Campi Elettromagnetici

Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

a.a. 2019-2020 - Laurea “Triennale” – Secondo semestre - Secondo anno

Università degli Studi di Napoli “Parthenope”

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Color legend

New formulas, important considerations, important formulas, important concepts

Very important for the discussion

Memo

Mathematical tools to be exploited

Mathematics

Mathematical tools that we will exploit today

$$\nabla \cdot \left(\nabla \times \vec{\mathbf{A}}(\vec{\mathbf{r}}) \right) = 0$$

Maxwell equations

Time domain - Differential form

$$\begin{cases} \nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\ \nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0 \end{cases}$$

James Clerk Maxwell 1831-1879

Maxwell equations

Time domain - Differential form

$$
\begin{cases}
\nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\
\nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0
\end{cases}
$$

James Clerk Maxwell 1831-1879

Maxwell equations

$$
\begin{cases}
\nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\
\nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0
\end{cases}
$$

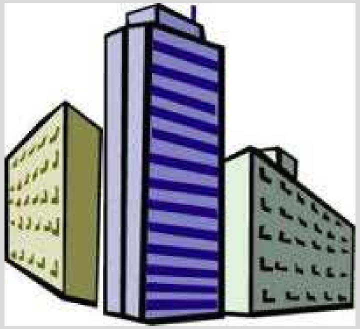
$$
\begin{cases}
\vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\
\rho_0(\vec{\mathbf{r}},t)
\end{cases}
$$

Prescribed sources

$$
\begin{cases}
\vec{\mathbf{j}}(\vec{\mathbf{r}},t) \\
\rho(\vec{\mathbf{r}},t)
\end{cases}
$$

Induced sources

Complex scenario



The independence of the Maxwell equations

Time domain - Differential form

$$\begin{cases} \nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\ \nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0 \end{cases}$$

$$\nabla \cdot \left(\nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t)\right) = \nabla \cdot \left(-\frac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t}\right)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$0 \qquad = -\frac{\partial\left(\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t)\right)}{\partial t}$$

$$\nabla \cdot \left(\nabla \times \vec{\mathbf{A}}(\vec{\mathbf{r}})\right) = 0$$

The independence of the Maxwell equations

Time domain - Differential form

$$\begin{cases} \nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\ \nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0 \end{cases}$$

$$\nabla \cdot \left(\nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t)\right) = \nabla \cdot \left(-\frac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t}\right)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$0 = -\frac{\partial \left(\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t)\right)}{\partial t}$$

$\nabla \cdot \vec{\mathbf{b}}$ is independent of time. If the fields are equal to zero before a given time, then $\nabla \cdot \vec{\mathbf{b}} = 0$ for all times, thus recovering the last Maxwell equation

The independence of the Maxwell equations

Time domain - Differential form

$$\begin{cases} \nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\ \nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0 \end{cases}$$

Number of independent scalar equations:

3+3+1=7

Let us assume knowledge of the impressed sources $\vec{\mathbf{j}}_0(\vec{\mathbf{r}},t);\rho_0(\vec{\mathbf{r}},t)$

Number of unknown scalar quantities:

$\vec{\mathbf{e}}(\vec{\mathbf{r}},t);\vec{\mathbf{d}}(\vec{\mathbf{r}},t);\vec{\mathbf{h}}(\vec{\mathbf{r}},t);\vec{\mathbf{b}}(\vec{\mathbf{r}},t);\vec{\mathbf{j}}(\vec{\mathbf{r}},t);\rho(\vec{\mathbf{r}},t)$

3 +3 +3 +3 +3 +1

16

The independence of the Maxwell equations

Time domain - Differential form

$$
\begin{cases}
\nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\
\nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0
\end{cases}
$$

Even assuming knowledge of the impressed sources $\vec{\mathbf{j}}_0(\vec{\mathbf{r}},t);\rho_0(\vec{\mathbf{r}},t)$

Number of independent scalar equations: 7

Number of unknown scalar quantities: 16

Maxwell equations involve a number of unknowns larger than the number of equations!

Constitutive relationships

Time domain - Differential form

$$\begin{cases} \nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\ \nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0 \end{cases}$$

Even assuming knowledge of the impressed sources $\vec{\mathbf{j}}_0(\vec{\mathbf{r}},t); \rho_0(\vec{\mathbf{r}},t)$

Number of independent scalar equations: 7

Number of unknown scalar quantities: 16

Maxwell equations involve a number of unknowns larger than the number of equations!

The additional missing equations are provided by the **constitutive relationships**, which describe interaction of fields and matter from a macroscopic point of view

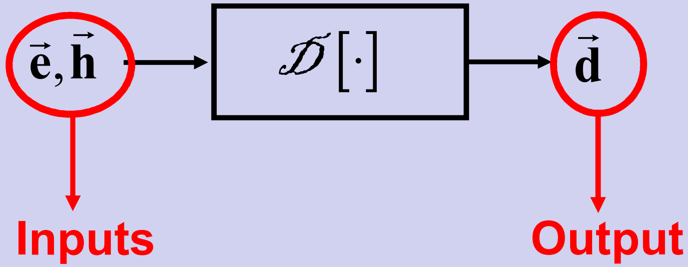
Constitutive relationships

Inductions and currents must be represented in terms of fields

$$\vec{\mathbf{d}} = \mathscr{D}\left[\vec{\mathbf{e}},\vec{\mathbf{h}}\right]$$

$$\vec{\mathbf{b}} = \mathscr{B}\left[\vec{\mathbf{e}},\vec{\mathbf{h}}\right]$$

$$\vec{\mathbf{j}} = \mathscr{J}\left[\vec{\mathbf{e}},\vec{\mathbf{h}}\right]$$



$\mathscr{D}[\cdot], \mathscr{B}[\cdot]$ and $\mathscr{J}[\cdot]$ are functionals which depend upon the medium in which the electromagnetic field is considered and upon the fields themselves.

Constitutive relationships

Linear media

$$\vec{\mathbf{d}}_1 = \mathscr{D}\left[\vec{\mathbf{e}}_1, \vec{\mathbf{h}}_1\right]$$

$$\vec{\mathbf{d}}_2 = \mathscr{D}\left[\vec{\mathbf{e}}_2, \vec{\mathbf{h}}_2\right]$$

$$\Rightarrow \quad \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 = \mathscr{D}\left[\vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_2, \vec{\mathbf{h}}_1 + \vec{\mathbf{h}}_2\right]$$

Constitutive relationships

In the following we will consider linear media

Constitutive relationships

Linear media

Example 1

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)+\boldsymbol{\chi}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{h}}(\vec{\mathbf{r}},t)$$

where

$\vec{\mathbf{d}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{e}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{h}}(\vec{\mathbf{r}},t)$: 3x1 column vectors

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t);\ \boldsymbol{\chi}(\vec{\mathbf{r}},t)$: 3x3 matrices

- **Local (non-dispersive) media**
- **Bianisotropic media**

Constitutive relationships

Linear media

Example 2

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

where

$\vec{\mathbf{d}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{e}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{h}}(\vec{\mathbf{r}},t):$ 3x1 column vectors

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t):$ 3x3 matrices

- **Local (non-dispersive) media**
- **Anisotropic media**

Constitutive relationships

Linear media

Example 2

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$$\vec{\mathbf{b}}(\vec{\mathbf{r}},t) = \boldsymbol{\mu}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{h}}(\vec{\mathbf{r}},t)$$

$$\vec{\mathbf{j}}(\vec{\mathbf{r}},t) = \boldsymbol{\sigma}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

- **Local (non-dispersive) media**
- **Anisotropic media**

where

$\vec{\mathbf{d}}(\vec{\mathbf{r}},t);\vec{\mathbf{b}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{j}}(\vec{\mathbf{r}},t);\vec{\mathbf{e}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{h}}(\vec{\mathbf{r}},t):$ 3x1 column vectors

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t);\boldsymbol{\mu}(\vec{\mathbf{r}},t);\boldsymbol{\sigma}(\vec{\mathbf{r}},t):$ 3x3 matrices

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)$:permittivity [Farad/m]

$\boldsymbol{\mu}(\vec{\mathbf{r}},t)$:permeability [Henry/m]

$\boldsymbol{\sigma}(\vec{\mathbf{r}},t)$:conductivity [Siemens/m]

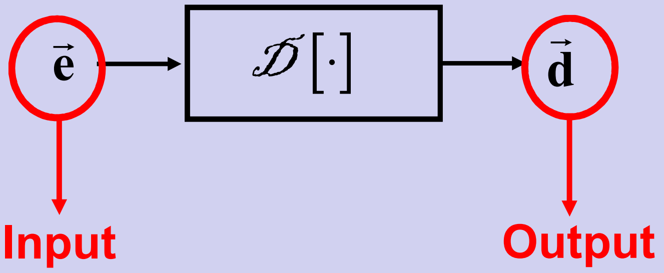
Constitutive relationships

Linear & Anisotropic media

$$\vec{\mathbf{d}} = \mathscr{D}\left[\vec{\mathbf{e}}\right]$$

$$\vec{\mathbf{b}} = \mathscr{B}\left[\vec{\mathbf{h}}\right]$$

$$\vec{\mathbf{j}} = \mathscr{J}\left[\vec{\mathbf{e}}\right]$$



Constitutive relationships

Linear media

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)$: 3x3 matrix

- **Local (non-dispersive) media**
- **Anisotropic media**

Class	**Property**	**Effect on the I-O relation**
Isotropic	A **rotation** of the input implies **the same rotation** of the output	$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)$ becomes scalar. It is not a matrix anymore!

Constitutive relationships

Linear media

- **Local (non-dispersive) media**
- **Anisotropic media**

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$$\vec{\mathbf{b}}(\vec{\mathbf{r}},t)=\boldsymbol{\mu}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{h}}(\vec{\mathbf{r}},t)$$

$$\vec{\mathbf{j}}(\vec{\mathbf{r}},t)=\boldsymbol{\sigma}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t);\boldsymbol{\mu}(\vec{\mathbf{r}},t);\boldsymbol{\sigma}(\vec{\mathbf{r}},t):$ 3x3 matrices

- **Local (non-dispersive) media**
- **Isotropic media**

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\varepsilon(\vec{\mathbf{r}},t)\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$$\vec{\mathbf{b}}(\vec{\mathbf{r}},t)=\mu(\vec{\mathbf{r}},t)\vec{\mathbf{h}}(\vec{\mathbf{r}},t)$$

$$\vec{\mathbf{j}}(\vec{\mathbf{r}},t)=\sigma(\vec{\mathbf{r}},t)\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$\varepsilon(\vec{\mathbf{r}},t);\mu(\vec{\mathbf{r}},t);\sigma(\vec{\mathbf{r}},t):$ scalar functions

Constitutive relationships

Linear media

Class	Property	Effect on the I-O relation
Time-dispersive		
Time-nondispersive		
Time-variant		
Time-invariant		

Constitutive relationships

Linear media

Class	Property	Effect on the I-O relation
Space-dispersive		
Space-nondispersive		
Space-variant		
Space-invariant		

Constitutive relationships

Linear media

Class	Property	Effect on the I-O relation
Time-dispersive		
Time-nondispersive	The output **at time t depends only** on the value of the input **at the same time t**	
Time-variant		
Time-invariant		

Constitutive relationships

Linear media

Class	Property	Effect on the I-O relation
Time-dispersive **Time-nondispersive** **Time-variant** **Time-invariant**	The output **at time t depends** on the values of the input **throughout a time-interval**.	

Constitutive relationships

Linear media

Class

Time-dispersive

Time-nondispersive

Time-variant

Time-invariant

Property

A **time translation** of the input implies **the same translation** of the output

Effect on the I-O relation

Constitutive relationships

Linear media

Class	Property	Effect on the I-O relation
Time-dispersive **Time-nondispersive** **Time-variant** **Time-invariant**	A **time translation** of the input **does not** imply **the same translation** of the output	

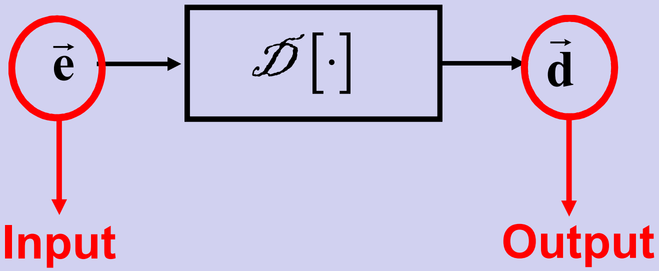
Constitutive relationships

Linear & Anisotropic media

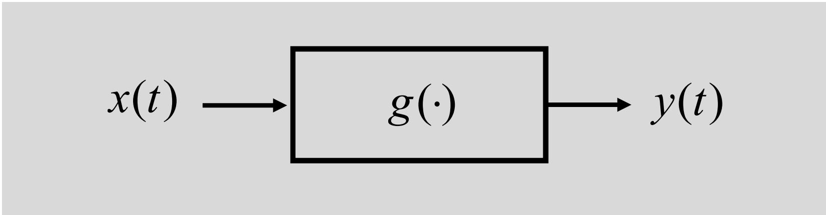
$$\vec{\mathbf{d}} = \mathscr{D}\left[\vec{\mathbf{e}}\right]$$

$$\vec{\mathbf{b}} = \mathscr{B}\left[\vec{\mathbf{h}}\right]$$

$$\vec{\mathbf{j}} = \mathscr{J}\left[\vec{\mathbf{e}}\right]$$



Memo: time-dispersive (TD) linear systems



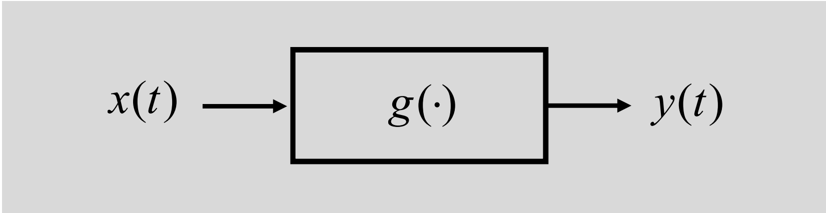
Effect on the I-O relation

$$y(t) = \int dt' \, g(t, t') \, x(t')$$

The output **at time t depends** on the values of the input **throughout a time-interval**.

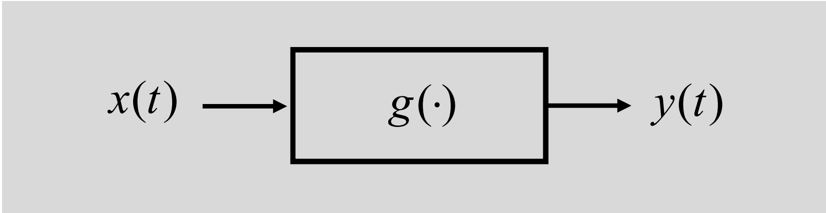
In the most general case, these systems possess an heredity: they are called **dispersive**

Memo: Time-nondispersive (TND) linear systems



TD $$y(t)=\int dt'\, g\left(t,t'\right)x\left(t'\right)$$

Memo: Time-nondispersive (TND) linear systems



TD $y(t) = \int dt' \, g(t,t') \, x(t')$

Property

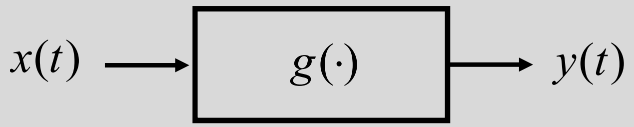
The output **at time *t* depends only** on the value of the input **at the same time *t***

Effect on the I-O relation

$$g(t,t') \rightarrow \delta(t-t') \, \tilde{g}(t')$$

$$y(t) = \int dt' \, \delta(t-t') \, \tilde{g}(t') \, x(t') = \tilde{g}(t) \, x(t)$$

Memo: Time-nondispersive (TND) linear systems



TD	$y(t)=\int dt'\, g(t,t')\,x(t')$
TND	$y(t)=\tilde{g}(t)\,x(t)$

Property

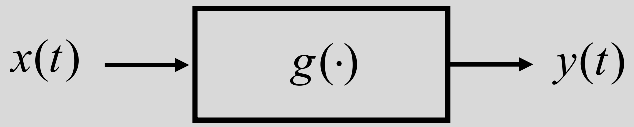
The output **at time t depends only** on the value of the input **at the same time t**

Effect on the I-O relation

$$g(t,t') \rightarrow \delta(t-t')\,\tilde{g}(t')$$

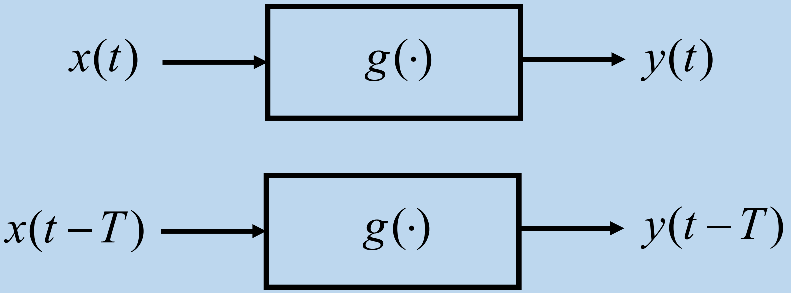
$$y(t)=\int dt'\,\delta(t-t')\,\tilde{g}(t')\,x(t') = \tilde{g}(t)\,x(t)$$

Memo: time-invariant (TI) linear systems



TD	$y(t)=\int dt'\, g(t,t')\,x(t')$
TND	$y(t)=\tilde{g}(t)\,x(t)$

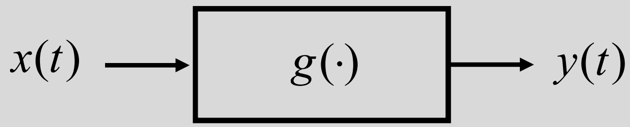
Property



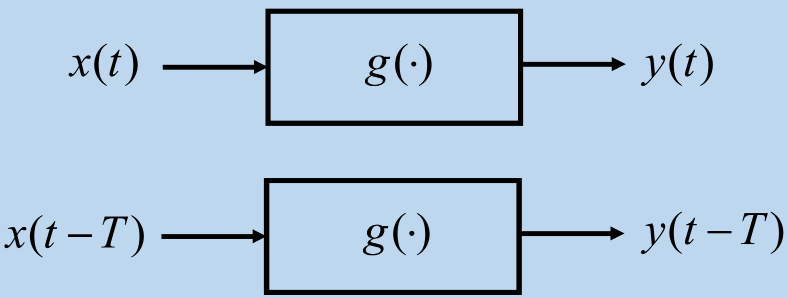
Effect on the I-O relation
TD case

Effect on the I-O relation
TND case

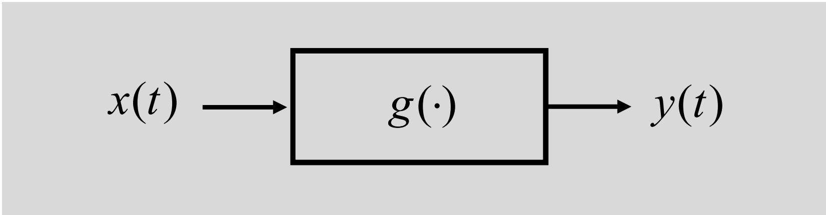
Memo: time-invariant (TI) linear systems



TD-TV	$y(t)=\int dt'\, g(t,t')\,x(t')$
TND-TV	$y(t)=\tilde{g}(t)\,x(t)$

Property	Effect on the I-O relation TD case	Effect on the I-O relation TND case
	$g(t,t') \rightarrow g(t-t')$ $y(t)=\int dt'\, g(t-t')\,x(t')$	$\tilde{g}(t) \rightarrow \tilde{g}$ $y(t)=\tilde{g}\,x(t)$

Memo: linear systems



$x(t)$ and $y(t)$ are scalar

	Time-dispersive (TD)	Time-nondispersive (TND)
Time-variant (TV)	$y(t)=\int dt'\, g(t,t')\, x(t')$	$y(t)=\tilde{g}(t)\, x(t)$
Time-invariant (TI)	$y(t)=\int dt'\, g(t-t')\, x(t')$	$y(t)=\tilde{g}\, x(t)$