Campi Elettromagnetici

Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

a.a. 2020-2021 - Laurea “Triennale” – Secondo semestre - Secondo anno

Università degli Studi di Napoli “Parthenope”

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Color legend

New formulas, important considerations,
important formulas, important concepts

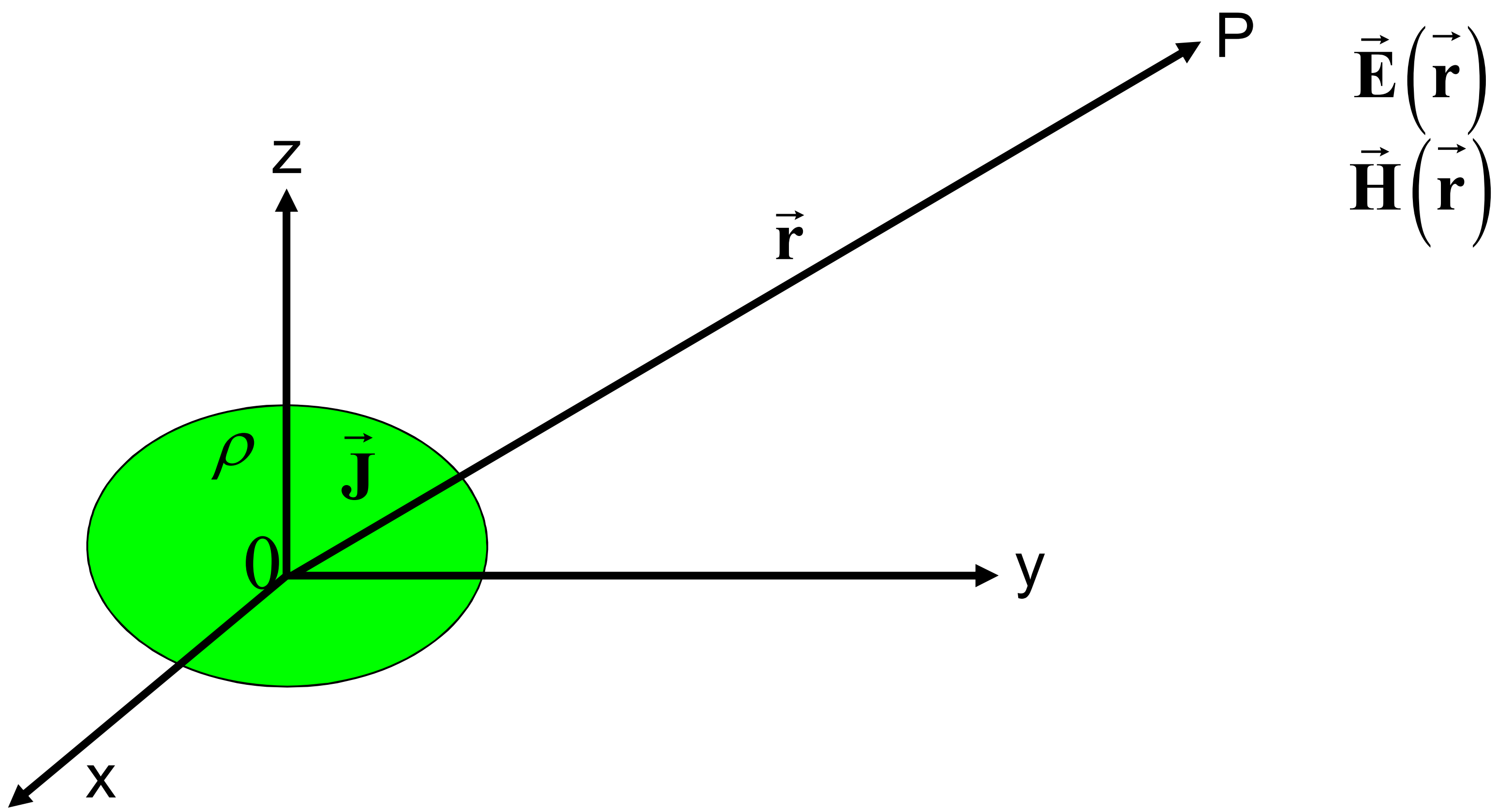
Very important for the discussion

Memo

Mathematical tools to be exploited

Mathematics

Radiation problem



Potentials

$\downarrow$ $\boldsymbol{J}(\boldsymbol{r})$

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ **Vector Helmholtz equation**

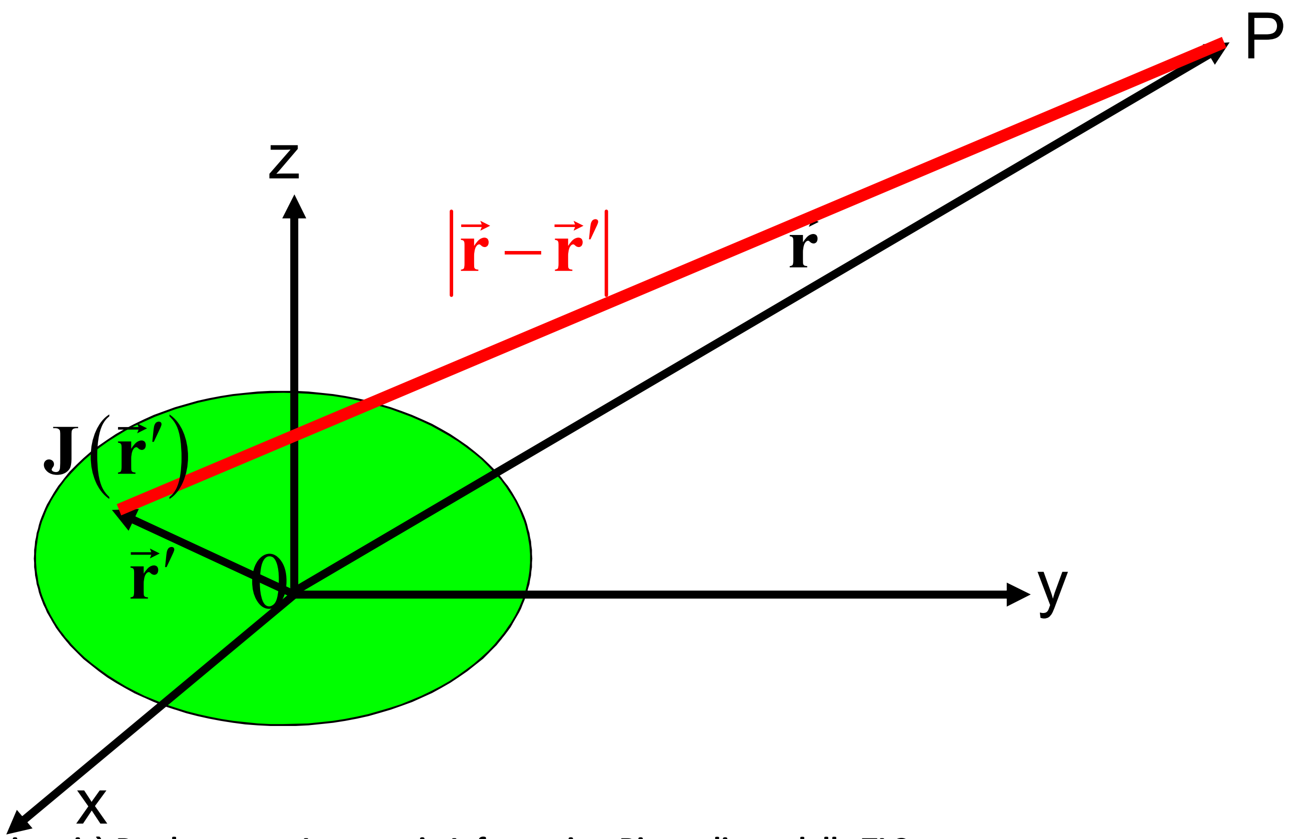
$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'$$

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla\nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$$

$\downarrow$ $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{H}(\boldsymbol{r})$

Potentials

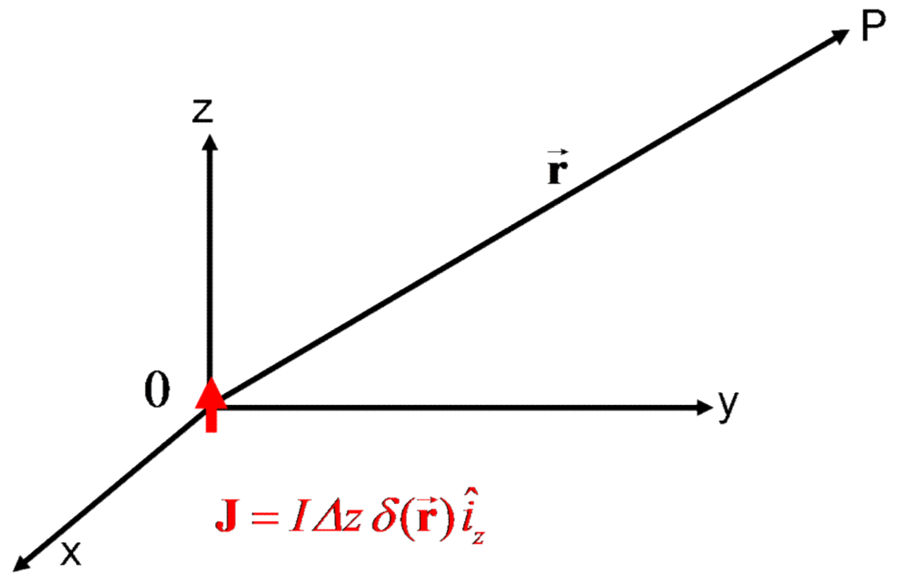
$$\mathbf{A}(\vec{\mathbf{r}}) = \frac{\mu}{4\pi} \int \mathbf{J}(\vec{\mathbf{r}}') \frac{e^{-jk|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} d\vec{\mathbf{r}}'$$



Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$



- Why are we interested in such a radiating element?
- Why is such a radiating element referred to as elementary electrical dipole?
- How can we physically approximate an elementary electrical dipole?

Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

$\boldsymbol{J}$ ↓

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ **Vector Helmholtz equation**

↓ $\boldsymbol{A}$

$$\begin{cases} \nabla^2 A_x + k^2 A_x = 0 & \Rightarrow A_x = 0 \\ \nabla^2 A_y + k^2 A_y = 0 & \Rightarrow A_y = 0 \\ \nabla^2 A_z + k^2 A_z = -\mu J_z = -\mu I \Delta z\,\delta(x)\delta(y)\delta(z) = -\mu I \Delta z\,\delta(\vec{\mathbf{r}}) \end{cases}$$

Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\,\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\,\hat{i}_z$$

↓ J

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ Vector Helmholtz equation

↓ $$\mathbf{A} = \frac{\mu}{4\pi} I\Delta z \frac{e^{-jkr}}{r}\hat{i}_z$$

$$\boldsymbol{H} = \frac{1}{\mu}\nabla\times\boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\varepsilon\mu}$$

↓ E H

Elementary electrical dipole

- A δ-source radiating element is also known as elementary electrical dipole.

$$\mathbf{J} = I\Delta z\,\delta(\vec{\mathbf{r}})\hat{i}_z = I\Delta z\,\delta(x)\delta(y)\delta(z)\hat{i}_z$$

The E.M. field radiated by the elementary electrical dipole

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_r(r,\vartheta)\hat{i}_r + \mathrm{E}_\vartheta(r,\vartheta)\hat{i}_\vartheta$$
$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_\varphi(r,\vartheta)\hat{i}_\varphi$$

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

Because of the problem symmetry there is no dependence on the azimuth angle φ.

Elementary electrical dipole: static case

The E.M. field radiated by the elementary electrical dipole

$$\begin{cases} E_r = \zeta \dfrac{jQ\Delta z}{2\pi}\left(\cancel{\dfrac{\omega}{r^2}} + \dfrac{\omega}{j\beta r^3}\right)\cos\vartheta \boxed{\exp(-j\beta r)} \\ E_\vartheta = \zeta \dfrac{jQ\Delta z}{4\pi}\left(\cancel{\dfrac{j\omega\beta}{r}} + \cancel{\dfrac{\omega}{r^2}} + \dfrac{\omega}{j\beta r^3}\right)\sin\vartheta \boxed{\exp(-j\beta r)} \\ H_\varphi = \dfrac{jQ\Delta z}{4\pi}\left(\cancel{\dfrac{j\omega\beta}{r}} + \cancel{\dfrac{\omega}{r^2}}\right)\sin\vartheta \boxed{\exp(-j\beta r)} \end{cases}$$

$\exp(-j\beta r) \rightarrow 1$

$$j\omega Q = I$$

… for ω=0 simplifies as

$$\begin{cases} E_r = \dfrac{\zeta}{\sqrt{\mu\varepsilon}} \dfrac{Q\Delta z}{2\pi} \dfrac{1}{r^3}\cos\vartheta = \dfrac{Q\Delta z}{2\pi} \dfrac{1}{\varepsilon r^3}\cos\vartheta \\ E_\vartheta = \dfrac{\zeta}{\sqrt{\mu\varepsilon}} \dfrac{Q\Delta z}{4\pi} \dfrac{1}{r^3}\sin\vartheta = \dfrac{Q\Delta z}{4\pi} \dfrac{1}{\varepsilon r^3}\sin\vartheta \\ H_\varphi = 0 \end{cases}$$

$$\zeta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\Rightarrow \quad \frac{\zeta}{\sqrt{\mu\varepsilon}} = \frac{1}{\varepsilon}$$

Elementary electrical dipole: far field

The E.M. field radiated by the elementary electrical dipole

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

… for r>>λ simplifies as

$$\begin{cases} E_r = 0 \\ E_\vartheta = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{E_\vartheta}{\zeta} \end{cases}$$

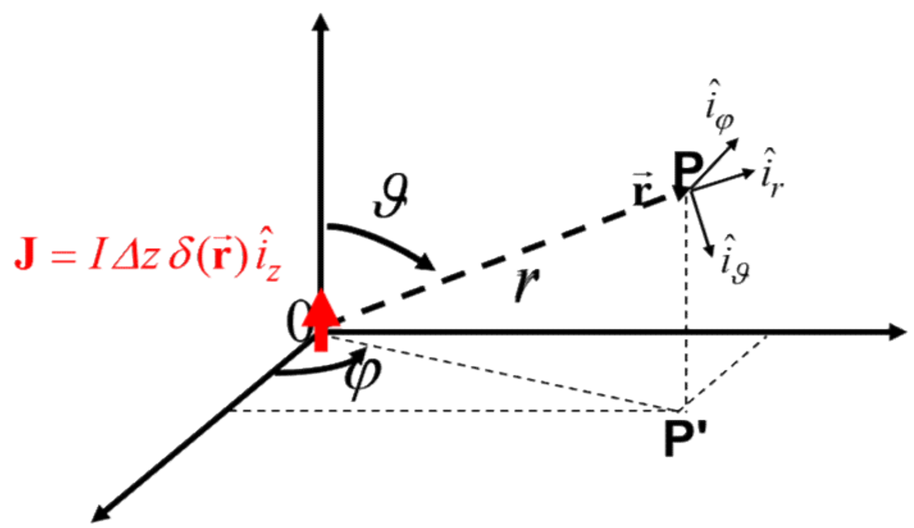
Elementary electrical dipole: far field

In the far-field case (r>>λ) the elementary electrical dipole behaves as follows

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_{\vartheta}(r,\vartheta)\hat{i}_{\vartheta}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_{\varphi}(r,\vartheta)\hat{i}_{\varphi}$$

$$\begin{cases} E_{\vartheta} = j\zeta \dfrac{I\Delta z}{2\lambda r}\sin\vartheta \exp(-j\beta r) \\ H_{\varphi} = \dfrac{E_{\vartheta}}{\zeta} \end{cases}$$



- the e.m. field propagates along $\hat{i}_r$
- the e.m. field lies on the plane orthogonal to the propagation direction
- |E| and |H| exhibit the decaying factor 1/r
- |E| and |H| are proportional through ζ

$$\zeta \mathbf{H} = \hat{i}_r \times \mathbf{E}$$

$$\vec{\mathbf{S}} = \frac{1}{2\zeta}\left|\vec{\mathbf{E}}\right|^2 \hat{i}_r = \frac{\zeta}{2}\left|\vec{\mathbf{H}}\right|^2 \hat{i}_r$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

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$$P = \frac{1}{2}\oiint_S \left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{i}_r dS = \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta r^2 \sin\vartheta E_\vartheta H_\varphi^*$$

$$dS = r^2 \sin\vartheta d\vartheta d\varphi$$

$$\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{i}_r = \left[\left(E_\vartheta \hat{i}_\vartheta + E_r\hat{i}_r\right)\times\left(H_\varphi^*\hat{i}_\varphi\right)\right]\cdot\hat{i}_r = E_\vartheta H_\varphi^* \hat{i}_r\cdot\hat{i}_r = E_\vartheta H_\varphi^*$$

$$\hat{i}_\varphi = \hat{i}_r \times \hat{i}_\vartheta$$

$$\hat{i}_\vartheta = \hat{i}_\varphi \times \hat{i}_r$$

$$\hat{i}_r = \hat{i}_\vartheta \times \hat{i}_\varphi$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

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$$= \frac{1}{2}\left[\zeta \frac{|I|^2 \Delta z^2}{(4\pi)^2}\right]\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta r^2 \sin\vartheta \left[\sin^2\vartheta\right]\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)^*$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \frac{I\Delta z}{2\pi}\left(\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \frac{I\Delta z}{4\pi}\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \frac{I\Delta z}{4\pi}\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

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$$= \frac{1}{2}\left[\zeta \frac{|I|^2 \Delta z^2}{(4\pi)^2}\right] \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta r^2 \sin\vartheta \left[\sin^2\vartheta\right]\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)^*$$

$$= \frac{1}{2}\left[\zeta \frac{|I|^2 \Delta z^2}{(4\pi)^2}\right] r^2 \left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)^* \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^3\vartheta$$

$$\int_0^{2\pi} d\varphi = 2\pi$$

$$\int_0^{\pi} d\vartheta \sin^3\vartheta = \frac{4}{3}$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

$$P=\frac{1}{2}\oiint_S \left[\mathbf{E}\times\mathbf{H}^*\right]\cdot \hat{i}_r dS = \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta r^2 \sin\vartheta E_\vartheta H_\varphi^* = \frac{1}{2}\left[\zeta \frac{|I|^2 \Delta z^2}{(4\pi)^2}\right] r^2 \left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)^* 2\pi\frac{4}{3}$$

$$=\frac{1}{2}\left[\zeta \frac{|I|^2 \Delta z^2}{(4\pi)^2}\right] r^2 \frac{\beta^2}{r^2}\left[1-j\frac{1}{(\beta r)^3}\right] 2\pi\frac{4}{3} = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2\left[1-j\frac{1}{(\beta r)^3}\right]|I|^2$$

$$\left(\frac{j\beta}{r}+\frac{1}{r^2}+\frac{1}{j\beta r^3}\right)\left(\frac{-j\beta}{r}+\frac{1}{r^2}\right) = \left(\frac{\beta}{r}\right)^2 + \cancel{\frac{j\beta}{r^3}} - \cancel{\frac{j\beta}{r^3}} + \cancel{\frac{1}{r^4}} - \cancel{\frac{1}{r^4}} - j\frac{1}{\beta r^5} = \left(\frac{\beta}{r}\right)^2 - j\frac{1}{\beta r^5} = \frac{\beta^2}{r^2}\left[1-j\frac{1}{(\beta r)^3}\right]$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}+\dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r}+\dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

$$P = P_1 + jP_2$$

$$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 |I|^2$$

$$P_2 = -\frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \frac{1}{(\beta r)^3}|I|^2$$

$$P = \frac{1}{2}\oiint_S \left[\mathbf{E}\times\mathbf{H}^*\right]\cdot \hat{i}_r dS$$

$$= \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \left[1 - j\frac{1}{(\beta r)^3}\right]|I|^2$$

Elementary electrical dipole: power flux

$$\begin{cases} E_r = \zeta \dfrac{I\Delta z}{2\pi}\left(\dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\cos\vartheta \exp(-j\beta r) \\ E_\vartheta = \zeta \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2} + \dfrac{1}{j\beta r^3}\right)\sin\vartheta \exp(-j\beta r) \\ H_\varphi = \dfrac{I\Delta z}{4\pi}\left(\dfrac{j\beta}{r} + \dfrac{1}{r^2}\right)\sin\vartheta \exp(-j\beta r) \end{cases}$$

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- Note that in the far-field case only the first active power term exists and it does not depend on r
- Note that the real part of the power, in lossless medium, is independent of *r*, therefore if one consider two different spherical surfaces one gets the same result. Only the so-called radiative terms contribute.
- The reactive part depends on *r*. Its sign is negative showing that there is an excess of stored **<u>electric</u>** energy in the neighbor of the electrical dipole (see Poynting's theorem)

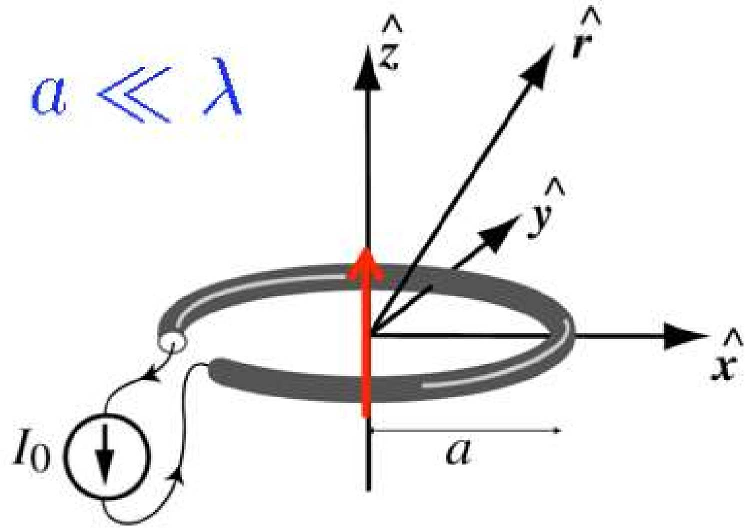
$$\int_0^{\pi} d\vartheta \sin^3\vartheta = \frac{4}{3}$$

$$\int_0^{\pi} d\vartheta \sin^3\vartheta = \int_0^{\pi} d\vartheta \sin\vartheta\left(1-\cos^2\vartheta\right) = \int_{-1}^{1} dx\left(1-x^2\right) = \left[x-\frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3}$$

$$x = \cos\vartheta$$
$$dx = -d\vartheta\sin\vartheta$$

Small loop antenna

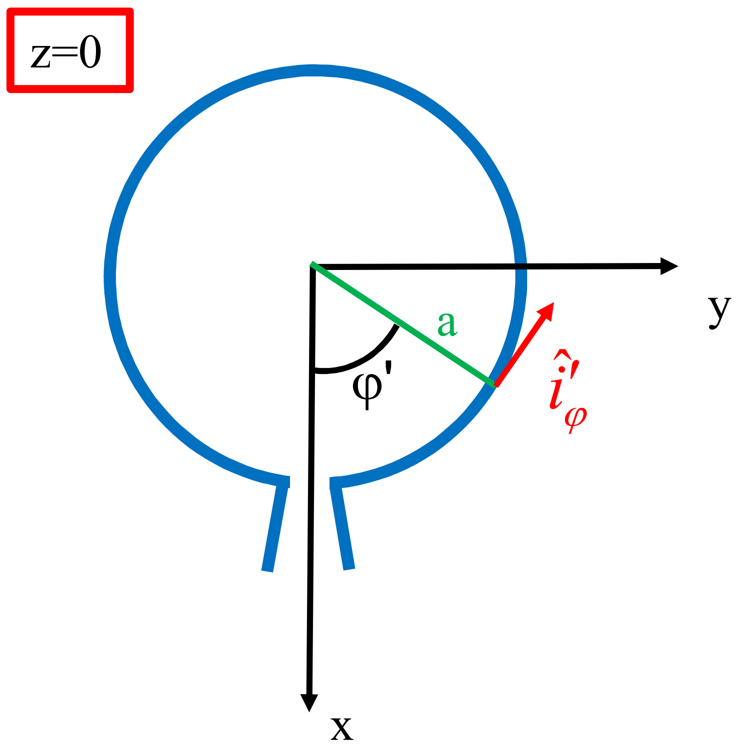
A simple and inexpensive antenna type is the loop antenna



$$\Delta S = \pi a^2$$

Small loop antenna

$$\mathbf{J} = I\delta(z)\delta(r-a)\hat{i}'_{\varphi}$$



↓ J

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ Vector Helmholtz equation

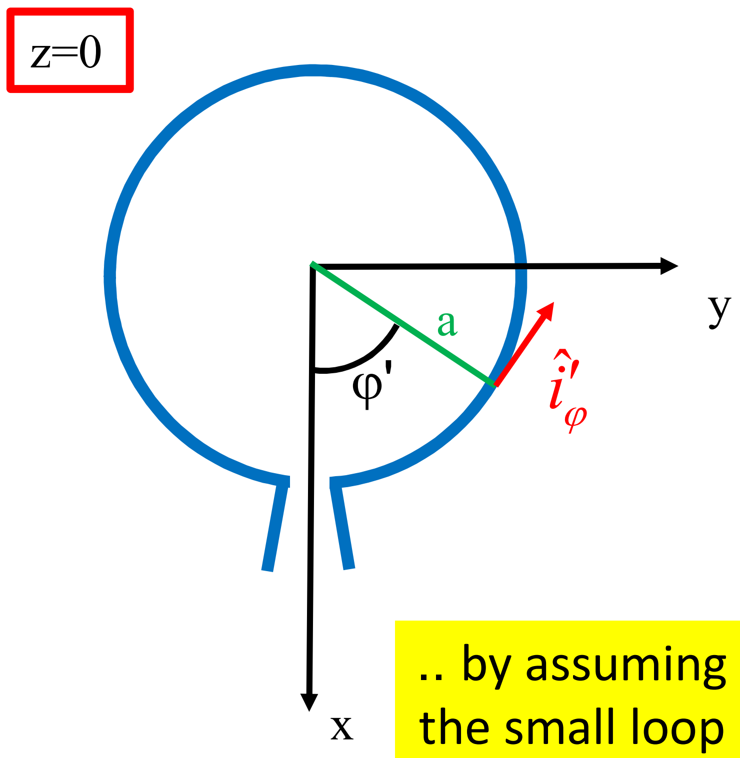
↓ $$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int \mathbf{J}(\mathbf{r}')\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla\nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$$

↓ $E(r)$
$H(r)$

Small loop antenna

$$\mathbf{J} = I\delta(z)\,\delta(r-a)\,\hat{i}'_\varphi$$



.. by assuming that the current I in the small loop is constant and that the radius of the loop a<<λ

J ↓

$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ Vector Helmholtz equation

↓ $\mathbf{A} \approx \dfrac{j\beta I \mu \Delta S}{4\pi} \dfrac{e^{-j\beta r}}{r} \left[1 + \dfrac{1}{j\beta r}\right] \sin\vartheta\, \hat{i}_\varphi$

$$\boldsymbol{H} = \frac{1}{\mu} \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla\nabla \cdot \boldsymbol{A}}{j\omega\varepsilon\mu}$$

↓ E H

Small loop antenna

The E.M. field radiated by the the small loop antenna

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_r(r,\vartheta)\hat{i}_r + \mathrm{H}_\vartheta(r,\vartheta)\hat{i}_\vartheta$$

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_\varphi(r,\vartheta)\hat{i}_\varphi$$

$$\begin{cases} H_r = \dfrac{I\Delta S}{2\pi}\dfrac{(j\beta)^2}{r}\left(\dfrac{1}{j\beta r} + \dfrac{1}{[j\beta r]^2}\right)\cos\vartheta\exp(-j\beta r) \\ H_\vartheta = \dfrac{I\Delta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1 + \dfrac{1}{j\beta r} + \dfrac{1}{[j\beta r]^2}\right)\sin\vartheta\exp(-j\beta r) \\ E_\varphi = -\dfrac{\zeta I\Delta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1 + \dfrac{1}{j\beta r}\right)\sin\vartheta\exp(-j\beta r) \end{cases}$$

Because of the problem symmetry there is no dependence on the azimuth angle φ.

Small loop antenna: far field

The E.M. field radiated by the the small loop antenna

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_r(r,\vartheta)\hat{i}_r + \mathrm{H}_\vartheta(r,\vartheta)\hat{i}_\vartheta$$

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… for r>>λ ($\beta r>>1$) simplifies as

$$\begin{cases} H_r = 0 \\ H_\vartheta = -\dfrac{\beta\Delta sI}{2\lambda r}\sin\vartheta\exp(-j\beta r) = \dfrac{-E_\varphi}{\zeta} \\ E_\varphi = \dfrac{\zeta\beta\Delta sI}{2\lambda r}\sin\vartheta\exp(-j\beta r) \end{cases}$$

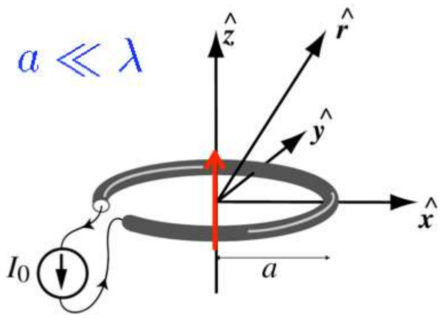
Small loop antenna: far field

In the far-field case (r>>λ) the small loop antenna behaves as follows

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_{\varphi}(r,\vartheta)\hat{i}_{\varphi}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_{\vartheta}(r,\vartheta)\hat{i}_{\vartheta}$$

$$\begin{cases} E_{\varphi} = \dfrac{\zeta\beta\Delta sI}{2\lambda r}\sin\vartheta\exp(-j\beta r) \\ H_{\vartheta} = -\dfrac{\beta\Delta sI}{2\lambda r}\sin\vartheta\exp(-j\beta r) = \dfrac{-E_{\varphi}}{\zeta} \end{cases}$$



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$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_\varphi(r,\vartheta)\hat{i}_\varphi$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_\vartheta(r,\vartheta)\hat{i}_\vartheta$$

$$\begin{cases} E_\varphi = \dfrac{\zeta\beta\Delta sI}{2\lambda r}\sin\vartheta\exp(-j\beta r) \\ H_\vartheta = -\dfrac{E_\varphi}{\zeta} \end{cases}$$

$$\zeta\vec{\mathbf{H}} = \zeta\mathrm{H}_\vartheta\hat{i}_\vartheta = -E_\varphi\hat{i}_\vartheta$$

$$\hat{i}_r \times \vec{\mathbf{E}} = \hat{i}_r \times E_\varphi\hat{i}_\varphi = -E_\varphi\hat{i}_\vartheta$$

$$\Rightarrow \quad \zeta\mathbf{H} = \hat{i}_r \times \mathbf{E}$$

$$\hat{i}_\varphi = \hat{i}_r \times \hat{i}_\vartheta$$
$$\hat{i}_\vartheta = \hat{i}_\varphi \times \hat{i}_r$$
$$\hat{i}_r = \hat{i}_\vartheta \times \hat{i}_\varphi$$

- the e.m. field propagates along $\hat{i}_r$
- the e.m. field lies on the plane orthogonal to the propagation direction
- |E| and |H| exhibit the decaying factor 1/r
- |E| and |H| are proportional through ζ

$$\zeta\mathbf{H} = \hat{i}_r \times \mathbf{E}$$

Small loop antenna: far field

In the far-field case (r>>λ) the small loop antenna behaves as follows

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \mathrm{E}_\varphi(r,\vartheta)\hat{i}_\varphi$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}) = \mathrm{H}_\vartheta(r,\vartheta)\hat{i}_\vartheta$$

$$\begin{cases} E_\varphi = \dfrac{\zeta\beta\Delta sI}{2\lambda r}\sin\vartheta\exp(-j\beta r) \\ H_\vartheta = -\dfrac{E_\varphi}{\zeta} \end{cases}$$

$$\vec{\mathbf{S}} = \frac{1}{2}\vec{\mathbf{E}}\times\vec{\mathbf{H}}^* = \frac{1}{2}E_\varphi\hat{i}_\varphi\times\left(H_\vartheta\hat{i}_\vartheta\right)^* = -\frac{1}{2}E_\varphi H_\vartheta^*\hat{i}_r$$

$$\hat{i}_\varphi = \hat{i}_r\times\hat{i}_\vartheta$$

$$\hat{i}_\vartheta = \hat{i}_\varphi\times\hat{i}_r$$

$$\hat{i}_r = \hat{i}_\vartheta\times\hat{i}_\varphi$$

$$-\frac{1}{2}E_\varphi H_\vartheta^*\hat{i}_r = \frac{1}{2\zeta}E_\varphi E_\varphi^*\hat{i}_r = \frac{1}{2\zeta}\left|E_\varphi\right|^2\hat{i}_r = \frac{1}{2\zeta}\left|\vec{\mathbf{E}}\right|^2\hat{i}_r$$

$$-\frac{1}{2}E_\varphi H_\vartheta^*\hat{i}_r = \frac{1}{2}\zeta H_\vartheta H_\vartheta^*\hat{i}_r = \frac{\zeta}{2}\left|H_\vartheta\right|^2\hat{i}_r = \frac{\zeta}{2}\left|\vec{\mathbf{H}}\right|^2\hat{i}_r$$

- the e.m. field propagates along $\hat{i}_r$
- the e.m. field lies on the plane orthogonal to the propagation direction
- |E| and |H| exhibit the decaying factor 1/r
- |E| and |H| are proportional through ζ

$$\zeta\mathbf{H} = \hat{i}_r\times\mathbf{E}$$

$$\vec{\mathbf{S}} = \frac{1}{2\zeta}\left|\vec{\mathbf{E}}\right|^2\hat{i}_r = \frac{\zeta}{2}\left|\vec{\mathbf{H}}\right|^2\hat{i}_r$$

Small loop antenna: power flux

$$\begin{cases} H_r = \dfrac{I\Delta S}{2\pi}\dfrac{(j\beta)^2}{r}\left(\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\cos\vartheta\exp(-j\beta r) \\ H_\vartheta = \dfrac{I\Delta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1+\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\sin\vartheta\exp(-j\beta r) \\ E_\varphi = -\dfrac{\zeta I\Delta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1+\dfrac{1}{j\beta r}\right)\sin\vartheta\exp(-j\beta r) \end{cases}$$

$$P = \frac{1}{2}\oiint_S \left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{i}_r dS = -\frac{1}{2}\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta r^2 \sin\vartheta E_\varphi H_\vartheta^*$$

$$dS = r^2\sin\vartheta d\vartheta d\varphi$$

$$\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{i}_r = \left[\left(E_\varphi\hat{i}_\varphi\right)\times\left(H_\vartheta\hat{i}_\vartheta + H_r\hat{i}_r\right)^*\right]\cdot\hat{i}_r = -E_\varphi H_\vartheta^*\hat{i}_r\cdot\hat{i}_r = -E_\varphi H_\vartheta^*$$

$$\hat{i}_\varphi = \hat{i}_r\times\hat{i}_\vartheta$$
$$\hat{i}_\vartheta = \hat{i}_\varphi\times\hat{i}_r$$
$$\hat{i}_r = \hat{i}_\vartheta\times\hat{i}_\varphi$$

Small loop antenna: power flux

$$\begin{cases} H_r = \dfrac{I\Delta S}{2\pi}\dfrac{(j\beta)^2}{r}\left(\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\cos\vartheta\exp(-j\beta r) \\ H_\vartheta = \dfrac{I\Delta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1+\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\sin\vartheta\exp(-j\beta r) \\ E_\varphi = -\dfrac{\zeta I\Delta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1+\dfrac{1}{j\beta r}\right)\sin\vartheta\exp(-j\beta r) \end{cases}$$

$$P = P_1 + jP_2$$

$$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta\Delta S}{\lambda}\right)^2 |I|^2$$

$$P_2 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta\Delta S}{\lambda}\right)^2 \frac{1}{(\beta r)^3}|I|^2$$

$$P = \frac{1}{2}\oiint_S \left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{i}_r dS = -\frac{1}{2}\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta r^2 \sin\vartheta E_\varphi H_\vartheta^* = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta\Delta S}{\lambda}\right)^2\left[1 + j\frac{1}{(\beta r)^3}\right]|I|^2$$

Small loop antenna: power flux

$$\begin{cases} H_r = \dfrac{I\Delta S}{2\pi}\dfrac{(j\beta)^2}{r}\left(\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\cos\vartheta\exp(-j\beta r) \\ H_\vartheta = \dfrac{I\Delta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1+\dfrac{1}{j\beta r}+\dfrac{1}{[j\beta r]^2}\right)\sin\vartheta\exp(-j\beta r) \\ E_\varphi = -\dfrac{\zeta I\Delta S}{4\pi}\dfrac{(j\beta)^2}{r}\left(1+\dfrac{1}{j\beta r}\right)\sin\vartheta\exp(-j\beta r) \end{cases}$$

$$P = P_1 + jP_2$$

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- Note that in the far-field case only the first active power term exists and it does not depend on r
- Note that the real part of the power, in lossless medium, is independent of *r*, therefore if one consider two different spherical surfaces one gets the same result. Only the so-called radiative terms contribute.
- The reactive part depends on *r*. The reactive part depends on r. Its sign is positive showing that there is an excess of stored **magnetic** energy in the neighbor of the magnetic dipole (see Poynting's theorem)

Elementary electrical dipole vs. small loop antenna

Elementary electrical dipole

$$P = P_1 + jP_2$$

$$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 |I|^2$$

$$P_2 = -\frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\Delta z}{\lambda}\right)^2 \frac{1}{(\beta r)^3}|I|^2$$

Small loop antenna

$$P = P_1 + jP_2$$

$$P_1 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta \Delta S}{\lambda}\right)^2 |I|^2$$

$$P_2 = \frac{1}{2}\frac{2\pi}{3}\zeta\left(\frac{\beta \Delta S}{\lambda}\right)^2 \frac{1}{(\beta r)^3}|I|^2$$

Small loop antenna

WHY?