Campi Elettromagnetici

Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

a.a. 2020-2021 - Laurea "Triennale" – Secondo semestre - Secondo anno

Università degli Studi di Napoli "Parthenope"

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Color legend

New formulas, important considerations,
important formulas, important concepts

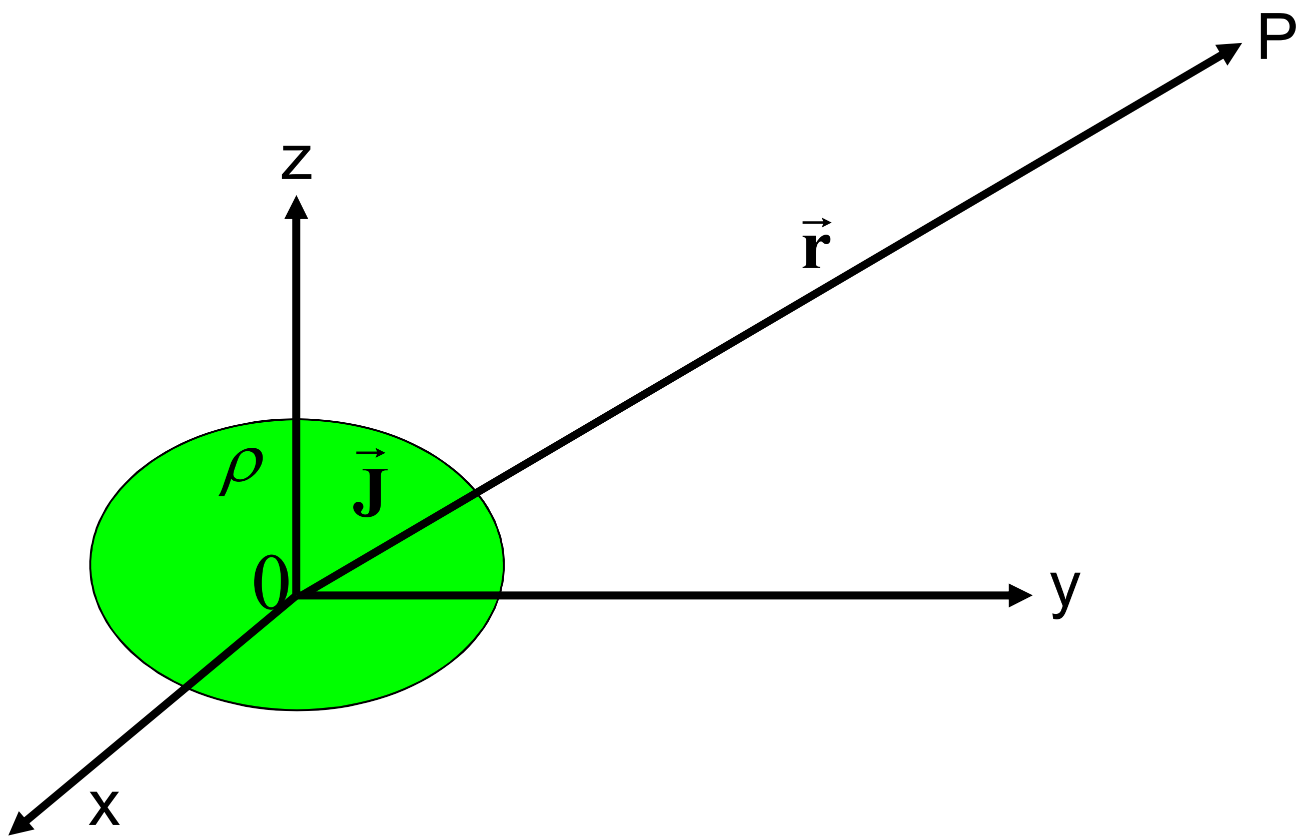
Very important for the discussion

Memo

Mathematical tools to be exploited

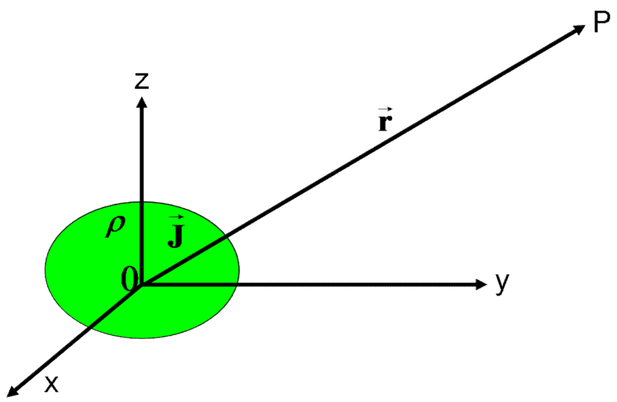
Mathematics

Radiation problem



$$\vec{\mathbf{E}}\left(\vec{\mathbf{r}}\right)$$
$$\vec{\mathbf{H}}\left(\vec{\mathbf{r}}\right)$$

An antenna (or aerial) is an electrical device which converts electric power into radio waves, and vice versa.



Potentials

$\boldsymbol{J}(\boldsymbol{r})$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

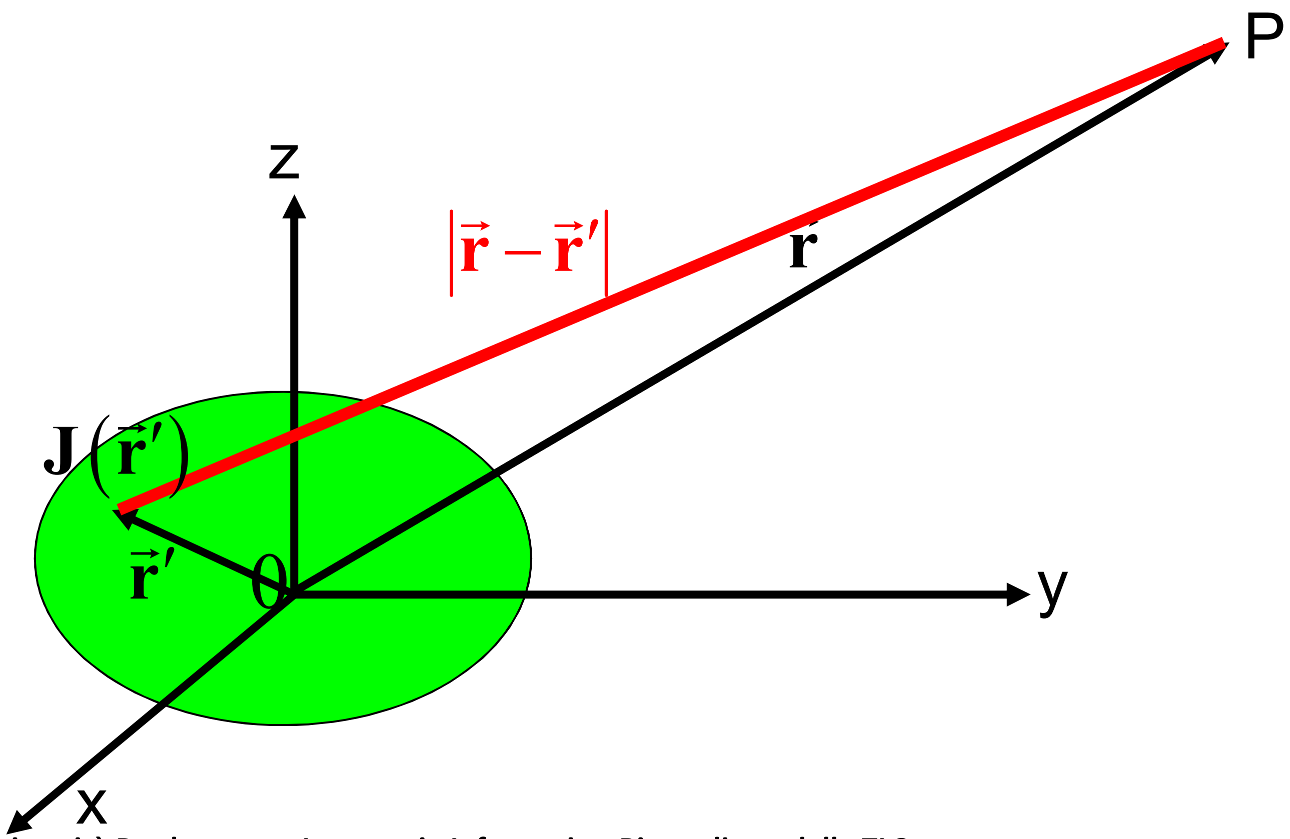
$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'$$

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla\nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$$

$\boldsymbol{E}(\boldsymbol{r})$
$\boldsymbol{H}(\boldsymbol{r})$

Potentials

$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi}\int \mathbf{J}(\vec{\mathbf{r}}')\frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\,\vec{\mathbf{r}}'$$



Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

$$\Big\downarrow \; \boldsymbol{J}$$

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ Vector Helmholtz equation

$$\Big\downarrow \; \mathbf{A} = \frac{\mu}{4\pi} I\Delta z \frac{e^{-jkr}}{r}\hat{i}_z$$

$\boldsymbol{H} = \frac{1}{\mu}\nabla\times\boldsymbol{A}$ and $\boldsymbol{E} = -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu}$

$$\Big\downarrow \; \boldsymbol{E},\ \boldsymbol{H}$$

Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\hat{i}_z$$

The E.M. field radiated by the elementary electrical dipole

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_r(r,\vartheta)\hat{i}_r + \mathrm{E}_\vartheta(r,\vartheta)\hat{i}_\vartheta$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_\varphi(r,\vartheta)\hat{i}_\varphi$$

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

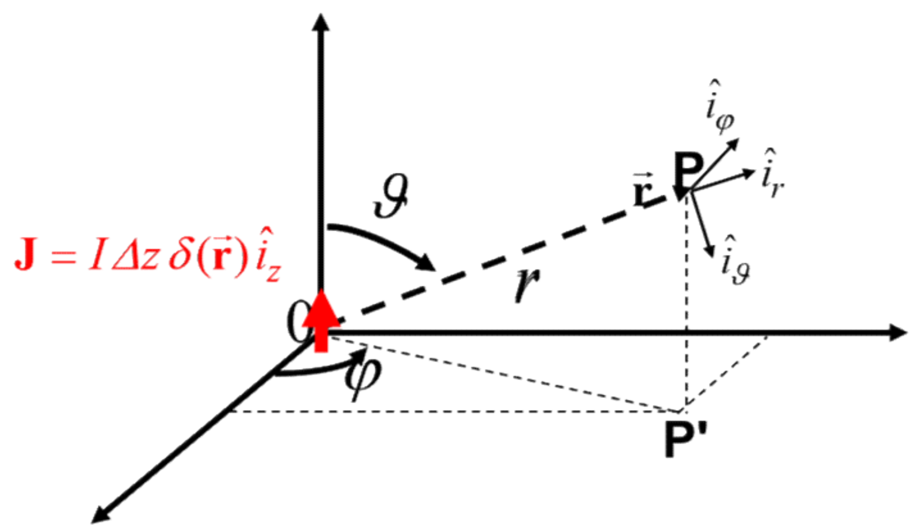
Because of the problem symmetry there is no dependence on the azimuth angle φ.

Elementary electrical dipole: far field

In the far-field case (r>>λ) the elementary electrical dipole behaves as follows

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = E_{\vartheta}(r,\vartheta)\hat{i}_{\vartheta} \qquad \begin{cases} E_{\vartheta} = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\ H_{\varphi} = \dfrac{E_{\vartheta}}{\zeta} \end{cases}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = H_{\varphi}(r,\vartheta)\hat{i}_{\varphi}$$



- the e.m. field propagates along $\hat{i}_r$
- the e.m. field lies on the plane orthogonal to the propagation direction
- |E| and |H| exhibit the decaying factor 1/r
- |E| and |H| are proportional through ζ

$$\zeta\mathbf{H} = \hat{i}_r \times \mathbf{E}$$

$$\vec{\mathbf{S}} = \frac{1}{2\zeta}\left|\vec{\mathbf{E}}\right|^2 \hat{i}_r = \frac{\zeta}{2}\left|\vec{\mathbf{H}}\right|^2 \hat{i}_r$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

$$P = P_1 + jP_2$$

$$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 |I|^2$$

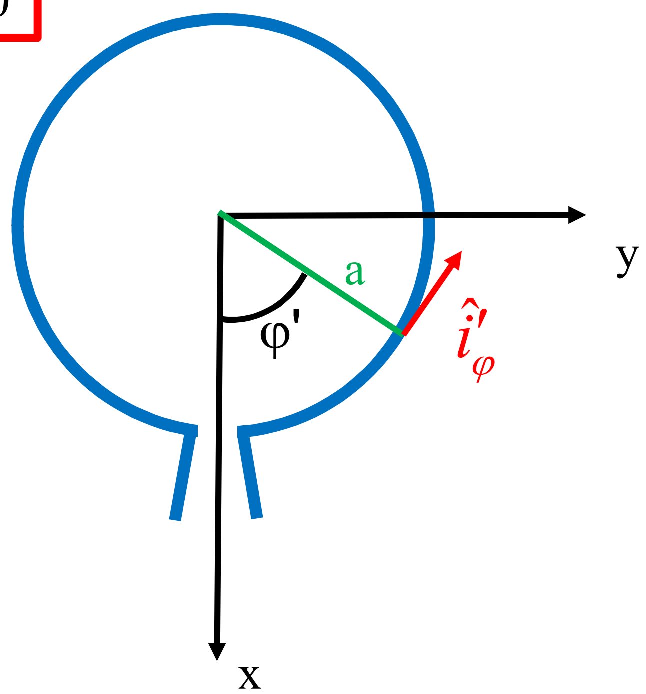
$$P_2 = -\frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \frac{1}{(\beta r)^3}|I|^2$$

- Note that in the far-field case only the first active power term exists and it does not depend on r
- Note that the real part of the power, in lossless medium, is independent of *r*, therefore if one consider two different spherical surfaces one gets the same result. Only the so-called radiative terms contribute.
- The reactive part depends on *r*. Its sign is negative showing that there is an excess of stored **electric** energy in the neighbor of the electrical dipole (see Poynting's theorem)

Small loop antenna

$$\mathbf{J} = I\delta(z)\delta(r-a)\hat{i}'_{\varphi}$$

z=0



$\downarrow J$

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ Vector Helmholtz equation

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int \mathbf{J}(\mathbf{r}')\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu}$$

$\downarrow$ $E(r)$ $H(r)$

Small loop antenna

The E.M. field radiated by the the small loop antenna

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_r(r,\vartheta)\hat{i}_r + \mathrm{H}_\vartheta(r,\vartheta)\hat{i}_\vartheta$$

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_\varphi(r,\vartheta)\hat{i}_\varphi$$

$$\begin{cases} H_r = \dfrac{I\varDelta S}{2\pi}\dfrac{(j\beta)^2}{r}\left(\dfrac{1}{j\beta r} + \dfrac{1}{[j\beta r]^2}\right)\cos\vartheta\exp(-j\beta r) \\ H_\vartheta = \dfrac{I\varDelta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1 + \dfrac{1}{j\beta r} + \dfrac{1}{[j\beta r]^2}\right)\sin\vartheta\exp(-j\beta r) \\ E_\varphi = -\dfrac{\zeta I\varDelta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1 + \dfrac{1}{j\beta r}\right)\sin\vartheta\exp(-j\beta r) \end{cases}$$

Because of the problem symmetry there is no dependence on the azimuth angle φ.

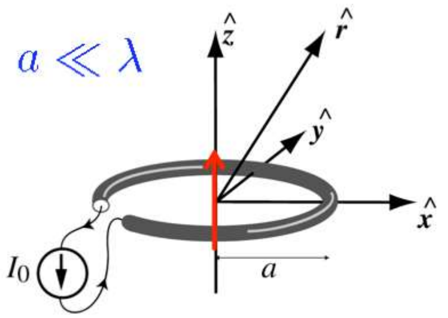
Small loop antenna: far field

In the far-field case (r>>λ) the small loop antenna behaves as follows

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_{\varphi}(r,\vartheta)\hat{i}_{\varphi}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_{\vartheta}(r,\vartheta)\hat{i}_{\vartheta}$$

$$\begin{cases} E_{\varphi} = \dfrac{\zeta\beta\Delta sI}{2\lambda r}\sin\vartheta\exp(-j\beta r) \\ H_{\vartheta} = -\dfrac{\beta\Delta sI}{2\lambda r}\sin\vartheta\exp(-j\beta r) = \dfrac{-E_{\varphi}}{\zeta} \end{cases}$$



$$\Delta S = \pi a^2$$

- the e.m. field propagates along $\hat{i}_r$
- the e.m. field lies on the plane orthogonal to the propagation direction
- |E| and |H| exhibit the decaying factor 1/r
- |E| and |H| are proportional through ζ

Small loop antenna: power flux

$$\begin{cases} H_r = \dfrac{I\Delta S}{2\pi}\dfrac{(j\beta)^2}{r}\left(\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\cos\vartheta\exp(-j\beta r) \\ H_\vartheta = \dfrac{I\Delta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1+\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\sin\vartheta\exp(-j\beta r) \\ E_\varphi = -\dfrac{\zeta I\Delta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1+\dfrac{1}{j\beta r}\right)\sin\vartheta\exp(-j\beta r) \end{cases}$$

$$P = P_1 + jP_2$$

$$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta\Delta S}{\lambda}\right)^2 |I|^2$$

$$P_2 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta\Delta S}{\lambda}\right)^2 \frac{1}{(\beta r)^3}|I|^2$$

- Note that in the far-field case only the first active power term exists and it does not depend on r
- Note that the real part of the power, in lossless medium, is independent of *r*, therefore if one consider two different spherical surfaces one gets the same result. Only the so-called radiative terms contribute.
- The reactive part depends on *r*. The reactive part depends on r. Its sign is positive showing that there is an excess of stored **magnetic** energy in the neighbor of the magnetic dipole (see Poynting's theorem)

Elementary electrical dipole vs. small loop antenna

Elementary electrical dipole

$$P = P_1 + jP_2$$

$$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 |I|^2$$

$$P_2 = -\frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \frac{1}{(\beta r)^3}|I|^2$$

Small loop antenna

$$P = P_1 + jP_2$$

$$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta \Delta S}{\lambda}\right)^2 |I|^2$$

$$P_2 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta \Delta S}{\lambda}\right)^2 \frac{1}{(\beta r)^3}|I|^2$$

Small loop antenna

WHY?

Small loop antenna

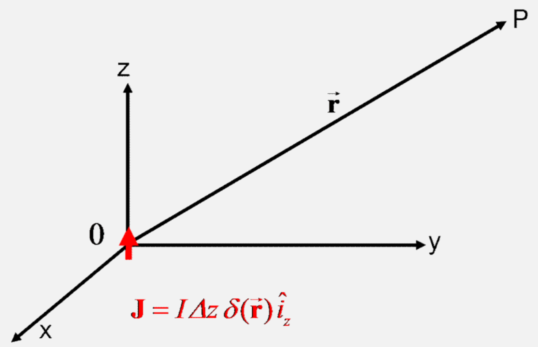
Elementary electrical dipole

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z$$

• Why are we interested in such a radiating element?

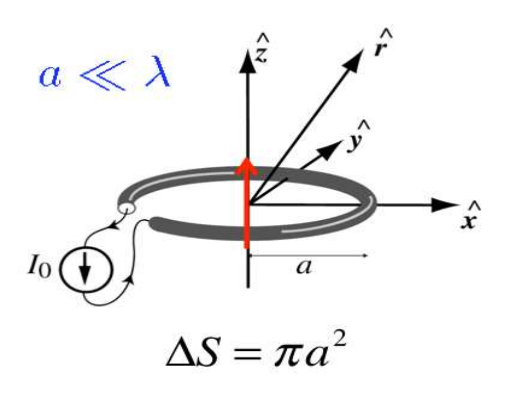
• Why is such a radiating element referred to as elementary electrical dipole?

• How can we physically approximate an elementary electrical dipole?



Small loop antenna

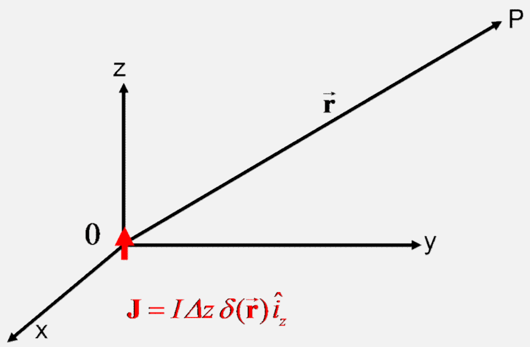
$$\mathbf{J} = I\delta(z)\,\delta(r-a)\hat{i}'_\varphi$$



Small loop antenna

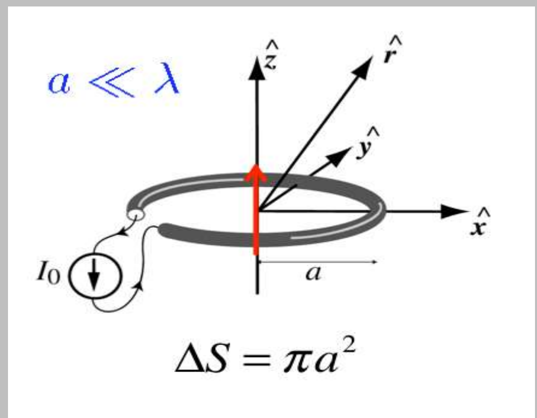
Elementary electrical dipole

$$\mathbf{J} = I\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z$$



Small loop antenna

$$\mathbf{J} = I\,\delta(z)\,\delta(r-a)\,\hat{i}'_\varphi$$

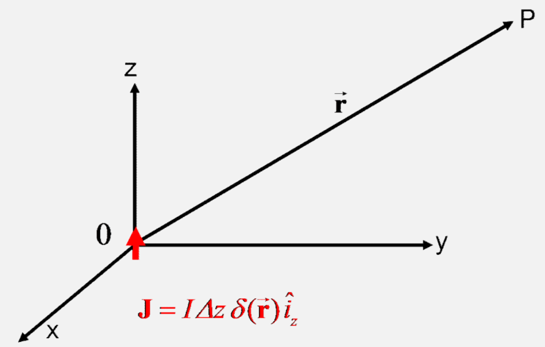


Elementary electrical dipole vs. small loop antenna

Elementary electrical dipole

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z$$

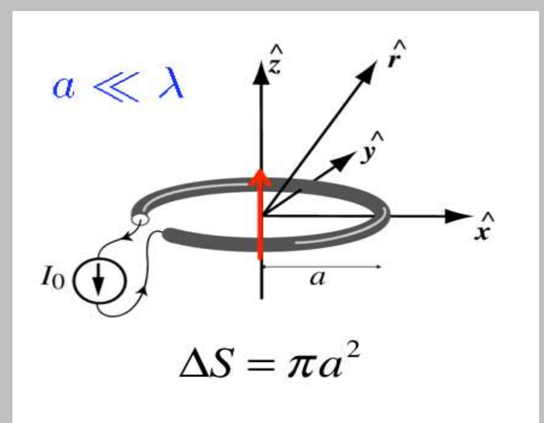
$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta\exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta\exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta\exp(-j\beta r) \end{cases}$$



Small loop antenna

$$\mathbf{J} = I\delta(z)\,\delta(r-a)\,\hat{i}'_\varphi$$

$$\begin{cases} H_r = \dfrac{I\Delta S}{2\pi}j\beta\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta\exp(-j\beta r) \\ H_\vartheta = \dfrac{I\Delta S}{4\pi}j\beta\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta\exp(-j\beta r) \\ E_\varphi = -\dfrac{\zeta I\Delta S}{4\pi}j\beta\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta\exp(-j\beta r) \end{cases}$$



Elementary electrical dipole vs. small loop antenna

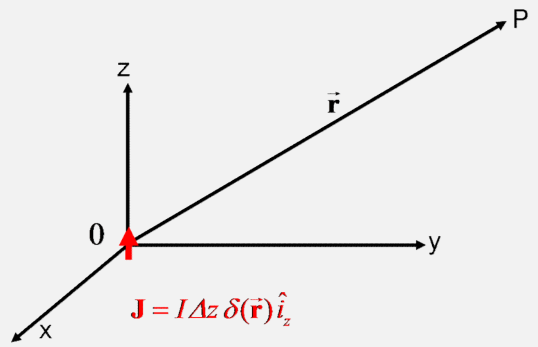
Elementary electrical dipole

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\hat{i}_z$$

for $r >> \lambda$

$$\mathbf{E} = \frac{j\zeta I}{2\lambda}\frac{\exp(-j\beta r)}{r}\Delta z \sin\vartheta\ \hat{i}_\vartheta$$

$$\zeta\mathbf{H} = \hat{i}_r \times \mathbf{E}$$



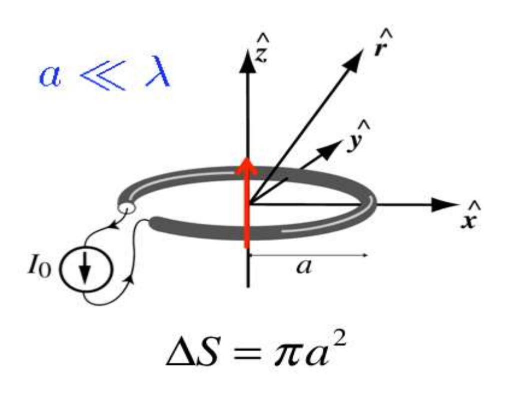
Small loop antenna

$$\mathbf{J} = I\delta(z)\delta(r-a)\hat{i}'_\varphi$$

for $r >> \lambda$

$$\mathbf{E} = \frac{\zeta\beta\Delta S I}{2\lambda}\frac{\exp(-j\beta r)}{r}\sin\vartheta\ \hat{i}_\varphi$$

$$\zeta\mathbf{H} = \hat{i}_r \times \mathbf{E}$$



Elementary electrical dipole vs. small loop antenna

$$P = \frac{1}{2} \oiint_S \left[\mathbf{E} \times \mathbf{H}^* \right] \cdot \hat{i}_r dS$$

Elementary electrical dipole	Small loop antenna
$P = P_1 + jP_2$	$P = P_1 + jP_2$
$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \lvert I \rvert^2$	$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta \Delta S}{\lambda}\right)^2 \lvert I \rvert^2$
$P_2 = -\frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \frac{1}{(\beta r)^3} \lvert I \rvert^2$	$P_2 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta \Delta S}{\lambda}\right)^2 \frac{1}{(\beta r)^3} \lvert I \rvert^2$

Small loop antenna

WHY?

Magnetic Sources

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \\ \nabla \cdot \varepsilon\mathbf{E} = \rho \\ \nabla \cdot \mu\mathbf{H} = 0 \end{cases}$$

Magnetic Sources

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{J}_{\mathrm{m}} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \\ \nabla \cdot \varepsilon\mathbf{E} = \rho \\ \nabla \cdot \mu\mathbf{H} = \rho_{\mathrm{m}} \end{cases}$$

Magnetic Sources

What is the relation between sources and fields in this case?

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{J}_{\mathrm{m}} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \\ \nabla \cdot \varepsilon\mathbf{E} = \rho \\ \nabla \cdot \mu\mathbf{H} = \rho_{\mathrm{m}} \end{cases}$$

Magnetic Sources

What is the relation between sources and fields in this case?

Let's simplify the question. What is the relation between sources and fields in this case?

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{J}_m \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} \\ \nabla \cdot \varepsilon\mathbf{E} = 0 \\ \nabla \cdot \mu\mathbf{H} = \rho_m \end{cases}$$

Magnetic Sources

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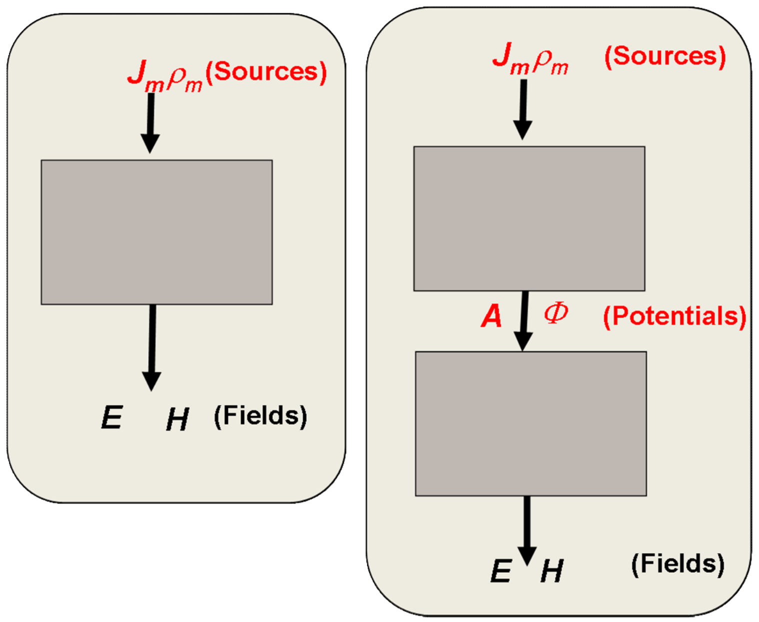


In principle, we could replace the same approach as that exploited for the electric sources

Magnetic Sources

What is the relation between sources and fields in this case?

Let's simplify the question. What is the relation between sources and fields in this case?

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{J}_{\mathrm{m}} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} \\ \nabla \cdot \varepsilon\mathbf{E} = 0 \\ \nabla \cdot \mu\mathbf{H} = \rho_{\mathrm{m}} \end{cases}$$

In practice, we follow an easier way, provided by the **duality theorem**

Magnetic Sources

What is the relation between sources and fields in this case?

Let's simplify the question. What is the relation between sources and fields in this case?

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{J}_{\mathrm{m}} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} \\ \nabla \cdot \varepsilon\mathbf{E} = 0 \\ \nabla \cdot \mu\mathbf{H} = \rho_{\mathrm{m}} \end{cases}$$

In practice, we follow an easier way, provided by the **duality theorem**

$$\begin{array}{c} \mathbf{J} \;\; \rho \\ \downarrow \\ \mathbf{E}, \mathbf{H} \end{array} \quad \Longrightarrow \quad \begin{array}{c} \mathbf{J}_{\mathrm{m}} = -\zeta\mathbf{J} \quad \rho_{\mathrm{m}} = -\zeta\rho \\ \downarrow \\ \mathbf{E}' = \zeta\mathbf{H} \;\; , \mathbf{H}' = -\dfrac{1}{\zeta}\mathbf{E} \end{array}$$

Duality Theorem

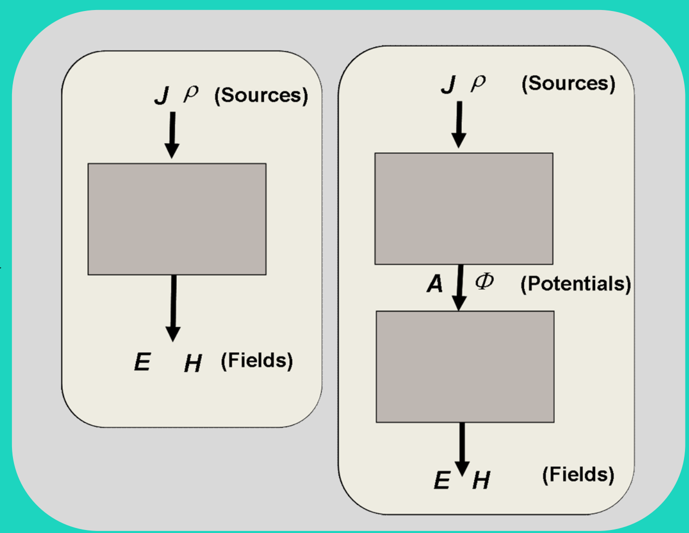
$$\mathbf{J}\ \ \rho \;\downarrow\; \mathbf{E},\mathbf{H} \quad \Longrightarrow \quad \boxed{\mathbf{J}_\mathrm{m} = -\zeta\mathbf{J} \quad \rho_\mathrm{m} = -\zeta\rho} \;\downarrow\; \boxed{\mathbf{E}' = \zeta\mathbf{H} \quad , \mathbf{H}' = -\frac{1}{\zeta}\mathbf{E}}$$

$$\mathbf{J}_\mathrm{m} \quad \rho_\mathrm{m}$$

$$\downarrow$$

$$\mathbf{J} = -\frac{1}{\zeta}\mathbf{J}_\mathrm{m} \qquad \rho = -\frac{1}{\zeta}\rho_\mathrm{m}$$

$$\rightarrow$$



$$\rightarrow$$

$$\mathbf{E}' = \zeta\mathbf{H}$$

$$\mathbf{H}' = -\frac{1}{\zeta}\mathbf{E}$$

Elementary electrical and magnetic dipoles

Elementary electrical dipole

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

$$I\Delta z = j\omega Q\Delta z = j\omega U$$

Elementary magnetic dipole

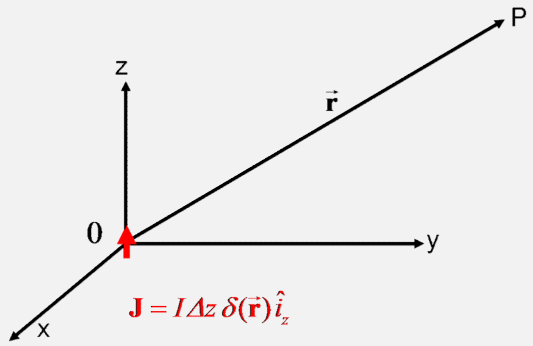
$$\mathbf{J}_m = I_m\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I_m\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

$$I_m\Delta z = j\omega U_m$$

Elementary electrical and magnetic dipoles

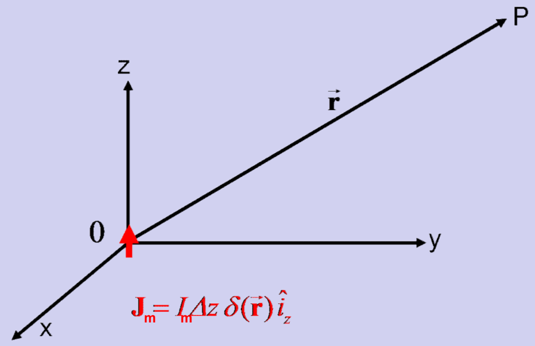
Elementary electrical dipole

$$\mathbf{J} = I\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z$$



Elementary magnetic dipole

$$\mathbf{J}_m = I_m\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z$$



Duality Theorem

$$\mathbf{J}\ \rho \;\longrightarrow\; \mathbf{E},\mathbf{H}$$

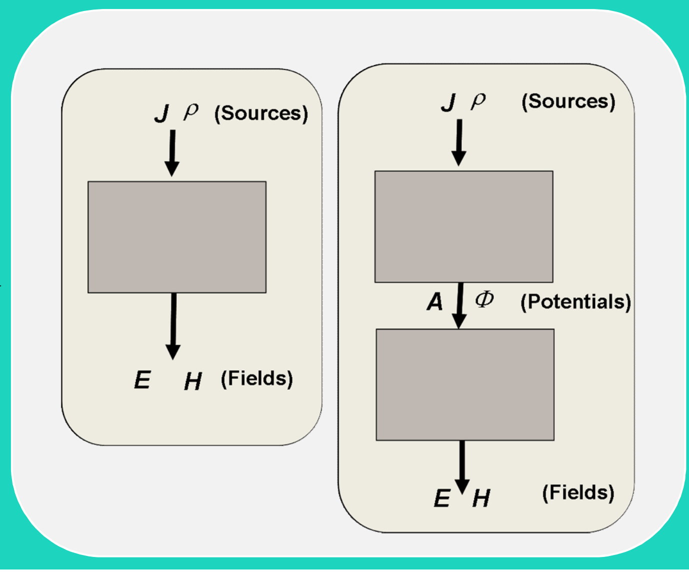
$$\Rightarrow$$

$$\mathbf{J}_{\mathrm{m}} = -\zeta\mathbf{J} \quad \rho_{\mathrm{m}} = -\zeta\rho \;\longrightarrow\; \mathbf{E}' = \zeta\mathbf{H}\ ,\ \mathbf{H}' = -\frac{1}{\zeta}\mathbf{E}$$

$$\mathbf{J}_{\mathrm{m}}\quad \rho_{\mathrm{m}}$$

$$\downarrow$$

$$\mathbf{J} = -\frac{1}{\zeta}\mathbf{J}_{\mathrm{m}} \qquad \rho = -\frac{1}{\zeta}\rho_{\mathrm{m}} \;\rightarrow$$



$$\rightarrow\; \mathbf{E}' = \zeta\mathbf{H} \qquad \mathbf{H}' = -\frac{1}{\zeta}\mathbf{E}$$

Elementary electrical and magnetic dipoles

Ampere equivalence theorem

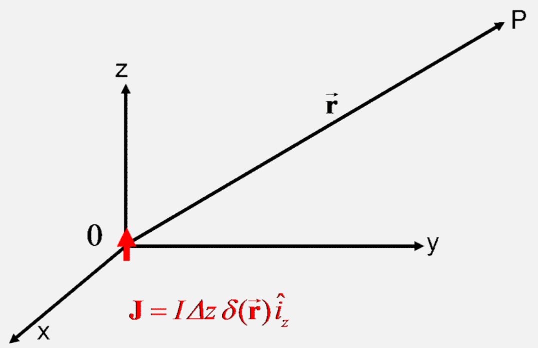
By applying the Duality theorem it turns out that the small loop antenna is equivalent to an elementary magnetic dipole, provided that:

$$U_m = \mu I \Delta S$$

Elementary electrical and magnetic dipoles

Elementary electrical dipole

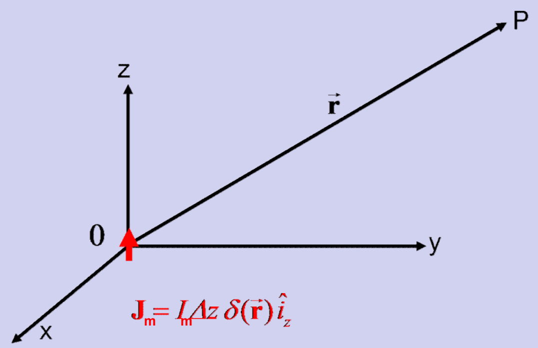
$$\mathbf{J} = I\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z$$



- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as elementary electrical dipole?
- How can we physically approximate an elementary electrical dipole?

Elementary magnetic dipole

$$\mathbf{J}_m = I_m\,\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z$$



- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as **elementary magnetic** dipole?
- How can we physically approximate an **elementary magnetic** dipole?

Small loop antenna

WHY?

Color legend

New formulas, important considerations, important formulas, important concepts

Very important for the discussion

Memo

Mathematical tools to be exploited

Mathematics

Outline

- Radiation problem for extended antennas
- Field regions

Extended antennas

$\downarrow J(r)$

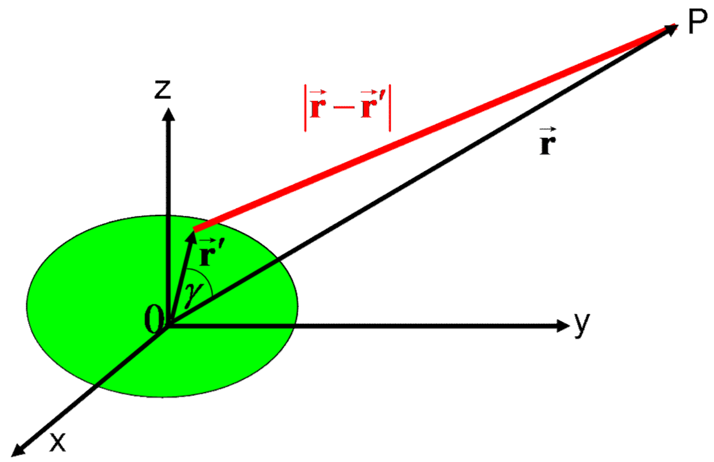
$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ Vector Helmholtz equation

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'$$

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla\nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$$

$\downarrow E(r)$

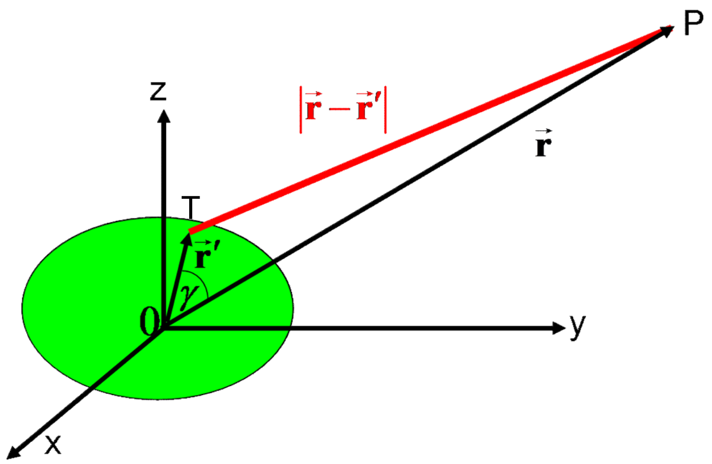
$H(r)$



Is it possible to simplify the expressions of the fields, possibly via proper approximation of the vector potential A?

Extended antennas

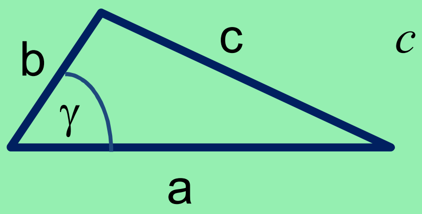
$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} \int \mathbf{J}(\vec{\mathbf{r}}') \frac{e^{-j\beta|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\,\vec{\mathbf{r}}'$$



$$|\vec{\mathbf{r}}| = r \qquad |\vec{\mathbf{r}}'| = r'$$

$$|\vec{\mathbf{r}}-\vec{\mathbf{r}}'| = \sqrt{r^2 + (r')^2 - 2rr'\cos\gamma} = \sqrt{r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\gamma\right]}$$

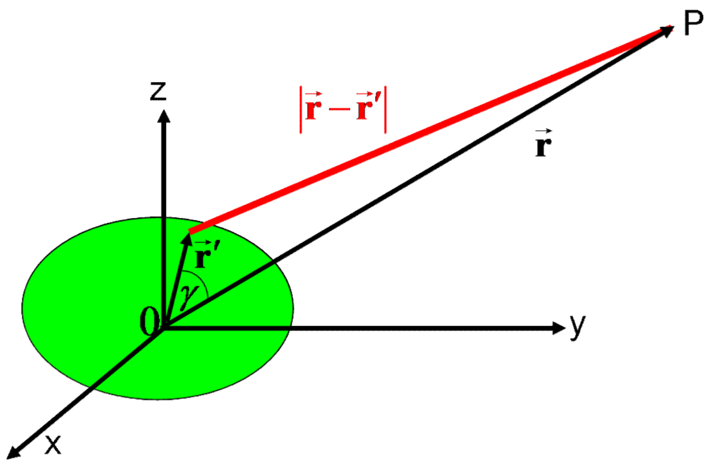
$$= r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\gamma}$$



$$c = \sqrt{a^2 + b^2 - 2ab\cos\gamma}$$

Extended antennas

$$\mathbf{A}(\vec{\mathbf{r}})=\frac{\mu}{4\pi}\int \mathbf{J}(\vec{\mathbf{r}}')\frac{e^{-j\beta|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\,\vec{\mathbf{r}}'$$



$$|\vec{\mathbf{r}}|=r \qquad |\vec{\mathbf{r}}'|=r'$$

$$|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|=\sqrt{r^2+(r')^2-2rr'\cos\gamma}=\sqrt{r^2\left[1+\left(\frac{r'}{r}\right)^2-2\frac{r'}{r}\cos\gamma\right]}$$

$$=r\sqrt{1+\left(\frac{r'}{r}\right)^2-2\frac{r'}{r}\cos\gamma}$$

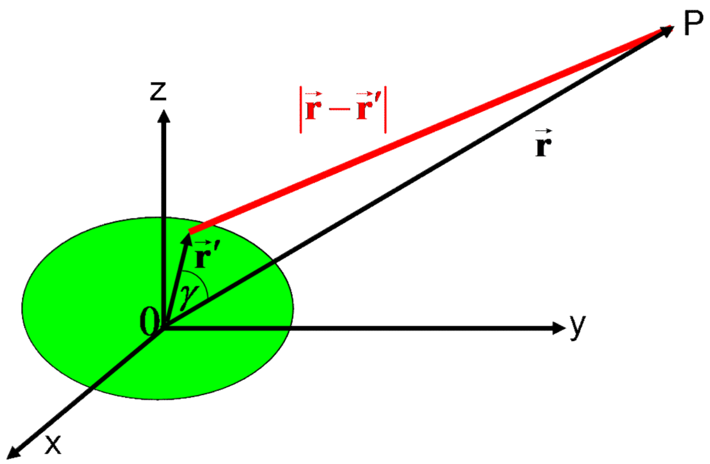
$$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$$

$$\sqrt{1+\boxed{\left(\frac{r'}{r}\right)^2-2\frac{r'}{r}\cos\gamma}}=1+\frac{1}{2}\left[\boxed{\left(\frac{r'}{r}\right)^2-2\frac{r'}{r}\cos\gamma}\right]-\frac{1}{8}\left[\boxed{\left(\frac{r'}{r}\right)^2-2\frac{r'}{r}\cos\gamma}\right]^2+\cdots=1+\frac{1}{2}\left(\frac{r'}{r}\right)^2-\frac{r'}{r}\cos\gamma-\frac{1}{8}\left[\cancel{\left(\frac{r'}{r}\right)^4}+4\left(\frac{r'}{r}\right)^2\cos^2\gamma-\cancel{4\left(\frac{r'}{r}\right)^3\cos\gamma}\right]+\cdots$$

$$=1+\frac{1}{2}\left(\frac{r'}{r}\right)^2-\frac{r'}{r}\cos\gamma-\frac{1}{2}\left(\frac{r'}{r}\right)^2\cos^2\gamma+\cdots=1-\frac{r'}{r}\cos\gamma+\frac{1}{2}\left(\frac{r'}{r}\right)^2\left(1-\cos^2\gamma\right)+\cdots=1-\frac{r'}{r}\cos\gamma+\frac{1}{2}\left(\frac{r'}{r}\right)^2\sin^2\gamma+\cdots$$

Extended antennas

$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi}\int \mathbf{J}(\vec{\mathbf{r}}')\frac{e^{-j\beta|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\vec{\mathbf{r}}' \qquad |\vec{\mathbf{r}}-\vec{\mathbf{r}}'| = r - r'\cos\gamma + \frac{(r')^2}{2r}\sin^2\gamma + \cdots$$



$$|\vec{\mathbf{r}}| = r \qquad |\vec{\mathbf{r}}'| = r'$$

$$|\vec{\mathbf{r}}-\vec{\mathbf{r}}'| = \sqrt{r^2 + (r')^2 - 2rr'\cos\gamma} = \sqrt{r^2\left[1+\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\gamma\right]}$$

$$= r\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\gamma} = r - r'\cos\gamma + \frac{(r')^2}{2r}\sin^2\gamma + \cdots$$

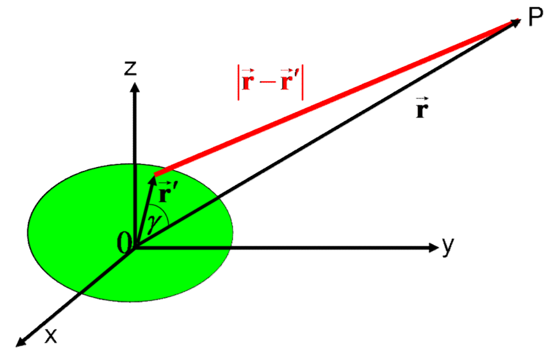
$$\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\gamma}$$

$$\boxed{= 1 - \frac{r'}{r}\cos\gamma + \frac{1}{2}\left(\frac{r'}{r}\right)^2\sin^2\gamma + \cdots}$$

Extended antennas

$$\boxed{\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi}\int \mathbf{J}(\vec{\mathbf{r}}')\frac{e^{-j\beta|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\,\vec{\mathbf{r}}'}$$

$$|\vec{\mathbf{r}}-\vec{\mathbf{r}}'| = r - r'\cos\gamma + \frac{(r')^2}{2r}\sin^2\gamma + \cdots$$



$$\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} =$$

$$e^{-j\beta|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} =$$

Extended antennas

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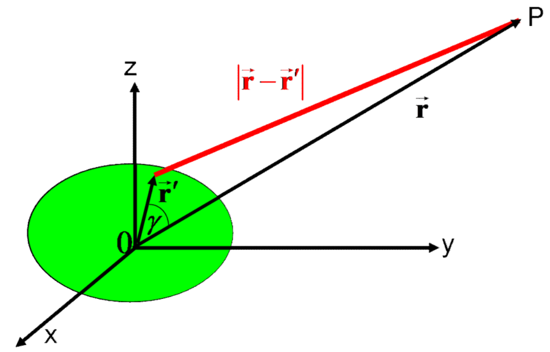
$$|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|=r-r'\cos\gamma+\frac{(r')^2}{2r}\sin^2\gamma+\cdots$$



$$\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}=\frac{1}{r-r'\cancel{\cos\gamma+\frac{(r')^2}{2r}\sin^2\gamma+\cdots}}\approx\frac{1}{r}$$

$$e^{-j\beta|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}=e^{-j\beta r}e^{j\beta r'\cos\gamma}e^{-j\beta\frac{(r')^2}{2r}\sin^2\gamma}\cdots$$

Extended antennas

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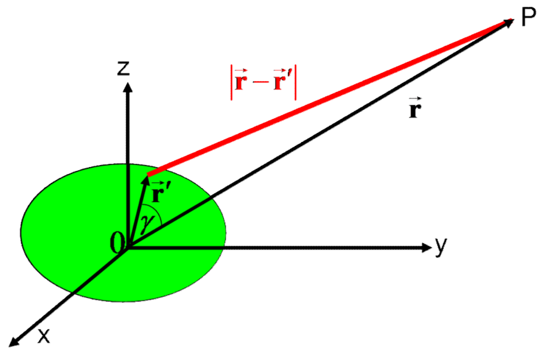
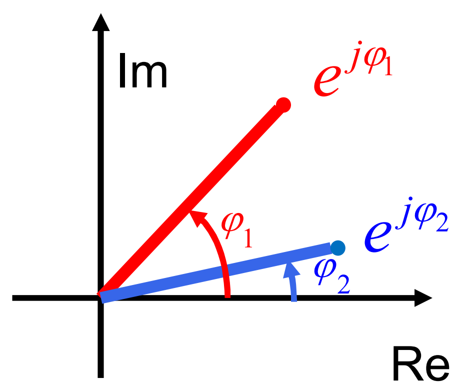


$$\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = \frac{1}{r - \cancel{r'\cos\gamma} + \cancel{\frac{(r')^2}{2r}\sin^2\gamma} + \cdots} \approx \frac{1}{r} \quad \textit{if} \quad r' << r$$

$$e^{-j\beta|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = e^{-j\beta r}\cancel{e^{j\beta r'\cos\gamma}}\,\cancel{e^{-j\beta\frac{(r')^2}{2r}\sin^2\gamma}}\cdots \approx e^{-j\beta r} \quad \textit{if} \quad r' << \lambda$$



$$e^{j\beta r'\cos\gamma} \approx 1 \quad \Rightarrow \quad \frac{2\pi}{\lambda}r' << 2\pi \quad \Longrightarrow \quad r' << \lambda$$

Extended antennas

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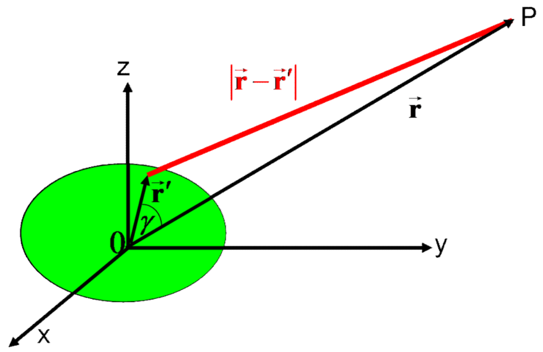

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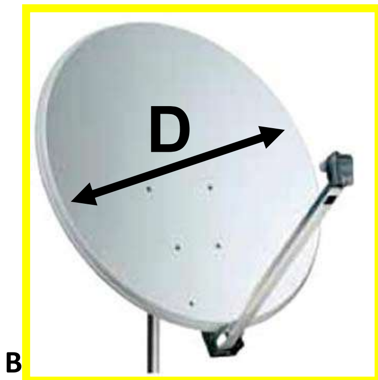

Extended antennas

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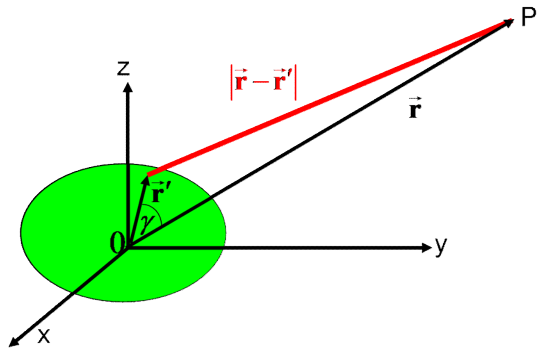

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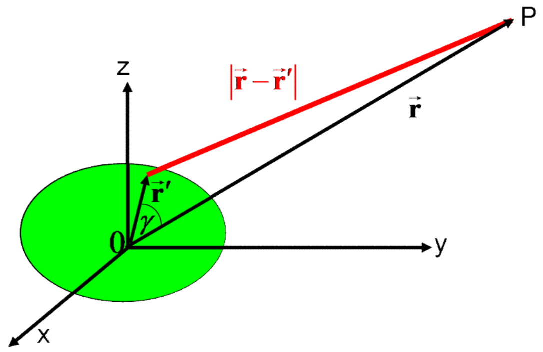

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When the antennas are **small** with respect to the wavelength **and** to the distance from the observation point

$$\frac{e^{-j\beta|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} \approx \frac{e^{-j\beta r}}{r} \quad \Longrightarrow \quad \mathbf{A}(\vec{\mathbf{r}}) \approx \frac{\mu}{4\pi}\frac{e^{-j\beta r}}{r}\int \mathbf{J}(\vec{\mathbf{r}}')d\vec{\mathbf{r}}'$$

Extended antennas

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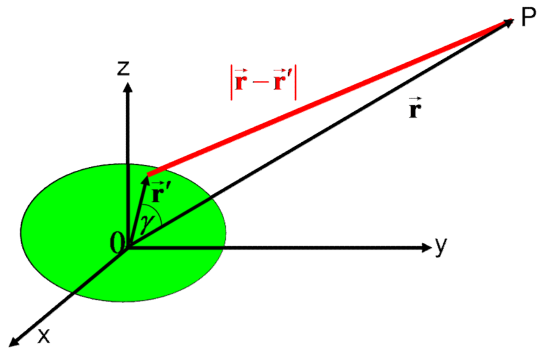

$$\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = \frac{1}{r - \cancel{r'\cos\gamma + \frac{(r')^2}{2r}\sin^2\gamma + \cdots}} \approx \frac{1}{r} \quad \text{if} \quad D << r$$

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Extended antennas

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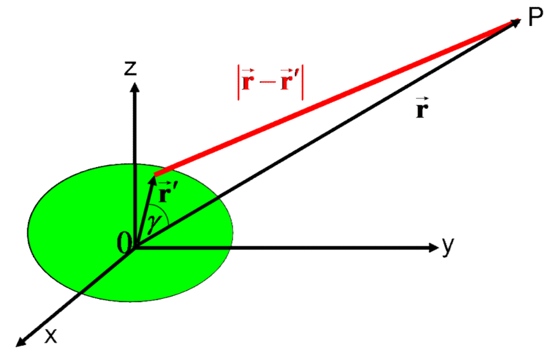

$$\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = \frac{1}{r - r'\cancel{\cos\gamma + \frac{(r')^2}{2r}\sin^2\gamma + \cdots}} \approx \frac{1}{r} \quad \textit{if} \quad D << r$$

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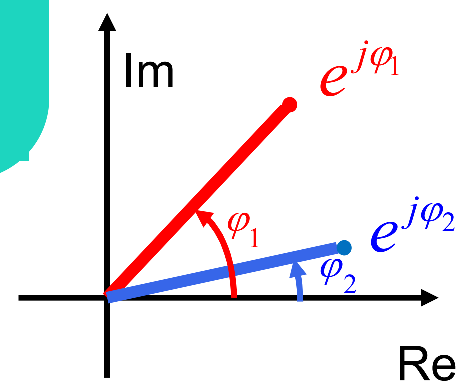

$$e^{-j\beta\frac{(r')^2}{2r}\sin^2\gamma} \approx 1 \;\Rightarrow\; \beta\frac{(r')^2}{2r}\sin^2\gamma << 2\pi \;\Rightarrow\; \frac{2\pi}{\lambda}\left(\frac{D}{2}\right)^2\frac{1}{2r} < \frac{\pi}{8} \;\Rightarrow\; r > \frac{2D^2}{\lambda}$$

Extended antennas

$$\boxed{\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi}\int \mathbf{J}(\vec{\mathbf{r}}')\frac{e^{-j\beta|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}d\vec{\mathbf{r}}'}$$

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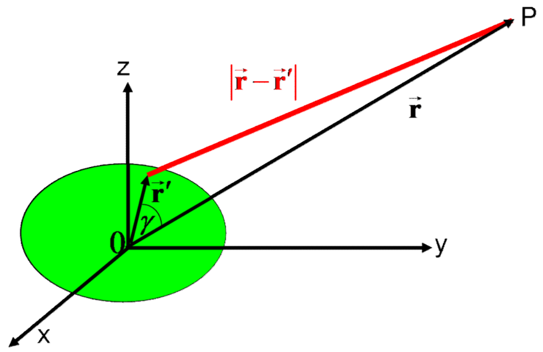

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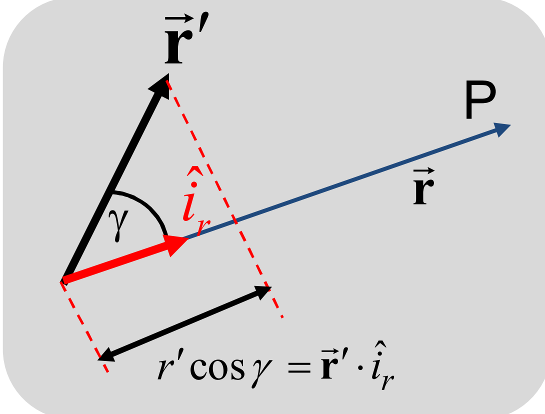

$$e^{-j\beta\frac{(r')^2}{2r}\sin^2\gamma} \approx 1 \Rightarrow \beta\frac{(r')^2}{2r}\sin^2\gamma << 2\pi \Rightarrow \frac{2\pi}{\lambda}\left(\frac{D}{2}\right)^2\frac{1}{2r} < \frac{\pi}{8} \Rightarrow r > \frac{2D^2}{\lambda}$$

Extended antennas

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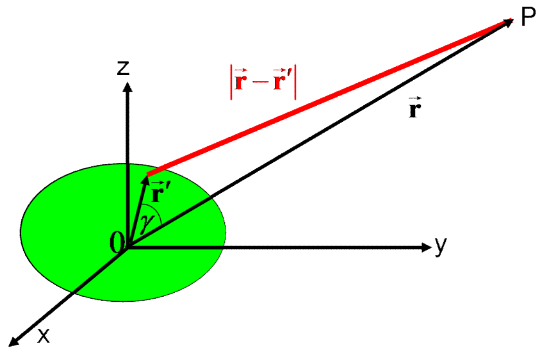
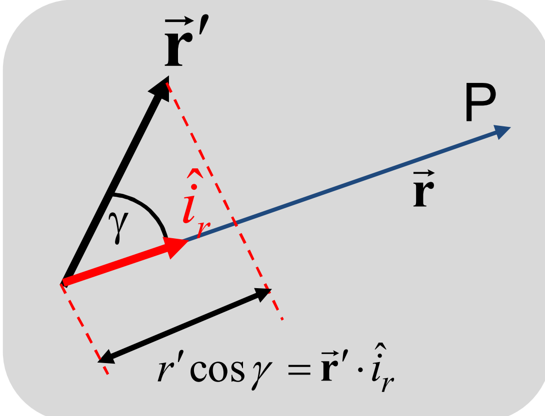


$$\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}=\frac{1}{r-\cancel{r'\cos\gamma}+\cancel{\frac{(r')^2}{2r}\sin^2\gamma}+\cdots}\approx\frac{1}{r}\quad \textit{if}\quad D<<r$$

$$e^{-j\beta|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}=e^{-j\beta r}e^{j\beta r'\cos\gamma}\cancel{e^{-j\beta\frac{(r')^2}{2r}\sin^2\gamma}}\cdots\approx e^{-j\beta r}e^{j\beta\vec{\mathbf{r}}'\cdot\hat{i}_r}\quad \textit{if}\quad r>\frac{2D^2}{\lambda}$$



$$e^{-j\beta\frac{(r')^2}{2r}\sin^2\gamma}\approx 1 \;\Rightarrow\; \beta\frac{(r')^2}{2r}\sin^2\gamma<<2\pi \;\Rightarrow\; \frac{2\pi}{\lambda}\left(\frac{D}{2}\right)^2\frac{1}{2r}<\frac{\pi}{8} \;\Rightarrow\; r>\frac{2D^2}{\lambda}$$