Campi Elettromagnetici

Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

a.a. 2020-2021 - Laurea “Triennale” – Secondo semestre - Secondo anno

Università degli Studi di Napoli “Parthenope”

Stefano Perna

Color legend

New formulas, important considerations,
important formulas, important concepts

Very important for the discussion

Memo

Mathematical tools to be exploited

Mathematics

Mathematical tools that we will exploit today

$$\nabla \cdot \left(\nabla \times \vec{\mathbf{A}}(\vec{\mathbf{r}}) \right) = 0$$

Maxwell equations

Time domain - Differential form

$$
\begin{cases}
\nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\
\nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0
\end{cases}
$$

James Clerk Maxwell 1831-1879

Maxwell equations

Time domain - Differential form

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\begin{cases}
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James Clerk Maxwell 1831-1879

Maxwell equations

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\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0
\end{cases}
$$

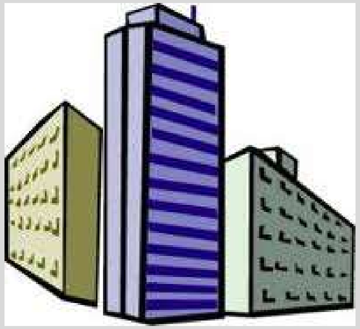
$$
\begin{cases}
\vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\
\rho_0(\vec{\mathbf{r}},t)
\end{cases}
$$

Prescribed sources

$$
\begin{cases}
\vec{\mathbf{j}}(\vec{\mathbf{r}},t) \\
\rho(\vec{\mathbf{r}},t)
\end{cases}
$$

Induced sources

Complex scenario



The independence of the Maxwell equations

Time domain - Differential form

$$\begin{cases} \nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\ \nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0 \end{cases}$$

$$\nabla \cdot \left(\nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t)\right) = \nabla \cdot \left(-\frac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t}\right)$$

⬇ ⬇

$$0 \qquad = -\frac{\partial\left(\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t)\right)}{\partial t}$$

$$\nabla \cdot \left(\nabla \times \vec{\mathbf{A}}(\vec{\mathbf{r}})\right) = 0$$

The independence of the Maxwell equations

Time domain - Differential form

$$\begin{cases} \nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\ \nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0 \end{cases}$$

$$\nabla \cdot \left(\nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t)\right) = \nabla \cdot \left(-\frac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t}\right)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$0 = -\frac{\partial\left(\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t)\right)}{\partial t}$$

$\nabla \cdot \vec{\mathbf{b}}$ is independent of time. If the fields are equal to zero before a given time, then $\nabla \cdot \vec{\mathbf{b}} = 0$ for all times, thus recovering the last Maxwell equation

The independence of the Maxwell equations

Time domain - Differential form

$$\begin{cases} \nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\ \nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\ \nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0 \end{cases}$$

Number of independent scalar equations:

3+3+1=7

Let us assume knowledge of the impressed sources $\vec{\mathbf{j}}_0(\vec{\mathbf{r}},t); \rho_0(\vec{\mathbf{r}},t)$

Number of unknown scalar quantities:

$\vec{\mathbf{e}}(\vec{\mathbf{r}},t); \vec{\mathbf{d}}(\vec{\mathbf{r}},t); \vec{\mathbf{h}}(\vec{\mathbf{r}},t); \vec{\mathbf{b}}(\vec{\mathbf{r}},t); \vec{\mathbf{j}}(\vec{\mathbf{r}},t); \rho(\vec{\mathbf{r}},t)$

3 +3 +3 +3 +3 +1

16

The independence of the Maxwell equations

Time domain - Differential form

$$
\begin{cases}
\nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\
\nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0
\end{cases}
$$

Even assuming knowledge of the impressed sources $\vec{\mathbf{j}}_0(\vec{\mathbf{r}},t); \rho_0(\vec{\mathbf{r}},t)$

Number of independent scalar equations: 7

Number of unknown scalar quantities: 16

Maxwell equations involve a number of unknowns larger than the number of equations!

Constitutive relationships

Time domain - Differential form

$$
\begin{cases}
\nabla \times \vec{\mathbf{e}}(\vec{\mathbf{r}},t) = -\dfrac{\partial \vec{\mathbf{b}}(\vec{\mathbf{r}},t)}{\partial t} \\
\nabla \times \vec{\mathbf{h}}(\vec{\mathbf{r}},t) = \dfrac{\partial \vec{\mathbf{d}}(\vec{\mathbf{r}},t)}{\partial t} + \vec{\mathbf{j}}(\vec{\mathbf{r}},t) + \vec{\mathbf{j}}_0(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{d}}(\vec{\mathbf{r}},t) = \rho(\vec{\mathbf{r}},t) + \rho_0(\vec{\mathbf{r}},t) \\
\nabla \cdot \vec{\mathbf{b}}(\vec{\mathbf{r}},t) = 0
\end{cases}
$$

Even assuming knowledge of the impressed sources $\vec{\mathbf{j}}_0(\vec{\mathbf{r}},t); \rho_0(\vec{\mathbf{r}},t)$

Number of independent scalar equations: 7

Number of unknown scalar quantities: 16

Maxwell equations involve a number of unknowns larger than the number of equations!

The additional missing equations are provided by the **constitutive relationships**, which describe interaction of fields and matter from a macroscopic point of view

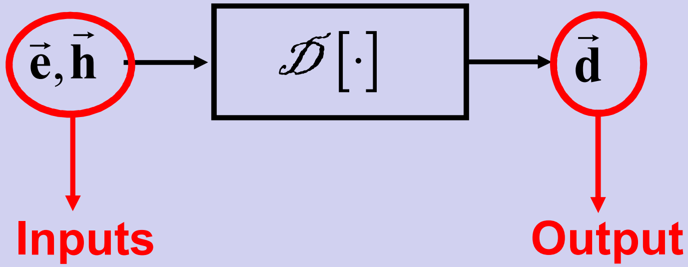
Constitutive relationships

Inductions and currents must be represented in terms of fields

$$\vec{\mathbf{d}} = \mathscr{D}\left[\vec{\mathbf{e}},\vec{\mathbf{h}}\right]$$

$$\vec{\mathbf{b}} = \mathscr{B}\left[\vec{\mathbf{e}},\vec{\mathbf{h}}\right]$$

$$\vec{\mathbf{j}} = \mathscr{J}\left[\vec{\mathbf{e}},\vec{\mathbf{h}}\right]$$



$\mathscr{D}[\cdot], \mathscr{B}[\cdot]$ and $\mathscr{J}[\cdot]$ are functionals which depend upon the medium in which the electromagnetic field is considered and upon the fields themselves.

Constitutive relationships

Linear media

$$\vec{\mathbf{d}}_1 = \mathscr{D}\left[\vec{\mathbf{e}}_1, \vec{\mathbf{h}}_1\right]$$

$$\vec{\mathbf{d}}_2 = \mathscr{D}\left[\vec{\mathbf{e}}_2, \vec{\mathbf{h}}_2\right]$$

$$\Rightarrow \quad \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 = \mathscr{D}\left[\vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_2, \vec{\mathbf{h}}_1 + \vec{\mathbf{h}}_2\right]$$

Constitutive relationships

In the following we will consider linear media

Constitutive relationships

Linear media

Example 1

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)+\boldsymbol{\chi}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{h}}(\vec{\mathbf{r}},t)$$

where

$\vec{\mathbf{d}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{e}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{h}}(\vec{\mathbf{r}},t)$: 3x1 column vectors

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t);\ \boldsymbol{\chi}(\vec{\mathbf{r}},t)$: 3x3 matrices

- **Local (non-dispersive) media**
- **Bianisotropic media**

Constitutive relationships

Linear media

Example 2

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

where

$\vec{\mathbf{d}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{e}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{h}}(\vec{\mathbf{r}},t):$ 3x1 column vectors

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t):$ 3x3 matrices

- **Local (non-dispersive) media**
- **Anisotropic media**

Constitutive relationships

Linear media

Example 2

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$
$$\vec{\mathbf{b}}(\vec{\mathbf{r}},t)=\boldsymbol{\mu}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{h}}(\vec{\mathbf{r}},t)$$
$$\vec{\mathbf{j}}(\vec{\mathbf{r}},t)=\boldsymbol{\sigma}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

- **Local (non-dispersive) media**
- **Anisotropic media**

where

$\vec{\mathbf{d}}(\vec{\mathbf{r}},t);\vec{\mathbf{b}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{j}}(\vec{\mathbf{r}},t);\vec{\mathbf{e}}(\vec{\mathbf{r}},t);\ \vec{\mathbf{h}}(\vec{\mathbf{r}},t)$: 3x1 column vectors

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t);\boldsymbol{\mu}(\vec{\mathbf{r}},t);\boldsymbol{\sigma}(\vec{\mathbf{r}},t)$: 3x3 matrices

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)$:permittivity [Farad/m]

$\boldsymbol{\mu}(\vec{\mathbf{r}},t)$:permeability [Henry/m]

$\boldsymbol{\sigma}(\vec{\mathbf{r}},t)$:conductivity [Siemens/m]

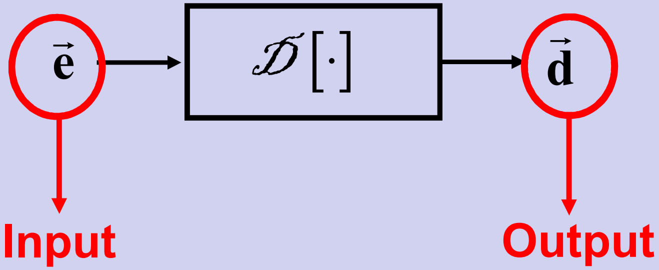
Constitutive relationships

Linear & Anisotropic media

$$\vec{\mathbf{d}} = \mathscr{D}\left[\vec{\mathbf{e}}\right]$$

$$\vec{\mathbf{b}} = \mathscr{B}\left[\vec{\mathbf{h}}\right]$$

$$\vec{\mathbf{j}} = \mathscr{J}\left[\vec{\mathbf{e}}\right]$$



Constitutive relationships

Linear media

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)$: 3x3 matrix

- **Local (non-dispersive) media**
- **Anisotropic media**

Class	Property	Effect on the I-O relation
Isotropic	A **rotation** of the input implies **the same rotation** of the output	$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)$ becomes scalar. It is not a matrix anymore!

Constitutive relationships

Linear media

- **Local (non-dispersive) media**
- **Anisotropic media**

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$$\vec{\mathbf{b}}(\vec{\mathbf{r}},t)=\boldsymbol{\mu}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{h}}(\vec{\mathbf{r}},t)$$

$$\vec{\mathbf{j}}(\vec{\mathbf{r}},t)=\boldsymbol{\sigma}(\vec{\mathbf{r}},t)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$\boldsymbol{\varepsilon}(\vec{\mathbf{r}},t);\boldsymbol{\mu}(\vec{\mathbf{r}},t);\boldsymbol{\sigma}(\vec{\mathbf{r}},t):$ 3x3 matrices

- **Local (non-dispersive) media**
- **Isotropic media**

$$\vec{\mathbf{d}}(\vec{\mathbf{r}},t)=\varepsilon(\vec{\mathbf{r}},t)\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$$\vec{\mathbf{b}}(\vec{\mathbf{r}},t)=\mu(\vec{\mathbf{r}},t)\vec{\mathbf{h}}(\vec{\mathbf{r}},t)$$

$$\vec{\mathbf{j}}(\vec{\mathbf{r}},t)=\sigma(\vec{\mathbf{r}},t)\vec{\mathbf{e}}(\vec{\mathbf{r}},t)$$

$\varepsilon(\vec{\mathbf{r}},t);\mu(\vec{\mathbf{r}},t);\sigma(\vec{\mathbf{r}},t):$ scalar functions

Constitutive relationships

Linear media

Class

Property

Effect on the I-O relation

Constitutive relationships

Linear media

Class	Property	Effect on the I-O relation
Time-dispersive		
Time-nondispersive		
Time-variant		
Time-invariant		

Constitutive relationships

Linear media

Class	Property	Effect on the I-O relation
Space-dispersive		
Space-nondispersive		
Space-variant		
Space-invariant		

Constitutive relationships

Linear media

Class	Property
Time-dispersive	
Time-nondispersive	The output **at time *t* depends only** on the value of the input **at the same time *t***
Time-variant	
Time-invariant	

Constitutive relationships

Linear media

Class	Property
Time-dispersive	The output **at time t depends** on the values of the input **throughout a time-interval**.
Time-nondispersive	
Time-variant	
Time-invariant	

Constitutive relationships

Linear media

Class

Time-dispersive

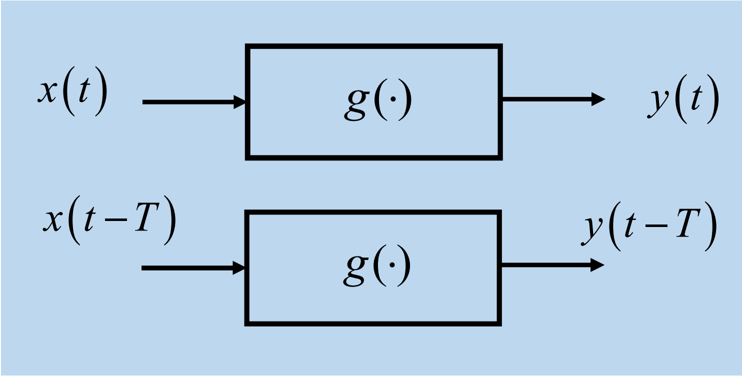
Time-nondispersive

Time-variant

Time-invariant

Property

A **time translation** of the input implies **the same translation** of the output



Constitutive relationships

Linear media

Class	Property
Time-dispersive	
Time-nondispersive	
Time-variant	A **time translation** of the input **does not** imply **the same translation** of the output
Time-invariant	

Constitutive relationships

Linear media

Class

Space-dispersive

Space-nondispersive

Space-variant

Space-invariant

Property

The output **at space $\vec{\mathbf{r}}$** **depends only** on the value of the input **at the same space $\vec{\mathbf{r}}$**

Constitutive relationships

Linear media

Class

Space-dispersive

Space-nondispersive

Space-variant

Space-invariant

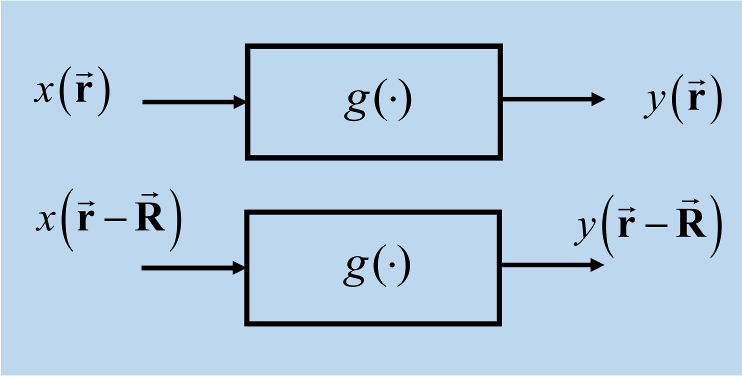
Property

The output **at space $\vec{\mathbf{r}}$ depends** on the values of the input **throughout a space-interval**

Constitutive relationships

Linear media

Class	Property
Space-dispersive	
Space-nondispersive	
Space-variant	
Space-invariant	A **space translation** of the input implies **the same translation** of the output



Constitutive relationships

Linear media

Class

Space-dispersive

Space-nondispersive

Space-variant

Space-invariant

Property

A **space translation** of the input **does not** imply **the same translation** of the output

Constitutive relationships

Linear media

Class	Property	Effect on the I-O relation

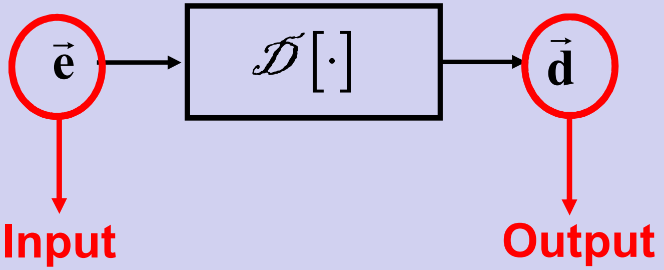
Constitutive relationships

Linear & Anisotropic media

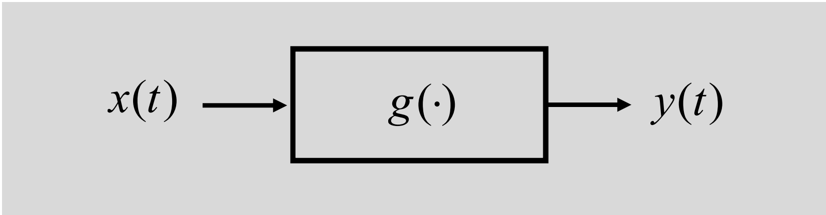
$$\vec{\mathbf{d}} = \mathscr{D}\left[\vec{\mathbf{e}}\right]$$

$$\vec{\mathbf{b}} = \mathscr{B}\left[\vec{\mathbf{h}}\right]$$

$$\vec{\mathbf{j}} = \mathscr{J}\left[\vec{\mathbf{e}}\right]$$



Memo: time-dispersive (TD) linear systems



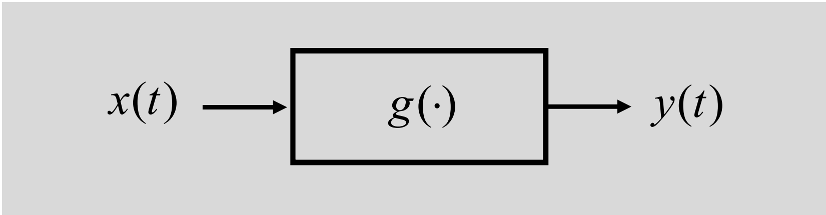
Effect on the I-O relation

$$y(t) = \int dt'\, g(t, t')\, x(t')$$

The output **at time** t **depends** on the values of the input **throughout a time-interval**.

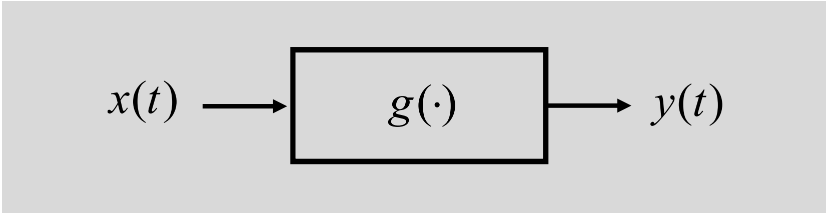
In the most general case, these systems possess an heredity: they are called **dispersive**

Memo: Time-nondispersive (TND) linear systems



TD $$y(t)=\int dt'\, g(t,t')\, x(t')$$

Memo: Time-nondispersive (TND) linear systems



TD $y(t)=\int dt'\, g(t,t')\, x(t')$

Property

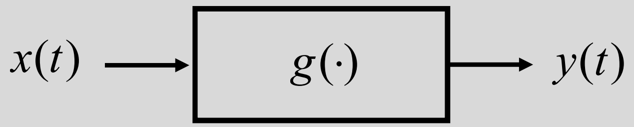
The output **at time t depends only** on the value of the input **at the same time t**

Effect on the I-O relation

$$g(t,t') \rightarrow \delta(t-t')\,\tilde{g}(t')$$

$$y(t)=\int dt'\,\delta(t-t')\,\tilde{g}(t')\,x(t') = \tilde{g}(t)\,x(t)$$

Memo: Time-nondispersive (TND) linear systems



TD	$y(t)=\int dt'\, g(t,t')\, x(t')$
TND	$y(t)=\tilde{g}(t)\, x(t)$

Property

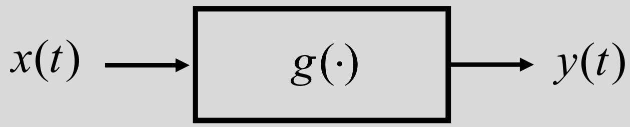
The output **at time t depends only** on the value of the input **at the same time t**

Effect on the I-O relation

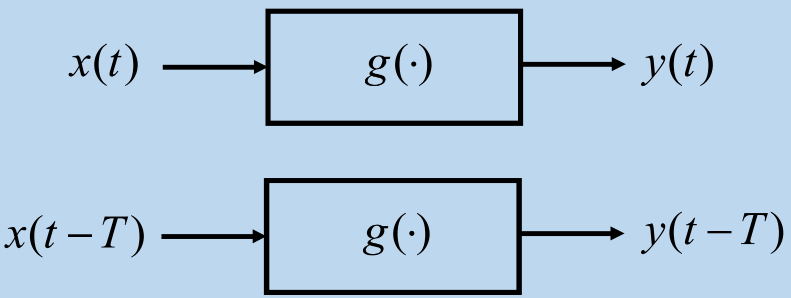
$$g(t,t') \rightarrow \delta(t-t')\,\tilde{g}(t')$$

$$y(t)=\int dt'\, \delta(t-t')\,\tilde{g}(t')\,x(t') = \tilde{g}(t)\,x(t)$$

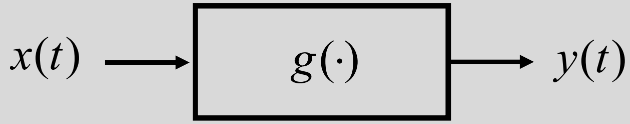
Memo: time-invariant (TI) linear systems



TD	$y(t)=\int dt'\, g(t,t')\,x(t')$
TND	$y(t)=\tilde{g}(t)\,x(t)$

Property	Effect on the I-O relation TD case	Effect on the I-O relation TND case
		

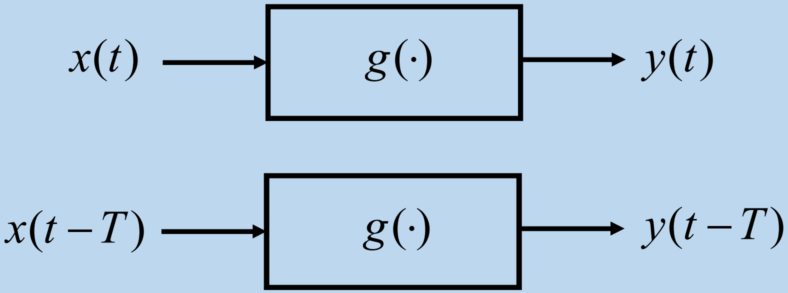
Memo: time-invariant (TI) linear systems



TD-TV $y(t)=\int dt'\, g(t,t')\, x(t')$

TND-TV $y(t)=\tilde{g}(t)\, x(t)$

Property



Effect on the I-O relation

TD case

$$g(t,t') \rightarrow g(t-t')$$

$$y(t)=\int dt'\, g(t-t')\, x(t')$$

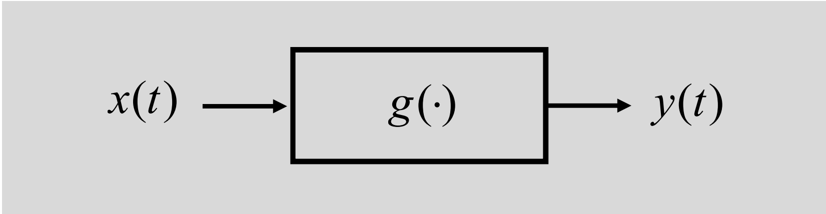
Effect on the I-O relation

TND case

$$\tilde{g}(t) \rightarrow \tilde{g}$$

$$y(t)=\tilde{g}\, x(t)$$

Memo: linear systems



$x(t)$ and $y(t)$ are scalar

	Time-dispersive (TD)	Time-nondispersive (TND)
Time-variant (TV)	$y(t)=\int dt'\, g(t,t')\,x(t')$	$y(t)=\tilde{g}(t)\,x(t)$
Time-invariant (TI)	$y(t)=\int dt'\, g(t-t')\,x(t')$	$y(t)=\tilde{g}\,x(t)$