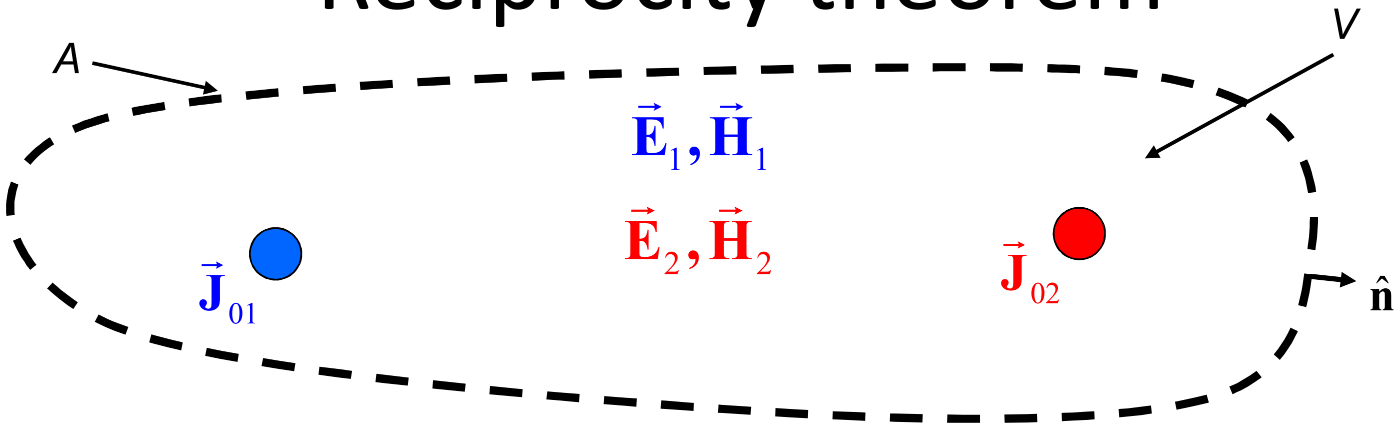
Consider a source distribution $\vec{J}_{01}$ with its associated electromagnetic field $\left(\vec{E}_1, \vec{H}_1\right)$

Consider a source distribution $\vec{J}_{02}$ with its associated electromagnetic field $\left(\vec{E}_2, \vec{H}_2\right)$

We define the mixed Poynting-like vector $\vec{S}_{12}$

$$\vec{S}_{12} = \vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1$$

Reciprocity theorem



Consider a source distribution $\vec{\mathbf{J}}_{01}$ with its associated electromagnetic field $\left(\vec{\mathbf{E}}_1, \vec{\mathbf{H}}_1\right)$

Consider a source distribution $\vec{\mathbf{J}}_{02}$ with its associated electromagnetic field $\left(\vec{\mathbf{E}}_2, \vec{\mathbf{H}}_2\right)$

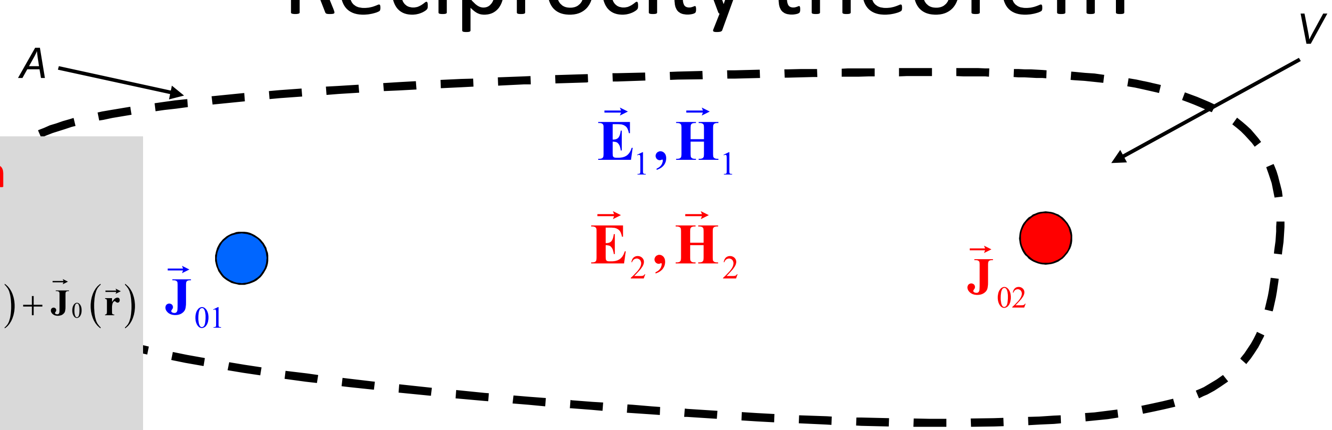
We define the mixed Poynting-like vector $\vec{\mathbf{S}}_{12}$

$$\vec{\mathbf{S}}_{12} = \vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2 - \vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1$$

The reciprocity theorem sates that

$$\oiint_A dA\, \vec{\mathbf{S}}_{12} \cdot \hat{\mathbf{n}} = \iiint_V dV \left[\vec{\mathbf{J}}_{01} \cdot \vec{\mathbf{E}}_2 - \vec{\mathbf{J}}_{02} \cdot \vec{\mathbf{E}}_1 \right]$$

Reciprocity theorem



Phasor domain

$$\nabla \times \vec{\mathbf{E}}(\vec{\mathbf{r}}) = -j\omega_0 \vec{\mathbf{B}}(\vec{\mathbf{r}})$$

$$\nabla \times \vec{\mathbf{H}}(\vec{\mathbf{r}}) = j\omega_0 \vec{\mathbf{D}}(\vec{\mathbf{r}}) + \vec{\mathbf{J}}(\vec{\mathbf{r}}) + \vec{\mathbf{J}}_0(\vec{\mathbf{r}})$$

$$\nabla \cdot \vec{\mathbf{D}}(\vec{\mathbf{r}}) = \rho(\vec{\mathbf{r}}) + \rho_0(\vec{\mathbf{r}})$$

$$\nabla \cdot \vec{\mathbf{B}}(\vec{\mathbf{r}}) = 0$$

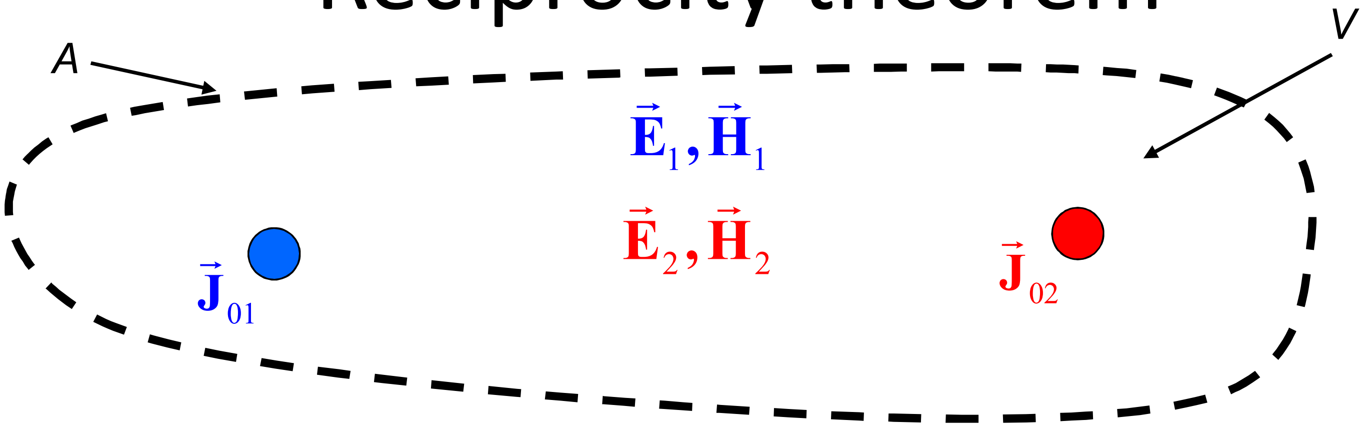
$$\vec{\mathbf{S}}_{12} = \vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2 - \vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1$$

$$\nabla \cdot \vec{\mathbf{S}}_{12} = \nabla \cdot \left(\vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2\right) - \nabla \cdot \left(\vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1\right) = \left[\vec{\mathbf{H}}_2 \cdot \left(\nabla \times \vec{\mathbf{E}}_1\right) - \vec{\mathbf{E}}_1 \cdot \left(\nabla \times \vec{\mathbf{H}}_2\right)\right] - \left[\vec{\mathbf{H}}_1 \cdot \left(\nabla \times \vec{\mathbf{E}}_2\right) - \vec{\mathbf{E}}_2 \cdot \left(\nabla \times \vec{\mathbf{H}}_1\right)\right]$$

$$= \left[\vec{\mathbf{H}}_2 \cdot \left(-j\omega_0 \vec{\mathbf{B}}_1\right) - \vec{\mathbf{E}}_1 \cdot \left(j\omega_0 \vec{\mathbf{D}}_2 + \vec{\mathbf{J}}_2 + \vec{\mathbf{J}}_{02}\right)\right] - \left[\vec{\mathbf{H}}_1 \cdot \left(-j\omega_0 \vec{\mathbf{B}}_2(\vec{\mathbf{r}})\right) - \vec{\mathbf{E}}_2 \cdot \left(j\omega_0 \vec{\mathbf{D}}_1 + \vec{\mathbf{J}}_1 + \vec{\mathbf{J}}_{01}\right)\right]$$

$$\nabla \cdot \left[\vec{\mathbf{A}}(\vec{\mathbf{r}}) \times \vec{\mathbf{B}}(\vec{\mathbf{r}})\right] = \vec{\mathbf{B}}(\vec{\mathbf{r}}) \cdot \left[\nabla \times \vec{\mathbf{A}}(\vec{\mathbf{r}})\right] - \vec{\mathbf{A}}(\vec{\mathbf{r}}) \cdot \left[\nabla \times \vec{\mathbf{B}}(\vec{\mathbf{r}})\right]$$

Reciprocity theorem



$$\vec{\mathbf{S}}_{12} = \vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2 - \vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1$$

$$\nabla \cdot \vec{\mathbf{S}}_{12} = \nabla \cdot \left(\vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2\right) - \nabla \cdot \left(\vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1\right) = \left[\vec{\mathbf{H}}_2 \cdot \left(\nabla \times \vec{\mathbf{E}}_1\right) - \vec{\mathbf{E}}_1 \cdot \left(\nabla \times \vec{\mathbf{H}}_2\right)\right] - \left[\vec{\mathbf{H}}_1 \cdot \left(\nabla \times \vec{\mathbf{E}}_2\right) - \vec{\mathbf{E}}_2 \cdot \left(\nabla \times \vec{\mathbf{H}}_1\right)\right]$$

$$= \left[\vec{\mathbf{H}}_2 \cdot \left(-j\omega_0 \vec{\mathbf{B}}_1\right) - \vec{\mathbf{E}}_1 \cdot \left(j\omega_0 \vec{\mathbf{D}}_2 + \vec{\mathbf{J}}_2 + \vec{\mathbf{J}}_{02}\right)\right] - \left[\vec{\mathbf{H}}_1 \cdot \left(-j\omega_0 \vec{\mathbf{B}}_2(\vec{\mathbf{r}})\right) - \vec{\mathbf{E}}_2 \cdot \left(j\omega_0 \vec{\mathbf{D}}_1 + \vec{\mathbf{J}}_1 + \vec{\mathbf{J}}_{01}\right)\right]$$

$$= \left[\vec{\mathbf{H}}_2 \cdot \left(-j\omega_0 \mu \vec{\mathbf{H}}_1\right) - \vec{\mathbf{E}}_1 \cdot \left(j\omega_0 \varepsilon \vec{\mathbf{E}}_2 + \sigma \vec{\mathbf{E}}_2 + \vec{\mathbf{J}}_{02}\right)\right] - \left[\vec{\mathbf{H}}_1 \cdot \left(-j\omega_0 \mu \vec{\mathbf{H}}_2\right) - \vec{\mathbf{E}}_2 \cdot \left(j\omega_0 \varepsilon \vec{\mathbf{E}}_1 + \sigma \vec{\mathbf{E}}_1 + \vec{\mathbf{J}}_{01}\right)\right]$$

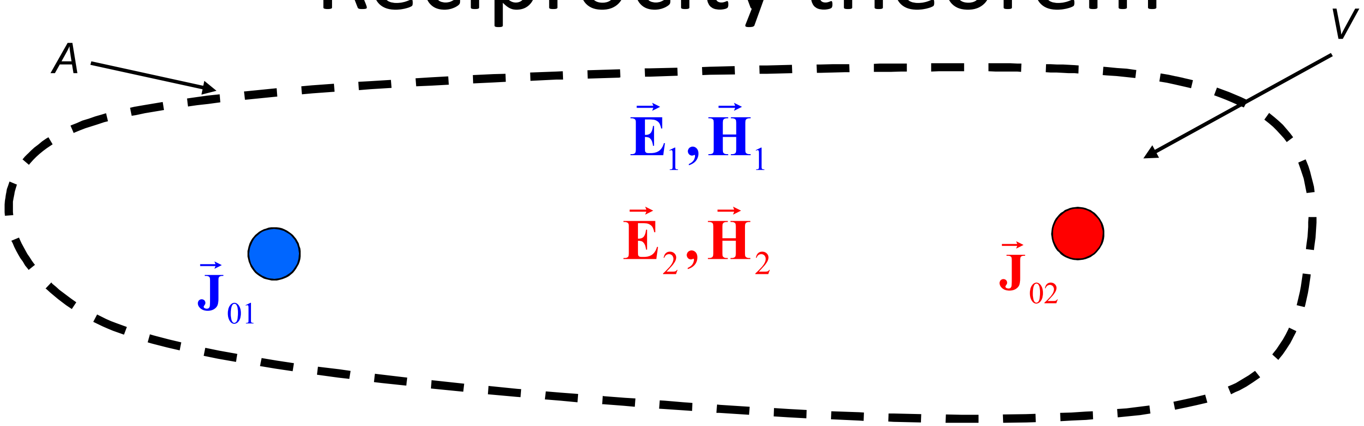
$$= \vec{\mathbf{H}}_2 \cdot \left(-j\omega_0 \mu \vec{\mathbf{H}}_1\right) - \vec{\mathbf{E}}_1 \cdot \left(j\omega_0 \varepsilon \vec{\mathbf{E}}_2 + \sigma \vec{\mathbf{E}}_2 + \vec{\mathbf{J}}_{02}\right) - \vec{\mathbf{H}}_1 \cdot \left(-j\omega_0 \mu \vec{\mathbf{H}}_2\right) + \vec{\mathbf{E}}_2 \cdot \left(j\omega_0 \varepsilon \vec{\mathbf{E}}_1 + \sigma \vec{\mathbf{E}}_1 + \vec{\mathbf{J}}_{01}\right)$$

Hypotheses on the medium (PD)

- Linear
- Isotropic
- Space-Nondispersive
- Time-invariant
- Time-Dispersive

$$\begin{cases} \vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}} \\ \vec{\mathbf{B}} = \mu \vec{\mathbf{H}} \\ \vec{\mathbf{J}} = \sigma \vec{\mathbf{E}} \end{cases}$$

Reciprocity theorem



$$\vec{S}_{12} = \vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1$$

$$\nabla \cdot \vec{S}_{12} = \nabla \cdot \left(\vec{E}_1 \times \vec{H}_2\right) - \nabla \cdot \left(\vec{E}_2 \times \vec{H}_1\right) = \left[\vec{H}_2 \cdot \left(\nabla \times \vec{E}_1\right) - \vec{E}_1 \cdot \left(\nabla \times \vec{H}_2\right)\right] - \left[\vec{H}_1 \cdot \left(\nabla \times \vec{E}_2\right) - \vec{E}_2 \cdot \left(\nabla \times \vec{H}_1\right)\right]$$

$$= \left[\vec{H}_2 \cdot \left(-j\omega_0 \vec{B}_1\right) - \vec{E}_1 \cdot \left(j\omega_0 \vec{D}_2 + \vec{J}_2 + \vec{J}_{02}\right)\right] - \left[\vec{H}_1 \cdot \left(-j\omega_0 \vec{B}_2(\vec{r})\right) - \vec{E}_2 \cdot \left(j\omega_0 \vec{D}_1 + \vec{J}_1 + \vec{J}_{01}\right)\right]$$

$$= \left[\vec{H}_2 \cdot \left(-j\omega_0 \mu \vec{H}_1\right) - \vec{E}_1 \cdot \left(j\omega_0 \varepsilon \vec{E}_2 + \sigma \vec{E}_2 + \vec{J}_{02}\right)\right] - \left[\vec{H}_1 \cdot \left(-j\omega_0 \mu \vec{H}_2\right) - \vec{E}_2 \cdot \left(j\omega_0 \varepsilon \vec{E}_1 + \sigma \vec{E}_1 + \vec{J}_{01}\right)\right]$$

$$= \vec{H}_2 \cdot \left(-j\omega_0 \mu \vec{H}_1\right) - \vec{E}_1 \cdot \left(j\omega_0 \varepsilon \vec{E}_2 + \sigma \vec{E}_2 + \vec{J}_{02}\right) - \vec{H}_1 \cdot \left(-j\omega_0 \mu \vec{H}_2\right) + \vec{E}_2 \cdot \left(j\omega_0 \varepsilon \vec{E}_1 + \sigma \vec{E}_1 + \vec{J}_{01}\right)$$

$$= -j\omega_0 \mu \vec{H}_2 \cdot \vec{H}_1 - j\omega_0 \varepsilon \vec{E}_1 \cdot \vec{E}_2 - \sigma \vec{E}_1 \cdot \vec{E}_2 - \vec{E}_1 \cdot \vec{J}_{02} + j\omega_0 \mu \vec{H}_1 \cdot \vec{H}_2 + j\omega_0 \varepsilon \vec{E}_2 \cdot \vec{E}_1 + \sigma \vec{E}_2 \cdot \vec{E}_1 + \vec{E}_2 \cdot \vec{J}_{01}$$

Hypotheses on the medium (PD)

- Linear
- Isotropic
- Space-Nondispersive
- Time-invariant
- Time-Dispersive

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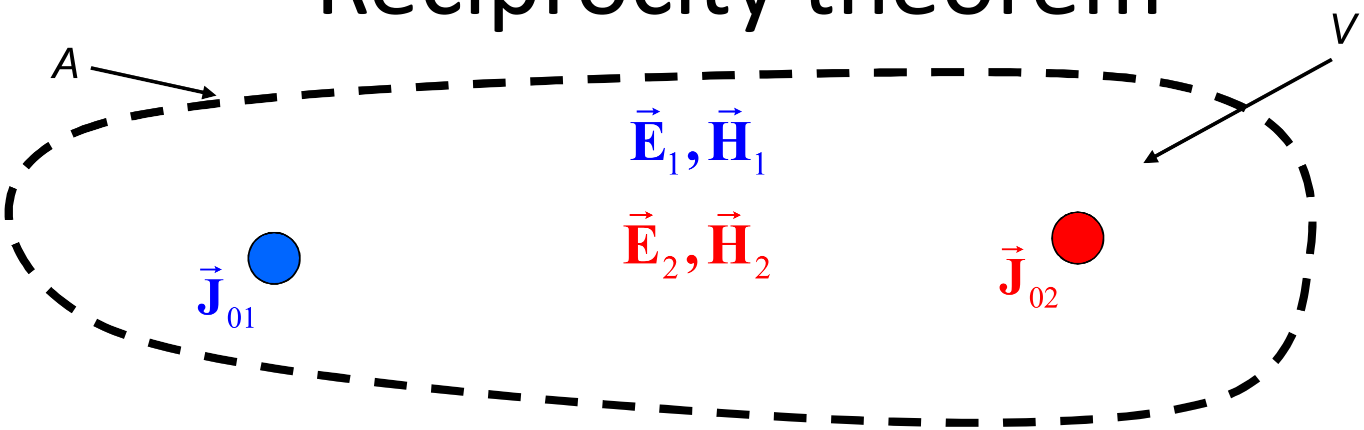
Reciprocity theorem



$$\vec{\mathbf{S}}_{12} = \vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2 - \vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1$$

$$\nabla \cdot \vec{\mathbf{S}}_{12} = -j\omega_0\mu\vec{\mathbf{H}}_2 \cdot \vec{\mathbf{H}}_1 - j\omega_0\varepsilon\vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 - \sigma\vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 - \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{J}}_{02} + j\omega_0\mu\vec{\mathbf{H}}_1 \cdot \vec{\mathbf{H}}_2 + j\omega_0\varepsilon\vec{\mathbf{E}}_2 \cdot \vec{\mathbf{E}}_1 + \sigma\vec{\mathbf{E}}_2 \cdot \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{J}}_{01}$$

$$= -\vec{\mathbf{E}}_1 \cdot \vec{\mathbf{J}}_{02} + \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{J}}_{01}$$

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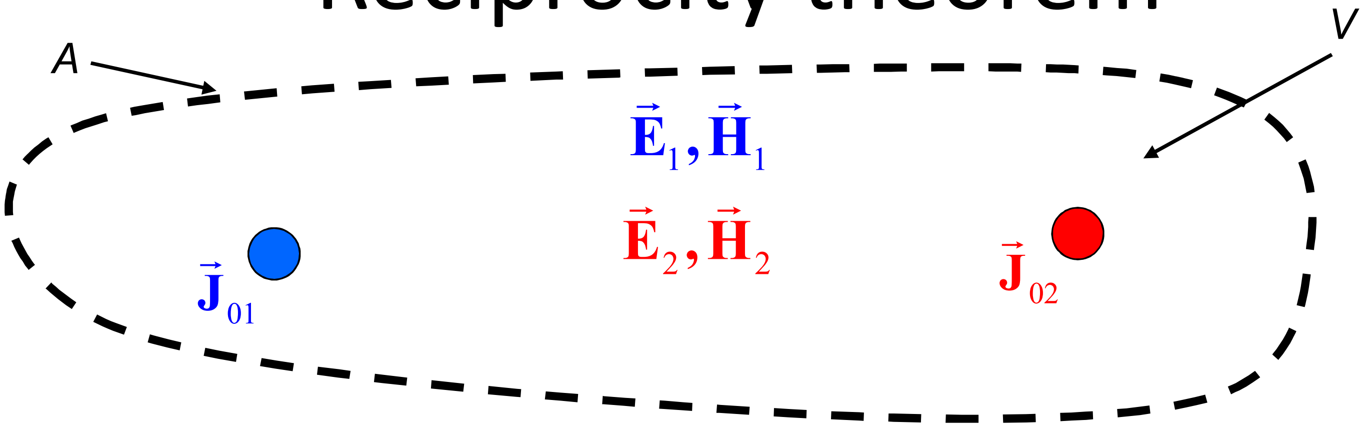

$$\vec{S}_{12} = \vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1$$

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$$= -\vec{E}_1 \cdot \vec{J}_{02} + \vec{E}_2 \cdot \vec{J}_{01}$$

$$\nabla \cdot \vec{S}_{12} = -\vec{E}_1 \cdot \vec{J}_{02} + \vec{E}_2 \cdot \vec{J}_{01}$$

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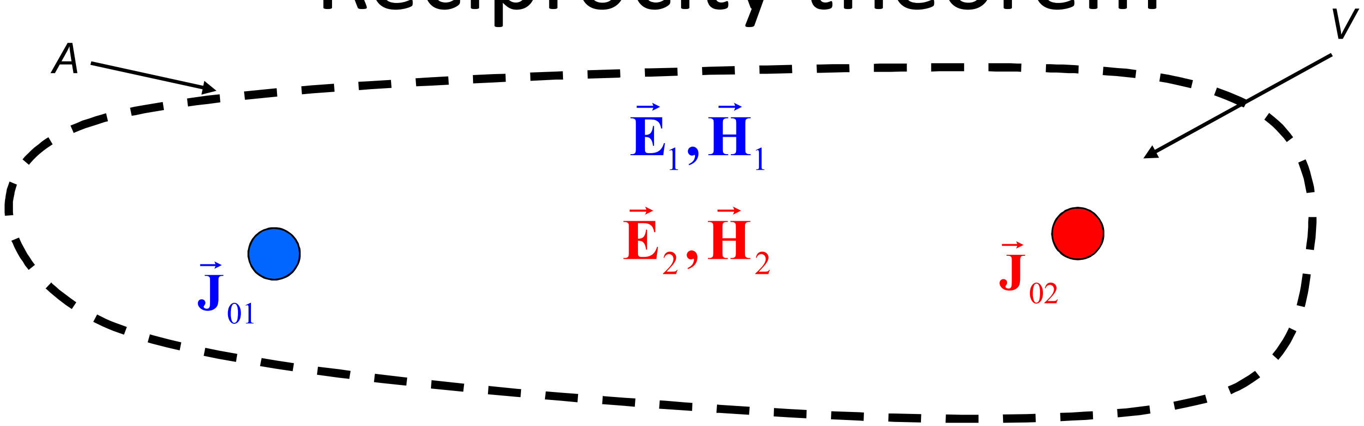

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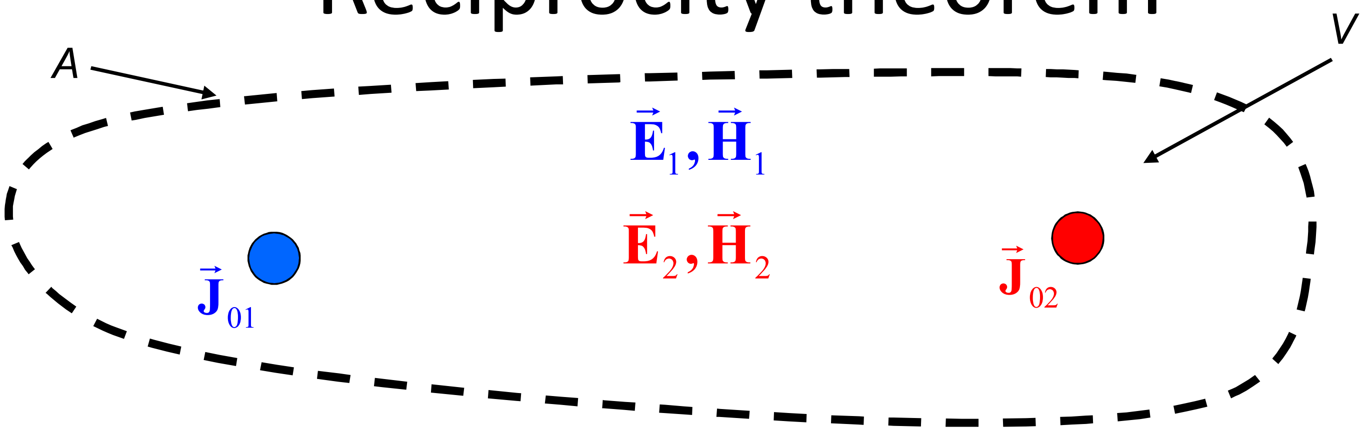

$$\vec{\mathbf{S}}_{12} = \vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2 - \vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1$$

$$\nabla \cdot \vec{\mathbf{S}}_{12} = -\vec{\mathbf{E}}_1 \cdot \vec{\mathbf{J}}_{02} + \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{J}}_{01} \Rightarrow \oiint_A dA\, \vec{\mathbf{S}}_{12} \cdot \hat{\mathbf{n}} = \iiint_V dV \left[\vec{\mathbf{J}}_{01} \cdot \vec{\mathbf{E}}_2 - \vec{\mathbf{J}}_{02} \cdot \vec{\mathbf{E}}_1 \right]$$

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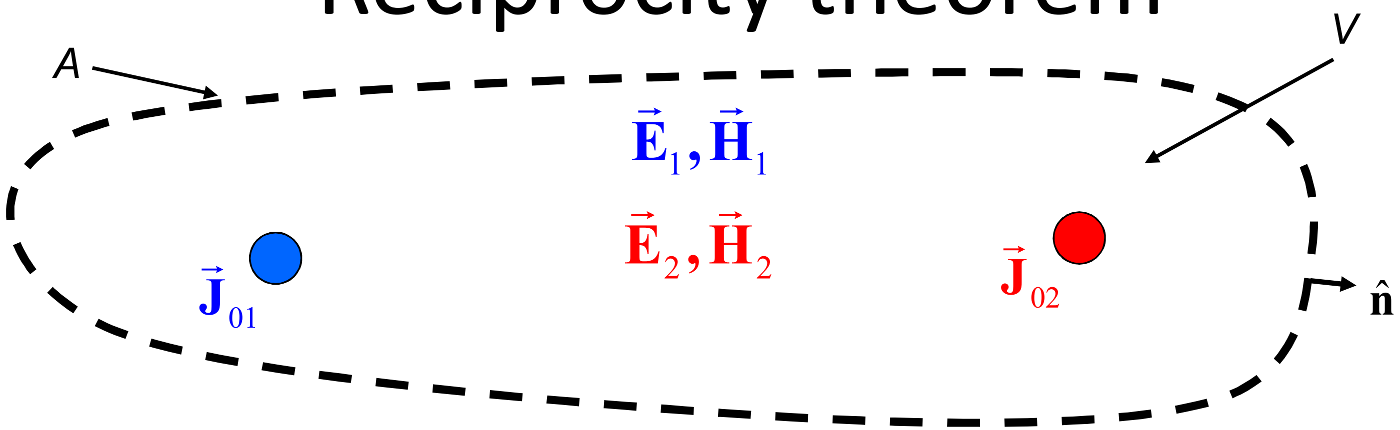
Reciprocity theorem



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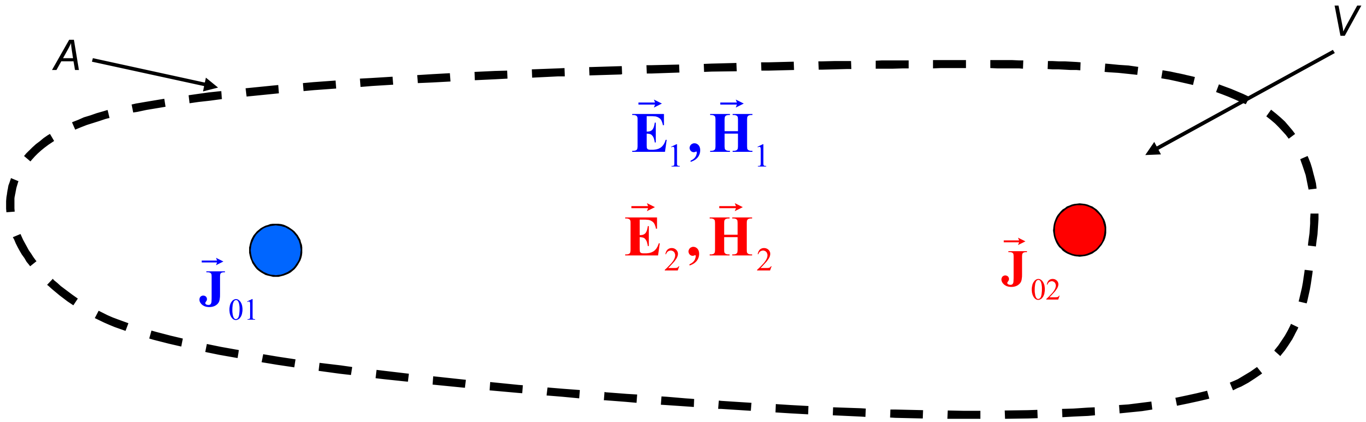
Reciprocity theorem: one consideration



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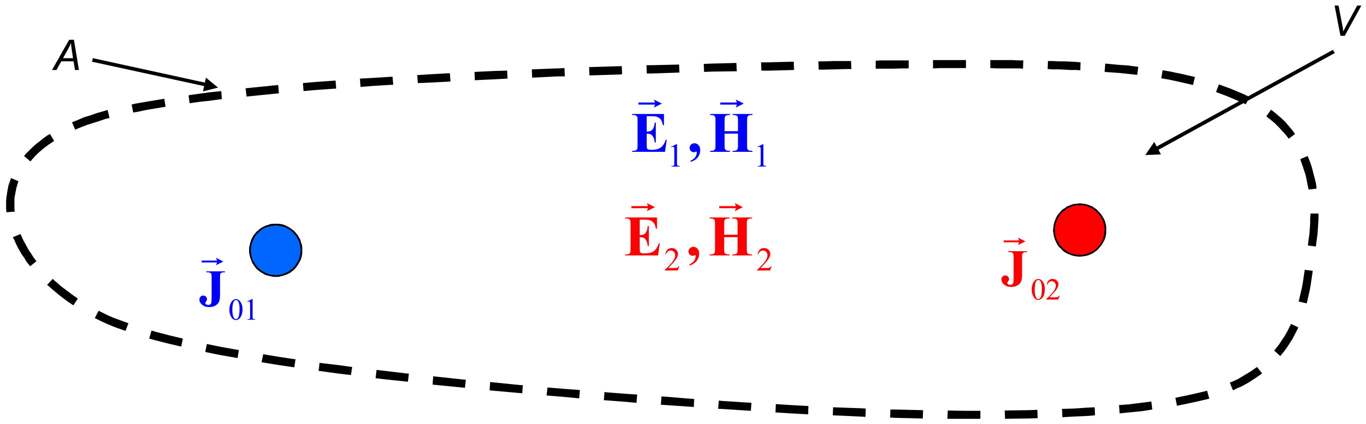

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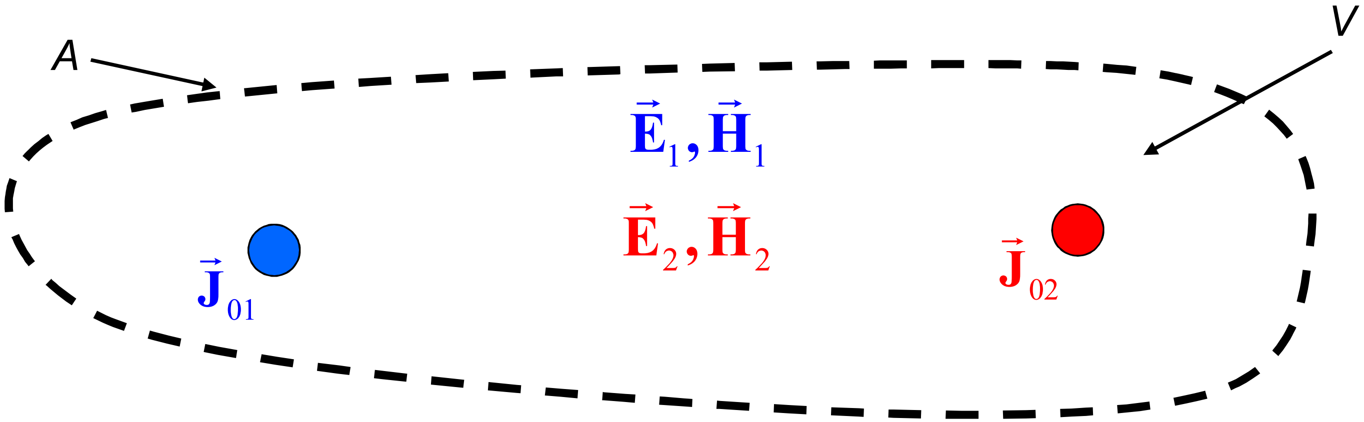
Reciprocity theorem: one consideration



$$\vec{\mathbf{S}}_{12} = \vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2 - \vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1$$

$$\nabla\cdot\vec{\mathbf{S}}_{12} = \nabla\cdot\left(\vec{\mathbf{E}}_1\times\vec{\mathbf{H}}_2\right) - \nabla\cdot\left(\vec{\mathbf{E}}_2\times\vec{\mathbf{H}}_1\right) = \left[\vec{\mathbf{H}}_2\cdot\left(\nabla\times\vec{\mathbf{E}}_1\right) - \vec{\mathbf{E}}_1\cdot\left(\nabla\times\vec{\mathbf{H}}_2\right)\right] - \left[\vec{\mathbf{H}}_1\cdot\left(\nabla\times\vec{\mathbf{E}}_2\right) - \vec{\mathbf{E}}_2\cdot\left(\nabla\times\vec{\mathbf{H}}_1\right)\right]$$

$$= \left[\vec{\mathbf{H}}_2\cdot\left(-j\omega_0\vec{\mathbf{B}}_1\right) - \vec{\mathbf{E}}_1\cdot\left(j\omega_0\vec{\mathbf{D}}_2 + \vec{\mathbf{J}}_2 + \vec{\mathbf{J}}_{02}\right)\right] - \left[\vec{\mathbf{H}}_1\cdot\left(-j\omega_0\vec{\mathbf{B}}_2(\vec{\mathbf{r}})\right) - \vec{\mathbf{E}}_2\cdot\left(j\omega_0\vec{\mathbf{D}}_1 + \vec{\mathbf{J}}_1 + \vec{\mathbf{J}}_{01}\right)\right]$$

$$= \left[\vec{\mathbf{H}}_2\cdot\left(-j\omega_0\mu\vec{\mathbf{H}}_1\right) - \vec{\mathbf{E}}_1\cdot\left(j\omega_0\varepsilon\vec{\mathbf{E}}_2 + \sigma\vec{\mathbf{E}}_2 + \vec{\mathbf{J}}_{02}\right)\right] - \left[\vec{\mathbf{H}}_1\cdot\left(-j\omega_0\mu\vec{\mathbf{H}}_2\right) - \vec{\mathbf{E}}_2\cdot\left(j\omega_0\varepsilon\vec{\mathbf{E}}_1 + \sigma\vec{\mathbf{E}}_1 + \vec{\mathbf{J}}_{01}\right)\right]$$

$$= \vec{\mathbf{H}}_2\cdot\left(-j\omega_0\mu\vec{\mathbf{H}}_1\right) - \vec{\mathbf{E}}_1\cdot\left(j\omega_0\varepsilon\vec{\mathbf{E}}_2 + \sigma\vec{\mathbf{E}}_2 + \vec{\mathbf{J}}_{02}\right) - \vec{\mathbf{H}}_1\cdot\left(-j\omega_0\mu\vec{\mathbf{H}}_2\right) + \vec{\mathbf{E}}_2\cdot\left(j\omega_0\varepsilon\vec{\mathbf{E}}_1 + \sigma\vec{\mathbf{E}}_1 + \vec{\mathbf{J}}_{01}\right)$$

$$= -j\omega_0\mu\vec{\mathbf{H}}_2\cdot\vec{\mathbf{H}}_1 - j\omega_0\varepsilon\vec{\mathbf{E}}_1\cdot\vec{\mathbf{E}}_2 - \sigma\vec{\mathbf{E}}_1\cdot\vec{\mathbf{E}}_2 - \vec{\mathbf{E}}_1\cdot\vec{\mathbf{J}}_{02} + j\omega_0\mu\vec{\mathbf{H}}_1\cdot\vec{\mathbf{H}}_2 + j\omega_0\varepsilon\vec{\mathbf{E}}_2\cdot\vec{\mathbf{E}}_1 + \sigma\vec{\mathbf{E}}_2\cdot\vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2\cdot\vec{\mathbf{J}}_{01}$$

Hypotheses on the medium (PD)

- **Linear**
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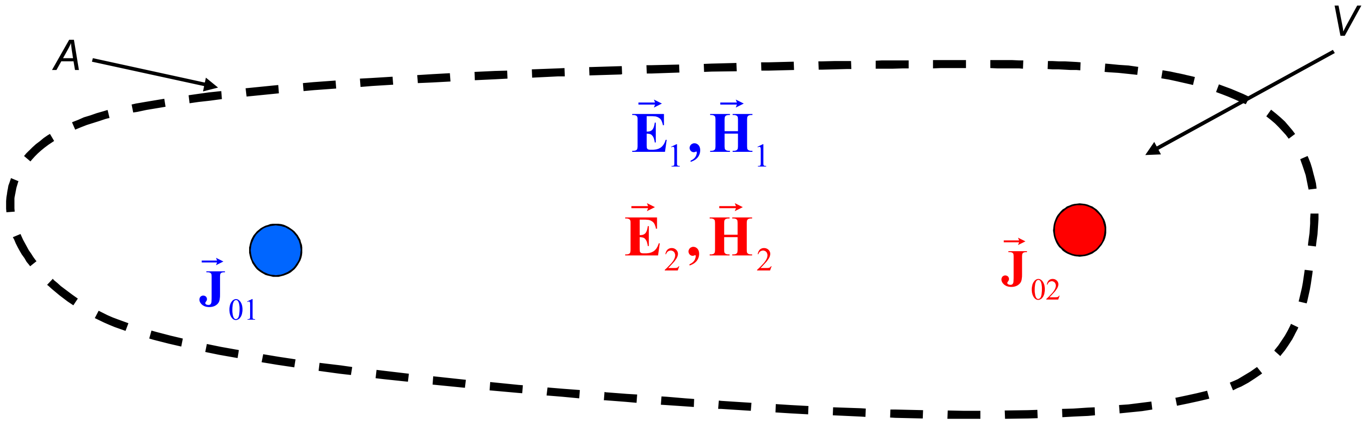
$$\begin{cases}\vec{\mathbf{D}} = \varepsilon\vec{\mathbf{E}}\\ \vec{\mathbf{B}} = \mu\vec{\mathbf{H}}\\ \vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}\end{cases}$$

Reciprocity theorem: one consideration



$$= \vec{H}_2 \cdot \left(-j\omega_0 \mu \vec{H}_1\right) - \vec{E}_1 \cdot \left(j\omega_0 \varepsilon \vec{E}_2 + \sigma \vec{E}_2 + \vec{J}_{02}\right) - \vec{H}_1 \cdot \left(-j\omega_0 \mu \vec{H}_2\right) + \vec{E}_2 \cdot \left(j\omega_0 \varepsilon \vec{E}_1 + \sigma \vec{E}_1 + \vec{J}_{01}\right)$$

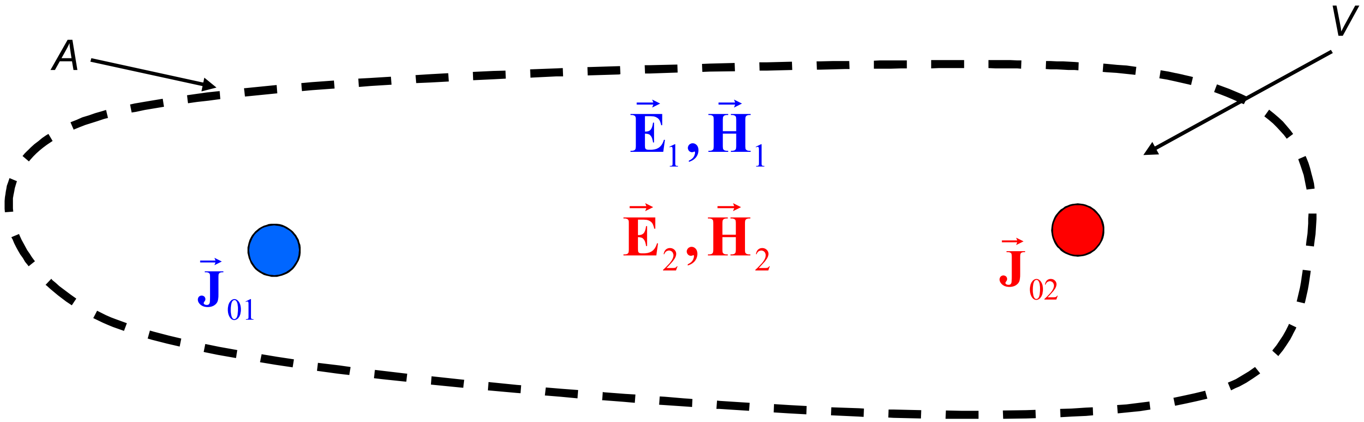
$$= -j\omega_0 \mu \vec{H}_2 \cdot \vec{H}_1 - j\omega_0 \varepsilon \vec{E}_1 \cdot \vec{E}_2 - \sigma \vec{E}_1 \cdot \vec{E}_2 - \vec{E}_1 \cdot \vec{J}_{02} + j\omega_0 \mu \vec{H}_1 \cdot \vec{H}_2 + j\omega_0 \varepsilon \vec{E}_2 \cdot \vec{E}_1 + \sigma \vec{E}_2 \cdot \vec{E}_1 + \vec{E}_2 \cdot \vec{J}_{01}$$

Hypotheses on the medium (PD)

- **Linear**
- **Isotropic**
- **Space-Nondispersive**
- **Time-invariant**
- **Time-Dispersive**

$$\begin{cases} \vec{D} = \varepsilon \vec{E} \\ \vec{B} = \mu \vec{H} \\ \vec{J} = \sigma \vec{E} \end{cases}$$

Reciprocity theorem: one consideration



$$\vec{\mathbf{H}}_1 \cdot \mu \vec{\mathbf{H}}_2 = \vec{\mathbf{H}}_2 \cdot \mu \vec{\mathbf{H}}_1$$

Hypotheses on the medium (PD)

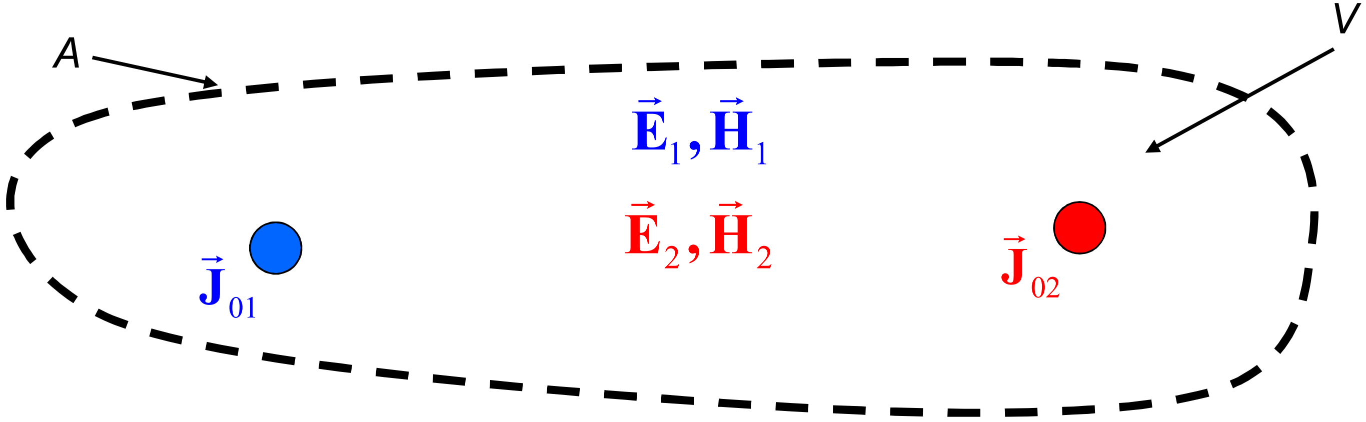
- Linear
- Isotropic
- Space-Nondispersive
- Time-invariant
- Time-Dispersive

$$\begin{cases} \vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}} \\ \vec{\mathbf{B}}(\vec{\mathbf{r}}) = \mu \vec{\mathbf{H}} \\ \vec{\mathbf{J}}(\vec{\mathbf{r}}) = \sigma \vec{\mathbf{E}} \end{cases}$$

$$= \vec{\mathbf{H}}_2 \cdot \left(-j\omega_0 \mu \vec{\mathbf{H}}_1\right) - \vec{\mathbf{E}}_1 \cdot \left(j\omega_0 \varepsilon \vec{\mathbf{E}}_2 + \sigma \vec{\mathbf{E}}_2 + \vec{\mathbf{J}}_{02}\right) - \vec{\mathbf{H}}_1 \cdot \left(-j\omega_0 \mu \vec{\mathbf{H}}_2\right) + \vec{\mathbf{E}}_2 \cdot \left(j\omega_0 \varepsilon \vec{\mathbf{E}}_1 + \sigma \vec{\mathbf{E}}_1 + \vec{\mathbf{J}}_{01}\right)$$

$$= -j\omega_0 \mu \vec{\mathbf{H}}_2 \cdot \vec{\mathbf{H}}_1 - j\omega_0 \varepsilon \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 - \sigma \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 - \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{J}}_{02} + j\omega_0 \mu \vec{\mathbf{H}}_1 \cdot \vec{\mathbf{H}}_2 + j\omega_0 \varepsilon \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{E}}_1 + \sigma \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{J}}_{01}$$

Reciprocity theorem: one consideration



Hypotheses on the medium (PD)

- Linear
- Anisotropic
- Space-Nondispersive
- Time-invariant
- Time-Dispersive

$$\begin{cases} \vec{\mathbf{D}} = \boldsymbol{\varepsilon} \cdot \vec{\mathbf{E}} \\ \vec{\mathbf{B}}(\vec{\mathbf{r}}) = \boldsymbol{\mu} \cdot \vec{\mathbf{H}} \\ \vec{\mathbf{J}}(\vec{\mathbf{r}}) = \sigma \vec{\mathbf{E}} \end{cases}$$

~~$\vec{\mathbf{H}}_1 \cdot \mu \vec{\mathbf{H}}_2 = \vec{\mathbf{H}}_2 \cdot \mu \vec{\mathbf{H}}_1$~~

$$\vec{\mathbf{H}}_1 \cdot \left[\boldsymbol{\mu} \cdot \vec{\mathbf{H}}_2\right] \neq \vec{\mathbf{H}}_2 \cdot \left[\boldsymbol{\mu} \cdot \vec{\mathbf{H}}_1\right]$$

$$\vec{\mathbf{E}}_1 \cdot \left[\boldsymbol{\varepsilon} \cdot \vec{\mathbf{E}}_2\right] \neq \vec{\mathbf{E}}_2 \cdot \left[\boldsymbol{\varepsilon} \cdot \vec{\mathbf{E}}_1\right]$$

Note however that when the matrixes μ and ε are symmetrical (reciprocal media):

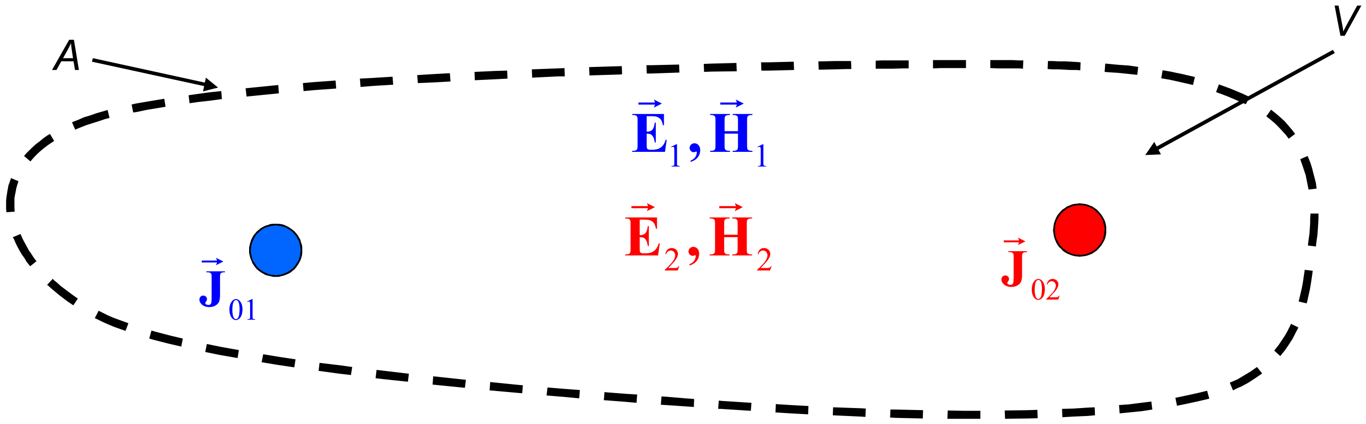
$$\vec{\mathbf{H}}_1 \cdot \left[\boldsymbol{\mu} \cdot \vec{\mathbf{H}}_2\right] = \vec{\mathbf{H}}_2 \cdot \left[\boldsymbol{\mu} \cdot \vec{\mathbf{H}}_1\right]$$

$$\vec{\mathbf{E}}_1 \cdot \left[\boldsymbol{\varepsilon} \cdot \vec{\mathbf{E}}_2\right] = \vec{\mathbf{E}}_2 \cdot \left[\boldsymbol{\varepsilon} \cdot \vec{\mathbf{E}}_1\right]$$

$$= \vec{\mathbf{H}}_2 \cdot \left(-j\omega_0 \mu \vec{\mathbf{H}}_1\right) - \vec{\mathbf{E}}_1 \cdot \left(j\omega_0 \varepsilon \vec{\mathbf{E}}_2 + \sigma \vec{\mathbf{E}}_2 + \vec{\mathbf{J}}_{02}\right) - \vec{\mathbf{H}}_1 \cdot \left(-j\omega_0 \mu \vec{\mathbf{H}}_2\right) + \vec{\mathbf{E}}_2 \cdot \left(j\omega_0 \varepsilon \vec{\mathbf{E}}_1 + \sigma \vec{\mathbf{E}}_1 + \vec{\mathbf{J}}_{01}\right)$$

$$= -j\omega_0 \mu \vec{\mathbf{H}}_2 \cdot \vec{\mathbf{H}}_1 - j\omega_0 \varepsilon \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 - \sigma \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 - \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{J}}_{02} + j\omega_0 \mu \vec{\mathbf{H}}_1 \cdot \vec{\mathbf{H}}_2 + j\omega_0 \varepsilon \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{E}}_1 + \sigma \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{J}}_{01}$$

Reciprocity theorem: one consideration



~~$\vec{\mathbf{H}}_1 \cdot \mu \vec{\mathbf{H}}_2 = \vec{\mathbf{H}}_2 \cdot \mu \vec{\mathbf{H}}_1$~~

$$\vec{\mathbf{H}}_1 \cdot \left[\boldsymbol{\mu} \cdot \vec{\mathbf{H}}_2 \right] \neq \vec{\mathbf{H}}_2 \cdot \left[\boldsymbol{\mu} \cdot \vec{\mathbf{H}}_1 \right]$$

$$\vec{\mathbf{E}}_1 \cdot \left[\boldsymbol{\varepsilon} \cdot \vec{\mathbf{E}}_2 \right] \neq \vec{\mathbf{E}}_2 \cdot \left[\boldsymbol{\varepsilon} \cdot \vec{\mathbf{E}}_1 \right]$$

Note however that when the matrixes μ and ε are symmetrical (reciprocal media):

$$\vec{\mathbf{H}}_1 \cdot \left[\boldsymbol{\mu} \cdot \vec{\mathbf{H}}_2 \right] = \vec{\mathbf{H}}_2 \cdot \left[\boldsymbol{\mu} \cdot \vec{\mathbf{H}}_1 \right]$$

$$\vec{\mathbf{E}}_1 \cdot \left[\boldsymbol{\varepsilon} \cdot \vec{\mathbf{E}}_2 \right] = \vec{\mathbf{E}}_2 \cdot \left[\boldsymbol{\varepsilon} \cdot \vec{\mathbf{E}}_1 \right]$$

Hypotheses on the medium (PD)

- **Linear**
- **Reciprocal**
- **Space-Nondispersive**
- **Time-invariant**
- **Time-Dispersive**

$$\begin{cases} \vec{\mathbf{D}} = \boldsymbol{\varepsilon} \cdot \vec{\mathbf{E}} \\ \vec{\mathbf{B}}(\vec{\mathbf{r}}) = \boldsymbol{\mu} \cdot \vec{\mathbf{H}} \\ \vec{\mathbf{J}}(\vec{\mathbf{r}}) = \sigma \vec{\mathbf{E}} \end{cases}$$

$$= \vec{\mathbf{H}}_2 \cdot \left(-j\omega_0 \mu \vec{\mathbf{H}}_1 \right) - \vec{\mathbf{E}}_1 \cdot \left(j\omega_0 \varepsilon \vec{\mathbf{E}}_2 + \sigma \vec{\mathbf{E}}_2 + \vec{\mathbf{J}}_{02} \right) - \vec{\mathbf{H}}_1 \cdot \left(-j\omega_0 \mu \vec{\mathbf{H}}_2 \right) + \vec{\mathbf{E}}_2 \cdot \left(j\omega_0 \varepsilon \vec{\mathbf{E}}_1 + \sigma \vec{\mathbf{E}}_1 + \vec{\mathbf{J}}_{01} \right)$$

$$= -j\omega_0 \mu \vec{\mathbf{H}}_2 \cdot \vec{\mathbf{H}}_1 - j\omega_0 \varepsilon \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 - \sigma \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{E}}_2 - \vec{\mathbf{E}}_1 \cdot \vec{\mathbf{J}}_{02} + j\omega_0 \mu \vec{\mathbf{H}}_1 \cdot \vec{\mathbf{H}}_2 + j\omega_0 \varepsilon \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{E}}_1 + \sigma \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 \cdot \vec{\mathbf{J}}_{01}$$

Reciprocity theorem

$$\vec{\mathbf{S}}_{12} = \vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2 - \vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1$$

$$\oiint_A dA\, \vec{\mathbf{S}}_{12} \cdot \hat{\mathbf{n}} = \iiint_V dV \left[\vec{\mathbf{J}}_{01} \cdot \vec{\mathbf{E}}_2 - \vec{\mathbf{J}}_{02} \cdot \vec{\mathbf{E}}_1 \right]$$

An interesting case

If

$$\oiint_A dA\, \vec{\mathbf{S}}_{12} \cdot \hat{\mathbf{n}} = 0$$

(f.i., when the surface material is a PEC or when the volume encompasses all the space), the reciprocity theorem simplifies as:

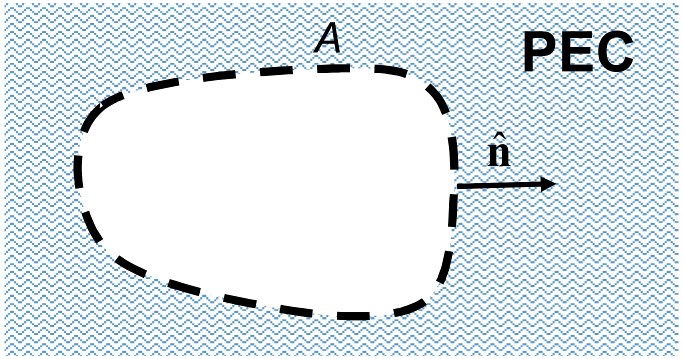
$$\iiint_V dV\, \vec{\mathbf{J}}_{01} \cdot \vec{\mathbf{E}}_2 = \iiint_V dV\, \vec{\mathbf{J}}_{02} \cdot \vec{\mathbf{E}}_1$$

Reciprocity theorem

$$\vec{\mathbf{S}}_{12} = \vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2 - \vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1$$

$$\oiint_A dA\, \vec{\mathbf{S}}_{12} \cdot \hat{\mathbf{n}} = \oiint_A dA \left[\vec{\mathbf{E}}_1 \times \vec{\mathbf{H}}_2 \right] \cdot \hat{\mathbf{n}} - \oiint_A dA \left[\vec{\mathbf{E}}_2 \times \vec{\mathbf{H}}_1 \right] \cdot \hat{\mathbf{n}} +$$

$$= \oiint_A dA \left[\hat{\mathbf{n}} \times \vec{\mathbf{E}}_1 \right] \cdot \vec{\mathbf{H}}_2 - \oiint_A dA \left[\hat{\mathbf{n}} \times \vec{\mathbf{E}}_2 \right] \cdot \vec{\mathbf{H}}_1 = 0$$



$$\vec{\mathbf{A}} \cdot \left[\vec{\mathbf{B}} \times \vec{\mathbf{C}} \right] = \vec{\mathbf{C}} \cdot \left[\vec{\mathbf{A}} \times \vec{\mathbf{B}} \right] = \vec{\mathbf{B}} \cdot \left[\vec{\mathbf{C}} \times \vec{\mathbf{A}} \right]$$