Campi Elettromagnetici

Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

a.a. 2020-2021 - Laurea “Triennale” – Secondo semestre - Secondo anno

Università degli Studi di Napoli “Parthenope”

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Color legend

New formulas, important considerations,
important formulas, important concepts

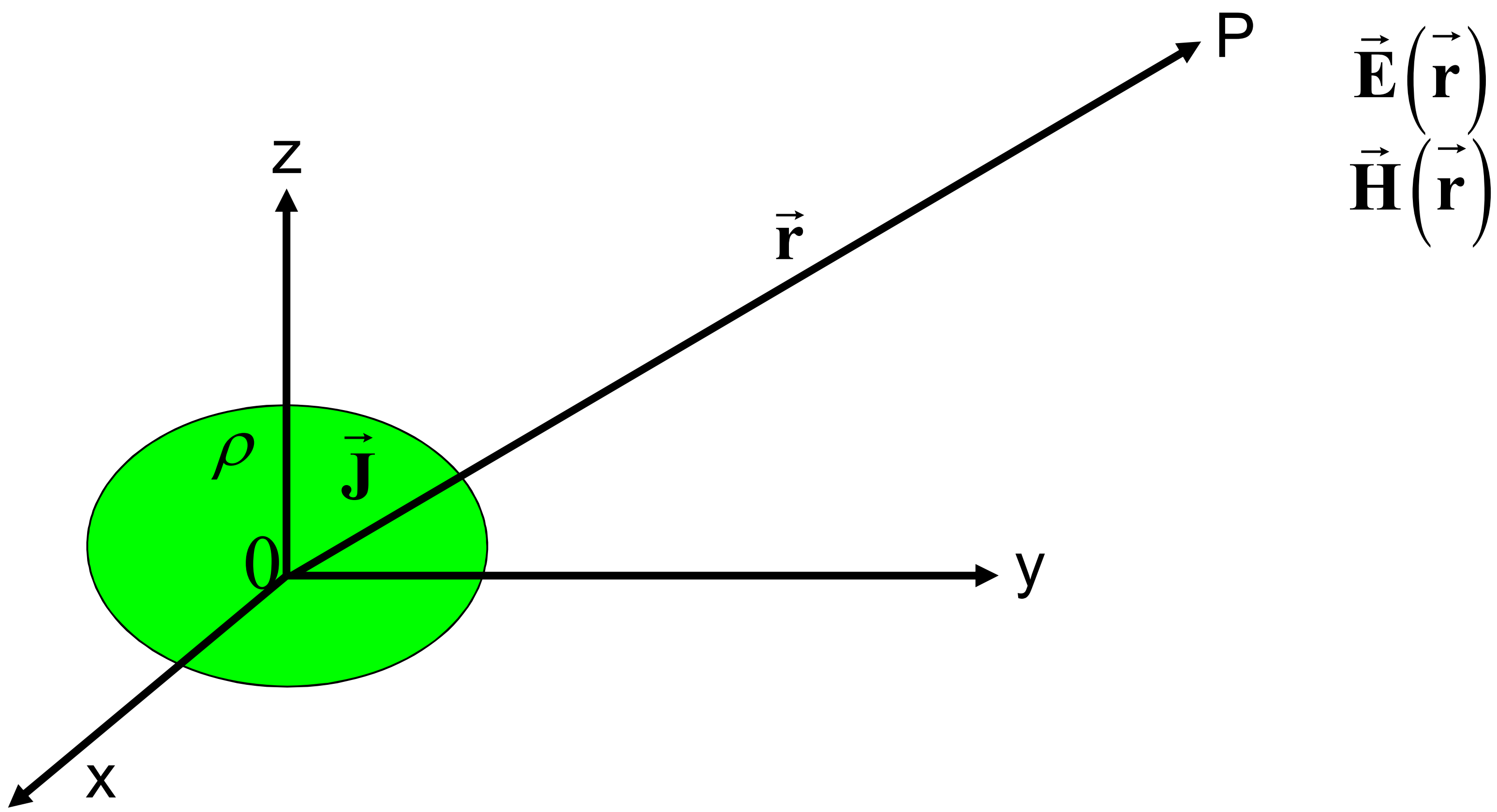
Very important for the discussion

Memo

Mathematical tools to be exploited

Mathematics

Radiation problem



Radiation problem

- We approach the radiation problem of an antenna assuming that the sources are known and the media in which propagation occurs is simple.
- Mathematically we exploit the potentials solution.

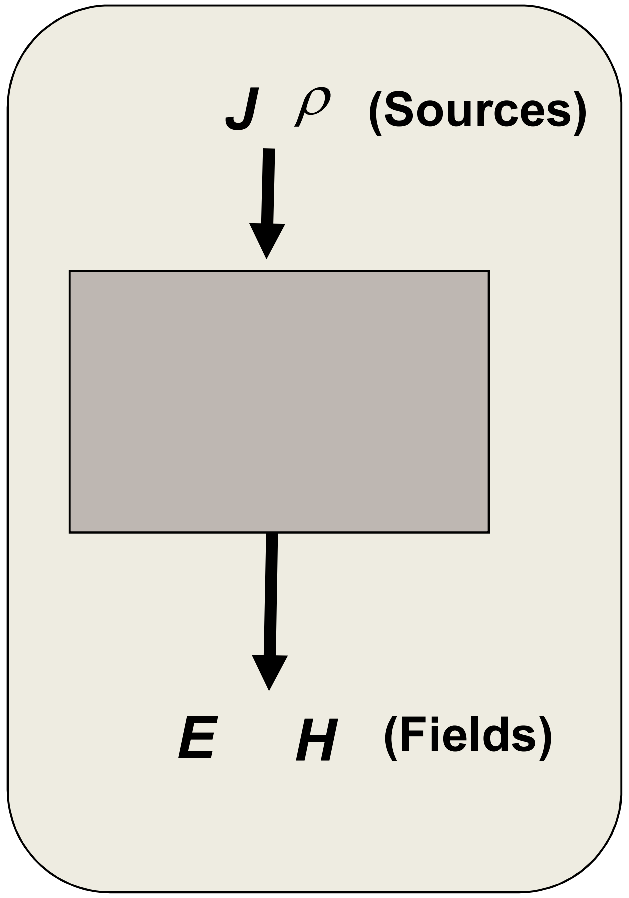
Medium: linear, isotropic, homogeneous (time-invariant and space-invariant), non-dispersive in space and time

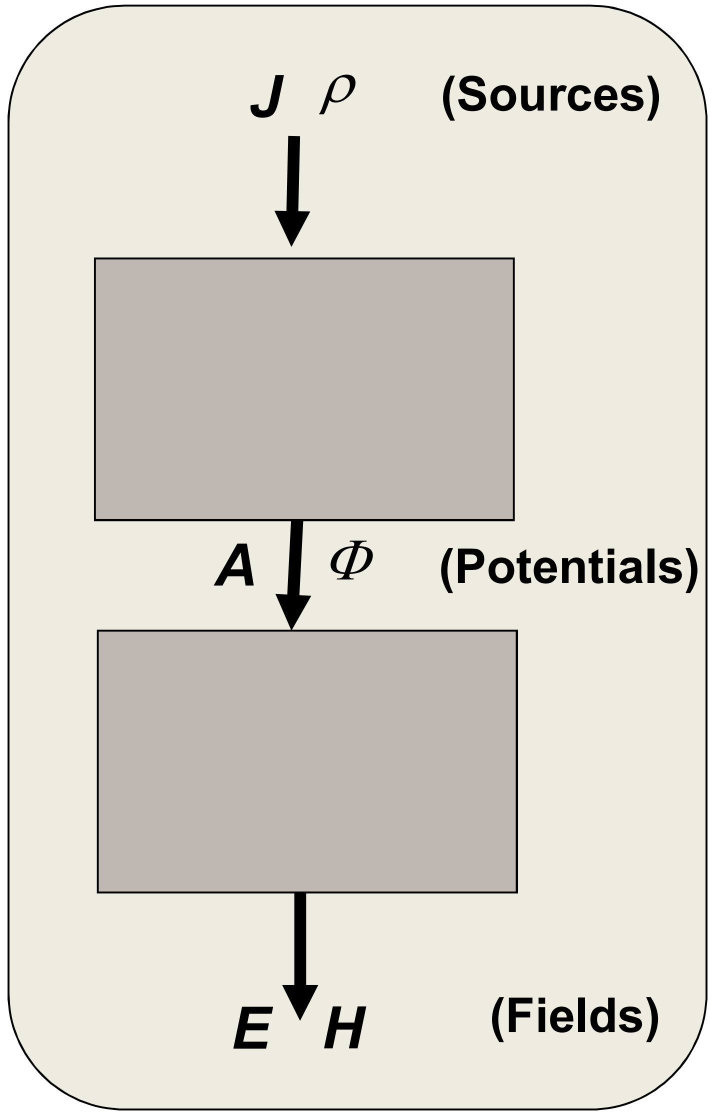
Phasor domain

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$

Radiation problem & potentials

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$





… mathematical tools that we will exploit today…

$$\nabla \times \mathbf{C} = \mathbf{0} \quad \Rightarrow \quad \exists\, \Phi \quad : \quad \mathbf{C} = \nabla \Phi$$

$$\nabla \cdot \mathbf{C} = 0 \quad \Rightarrow \quad \exists\, \mathbf{A} \quad : \quad \mathbf{C} = \nabla \times \mathbf{A}$$

Radiation problem & potentials

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon\boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon\boldsymbol{E} = \rho \\ \nabla \cdot \mu\boldsymbol{H} = 0 \end{cases}$$

$$\nabla \times \mathbf{C} = \mathbf{0} \quad \Rightarrow \quad \exists\, \Phi \quad : \quad \mathbf{C} = \nabla \Phi$$

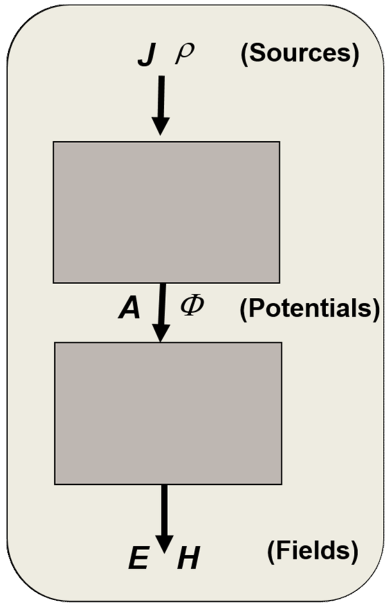
$$\nabla \cdot \mathbf{C} = 0 \quad \Rightarrow \quad \exists\, \mathbf{A} \quad : \quad \mathbf{C} = \nabla \times \mathbf{A}$$

$$\nabla \cdot \mu\mathbf{H} = 0 \quad \Rightarrow \mu\mathbf{H} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + j\omega\mu\mathbf{H} = 0 \quad \Rightarrow \nabla \times \mathbf{E} + j\omega\nabla \times \mathbf{A} = 0 \quad \Rightarrow \nabla \times \left[\mathbf{E} + j\omega\mathbf{A}\right] = 0 \quad \Rightarrow \left[\mathbf{E} + j\omega\mathbf{A}\right] = -\nabla\Phi$$

Radiation problem & potentials

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$

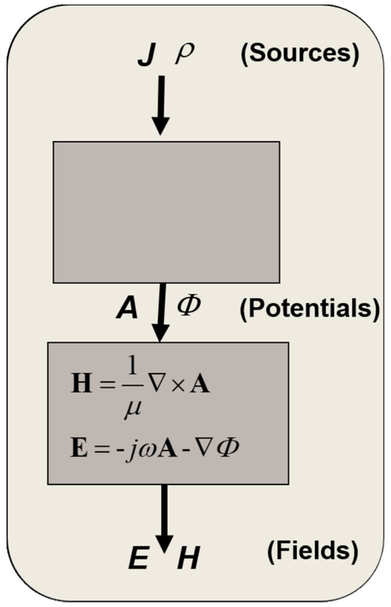


$$\nabla \cdot \mu \mathbf{H} = 0 \quad \Rightarrow \mu \mathbf{H} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + j\omega\mu \mathbf{H} = 0 \quad \Rightarrow \nabla \times \mathbf{E} + j\omega \nabla \times \mathbf{A} = 0 \quad \Rightarrow \nabla \times [\mathbf{E} + j\omega \mathbf{A}] = 0 \quad \Rightarrow [\mathbf{E} + j\omega \mathbf{A}] = -\nabla \Phi$$

Radiation problem & potentials

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$

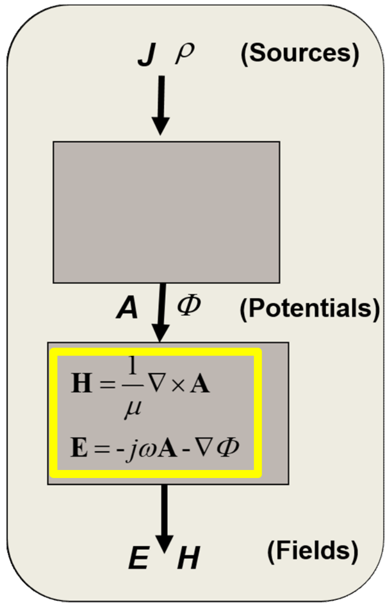


$$\nabla \cdot \mu \mathbf{H} = 0 \quad \Rightarrow \mu \mathbf{H} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + j\omega\mu \mathbf{H} = 0 \quad \Rightarrow \nabla \times \mathbf{E} + j\omega \nabla \times \mathbf{A} = 0 \quad \Rightarrow \nabla \times \left[\mathbf{E} + j\omega \mathbf{A}\right] = 0 \quad \Rightarrow \left[\mathbf{E} + j\omega \mathbf{A}\right] = -\nabla \Phi$$

Radiation problem & potentials

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$



$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\Phi\right)$$

$$\omega^2 \mu\varepsilon = k^2$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E} + \mathbf{J} \Rightarrow \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A}\right) = j\omega\varepsilon\left(-j\omega \mathbf{A} - \nabla\Phi\right) + \mathbf{J} \Rightarrow \nabla \times \left(\nabla \times \mathbf{A}\right) = j\omega\mu\varepsilon\left(-j\omega \mathbf{A} - \nabla\Phi\right) + \mu \mathbf{J}$$

$$\Downarrow$$

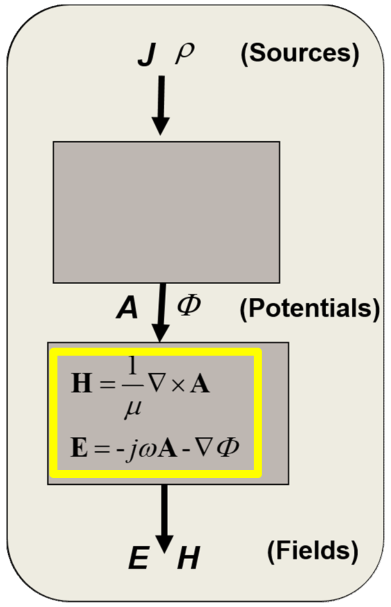
$$\nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} = \omega^2 \mu\varepsilon \mathbf{A} - j\omega\mu\varepsilon\nabla\Phi + \mu \mathbf{J}$$

$$\Rightarrow \nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\nabla\Phi \Rightarrow \nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\Phi\right)$$

$$\nabla \times \left(\nabla \times \mathbf{A}\right) = \nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}$$

Radiation problem & potentials

$$\begin{cases} \nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\ \nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\ \nabla \cdot \varepsilon \boldsymbol{E} = \rho \\ \nabla \cdot \mu \boldsymbol{H} = 0 \end{cases}$$



$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\Phi\right)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\Phi\right)$$

$$\omega^2 \mu\varepsilon = k^2$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \Rightarrow \nabla \cdot \left(-j\omega\mathbf{A} - \nabla\Phi\right) = \frac{\rho}{\varepsilon} \Rightarrow -\nabla \cdot \left(\nabla\Phi\right) - j\omega\nabla \cdot \mathbf{A} = \frac{\rho}{\varepsilon} \Rightarrow -\nabla^2\Phi - j\omega\nabla \cdot \mathbf{A} = \frac{\rho}{\varepsilon}$$

$$\Downarrow$$

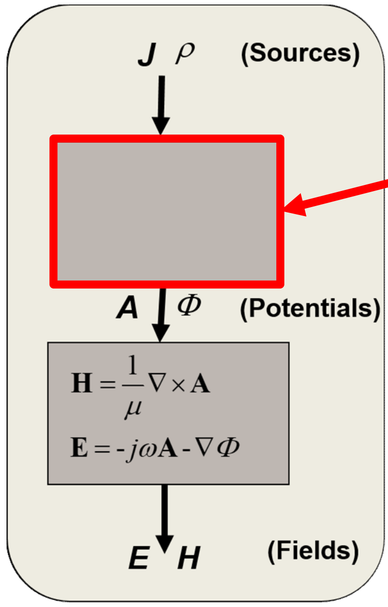
$$\nabla^2\Phi \;\; + k^2\Phi \;\; = -\frac{\rho}{\varepsilon} \;\; - j\omega\nabla \cdot \mathbf{A} - jj\omega\omega\mu\varepsilon\Phi$$

$$+\omega^2\mu\varepsilon\Phi \qquad\qquad +\omega^2\mu\varepsilon\Phi$$

$$\nabla \cdot \left(\nabla\Phi\right) = \nabla^2\Phi$$

Radiation problem & potentials

$$
\begin{cases}
\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \\
\nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \\
\nabla \cdot \varepsilon \boldsymbol{E} = \rho \\
\nabla \cdot \mu \boldsymbol{H} = 0
\end{cases}
$$



$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\Phi\right)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\Phi\right)$$

$$\omega^2 \mu\varepsilon = k^2$$

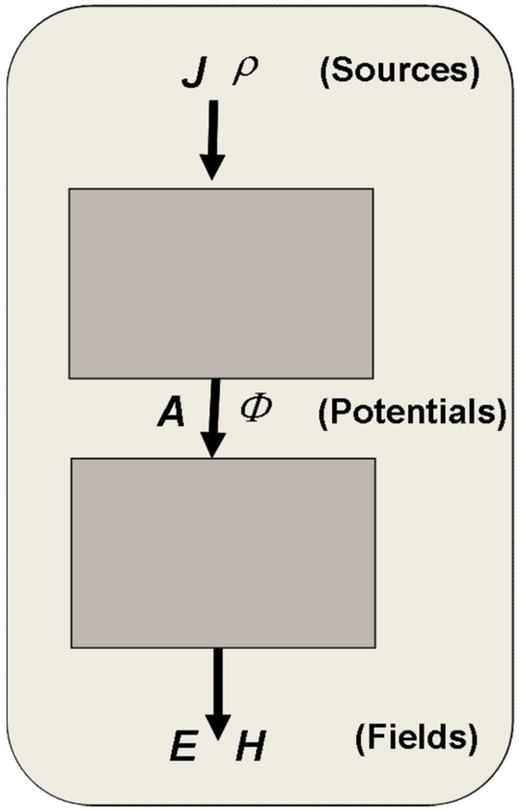
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \Rightarrow \nabla \cdot \left(-j\omega \mathbf{A} - \nabla\Phi\right) = \frac{\rho}{\varepsilon} \Rightarrow -\nabla \cdot \left(\nabla\Phi\right) - j\omega\nabla \cdot \mathbf{A} = \frac{\rho}{\varepsilon} \Rightarrow -\nabla^2\Phi - j\omega\nabla \cdot \mathbf{A} = \frac{\rho}{\varepsilon}$$

$$\Downarrow$$

$$\nabla^2\Phi \; + k^2\Phi \; + \omega^2\mu\varepsilon\Phi = -\frac{\rho}{\varepsilon} \; - j\omega\nabla \cdot \mathbf{A} - jj\omega\omega\mu\varepsilon\Phi + \omega^2\mu\varepsilon\Phi$$

$$\nabla \cdot \left(\nabla\Phi\right) = \nabla^2\Phi$$

Potentials



$\mathbf{J}$ ρ

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi)$$

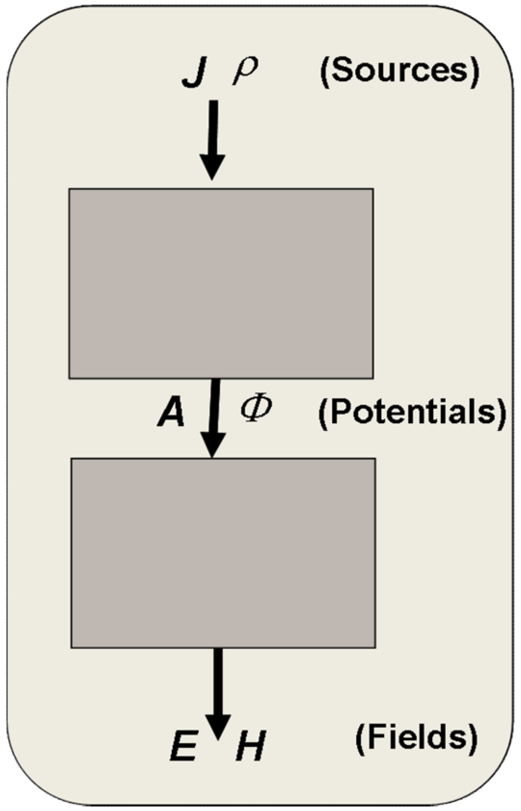
$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi\right)$$

$\mathbf{A}$ Φ

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

$\mathbf{E}$ $\mathbf{H}$

Potentials



$\mathbf{J}$ ρ

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi\right)$$

$\mathbf{A}$ Φ

Mathematical tools

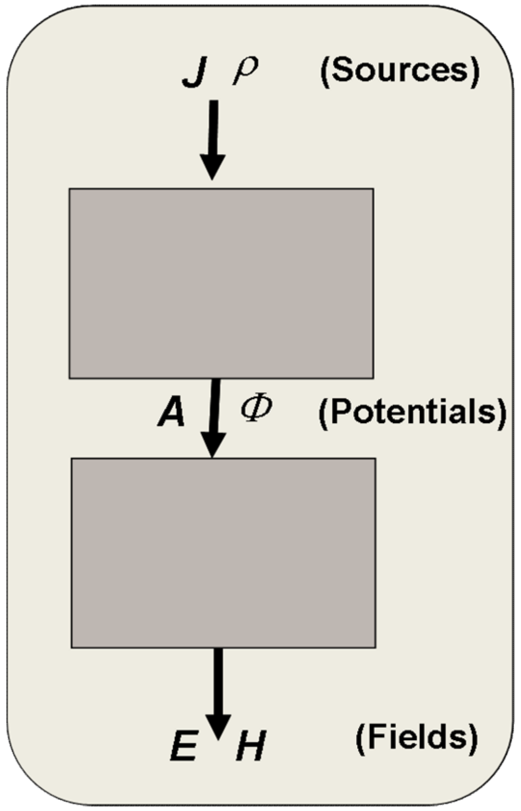
$$\mathbf{C} = \mathrm{C_x}(x,y,z)\hat{i}_x + \mathrm{C_y}(x,y,z)\hat{i}_y + \mathrm{C_z}(x,y,z)\hat{i}_z$$

$$\Phi = \Phi(x,y,z)$$

$$\nabla^2 \Phi = \nabla \cdot (\nabla \Phi) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

$$\nabla^2 \mathbf{C} = \nabla^2 \mathrm{C}_x \hat{i}_x + \nabla^2 \mathrm{C}_y \hat{i}_y + \nabla^2 \mathrm{C}_z \hat{i}_z$$

Potentials



$\mathbf{J} \quad \rho$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega(\nabla \cdot \mathbf{A} + j\omega\varepsilon\mu\Phi)$$

$\mathbf{A} \quad \Phi$

$$\nabla^2 \mathbf{C} = \nabla^2 C_x \hat{i}_x + \nabla^2 C_y \hat{i}_y + \nabla^2 C_z \hat{i}_z$$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = \cdots \quad \Rightarrow \quad \begin{cases} \nabla^2 A_x + k^2 A_x = \cdots \\ \nabla^2 A_y + k^2 A_y = \cdots \\ \nabla^2 A_z + k^2 A_z = \cdots \end{cases}$$

$$\nabla^2 \Phi + k^2 \Phi = \cdots$$

Mathematical tools

$$\mathbf{C} = \mathrm{C_x}(x,y,z)\hat{i}_x + \mathrm{C_y}(x,y,z)\hat{i}_y + \mathrm{C_z}(x,y,z)\hat{i}_z$$

$$\mathbf{A} = \mathrm{A_x}(x,y,z)\hat{i}_x + \mathrm{A_y}(x,y,z)\hat{i}_y + \mathrm{A_z}(x,y,z)\hat{i}_z$$

$$\Phi = \Phi(x,y,z)$$

I) $\nabla \cdot \mathbf{C} = 0 \quad \Rightarrow \quad \exists\, \mathbf{A} \quad : \quad \mathbf{C} = \nabla \times \mathbf{A}$

II) $\nabla \times \mathbf{C} = \mathbf{0} \quad \Rightarrow \quad \exists\, \Phi \quad : \quad \mathbf{C} = \nabla \Phi$

Potentials & uniqueness

$$\text{I)} \qquad \nabla \cdot \mathbf{C} = 0 \quad \Rightarrow \quad \exists\, \mathbf{A} \quad : \quad \mathbf{C} = \nabla \times \mathbf{A}$$

Let us suppose that a vector $\mathbf{A}_0$ exists such that $\nabla \times \mathbf{A}_0 = \mathbf{0}$

$$\nabla \times (\mathbf{A} + \mathbf{A}_0) = \nabla \times \mathbf{A} + \nabla \times \mathbf{A}_0 = \nabla \times \mathbf{A}$$

$$\text{I)} \quad \Rightarrow \quad \mathbf{C} = \nabla \times (\mathbf{A} + \mathbf{A}_0)$$

where $\nabla \times \mathbf{A}_0 = \mathbf{0}$

A is defined but for a vector $\mathbf{A}_o$ that is curl free.

Potentials & uniqueness

II) $\nabla \times \mathbf{C} = \mathbf{0} \quad \Rightarrow \quad \exists\, \Phi \quad : \quad \mathbf{C} = \nabla \Phi$

Let us suppose that a scalar Φ_0 exists such that $\nabla \Phi_0 = \mathbf{0}$

$$\nabla\left(\Phi + \Phi_0\right) = \nabla \Phi + \nabla \Phi_0 = \nabla \Phi$$

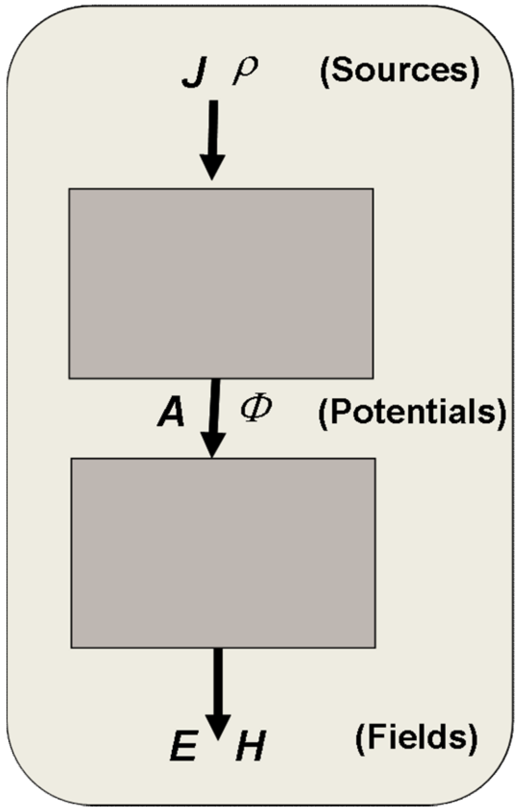
$$\text{II}) \quad \Rightarrow \quad \mathbf{C} = \nabla\left(\Phi + \Phi_0\right)$$

where $\nabla \Phi_0 = \mathbf{0}$

Φ is defined but for a scalar Φ_o that is gradient free.

Potentials

Amongst the infinite couples of potentials, is it possible to find a couple such that
$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0$ **?**



$\boldsymbol{J}$ ρ

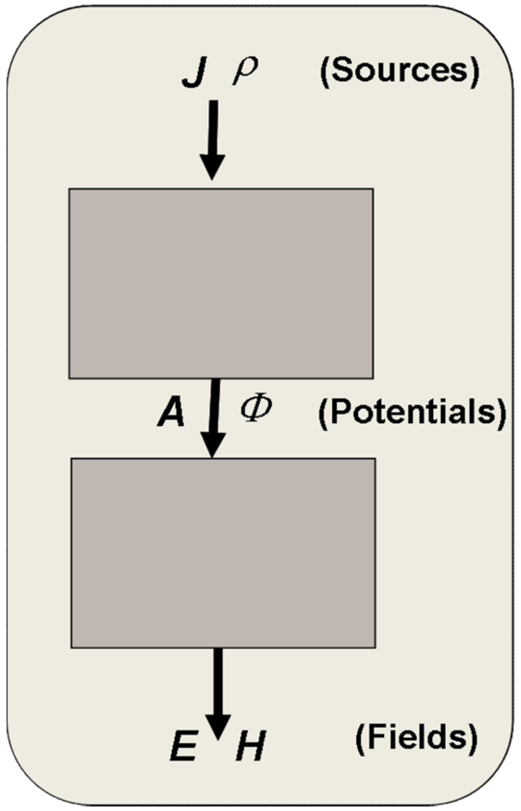
$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega\left(\nabla \cdot \boldsymbol{A} + \omega\varepsilon\mu\Phi\right)$$

$\boldsymbol{A}$ Φ

Potentials

$$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0$$ *Lorentz gauge*



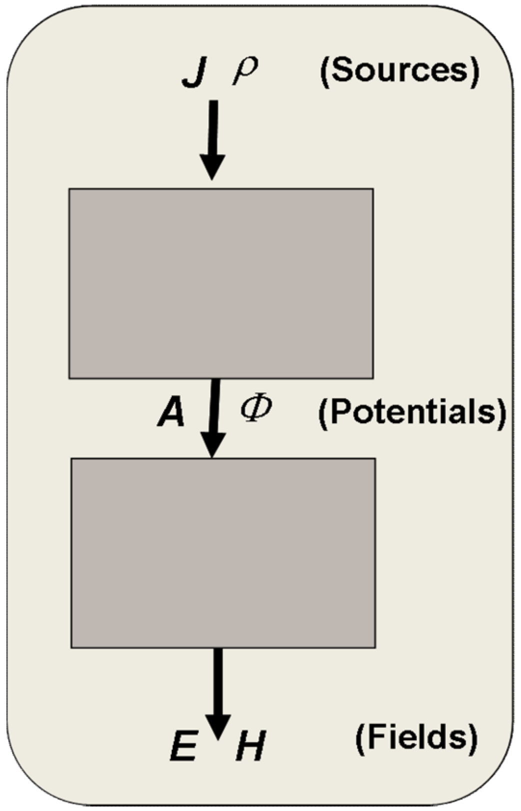
$\boldsymbol{J}$ ρ

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon} - j\omega(\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi)$$

$\boldsymbol{A}$ Φ

Potentials



$\boldsymbol{J}$ ρ

$$\begin{cases} \nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J} & \text{Vector Helmholtz equation} \\ \nabla^2 \Phi + k^2 \Phi = -\dfrac{\rho}{\varepsilon} & \text{Scalar Helmholtz equation} \end{cases}$$

$\boldsymbol{A}$ Φ

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega \mathbf{A} - \nabla \Phi$$

$\boldsymbol{E}$ $\boldsymbol{H}$

Potentials

$$\nabla \cdot \boldsymbol{A} + j\omega\varepsilon\mu\Phi = 0$$

Lorentz gauge

J ρ

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$ Vector Helmholtz equation

~~$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\varepsilon}$$~~ Scalar Helmholtz equation

A Φ

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega A - \nabla \Phi$$

$$\nabla\left(\frac{\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}\right)$$

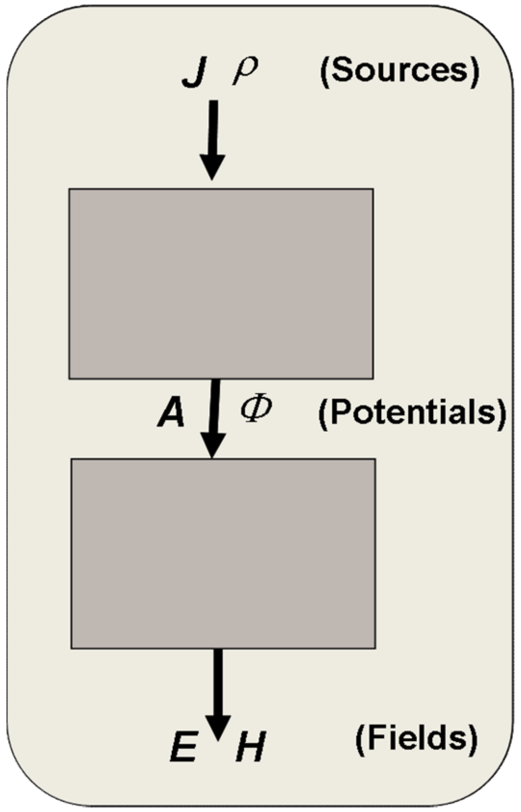
E H

Note that once $\boldsymbol{A}$ is calculated by solving the (vector) Helmholtz equation involving $\boldsymbol{A}$ and $\boldsymbol{J}$, subsequent calculation of Φ can be straightforwardly achieved by means of the *Lorentz gauge*

$$\Phi = -\frac{\nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}$$

thus rendering unnecessary the solution of the (scalar) Helmholtz equation relevant to Φ

Potentials



$\mathbf{J}$ ↓

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$ **Vector Helmholtz equation**

$\mathbf{A}$ ↓

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -j\omega \mathbf{A} + \frac{\nabla \nabla \cdot \mathbf{A}}{j\omega\varepsilon\mu}$$

↓ $\mathbf{E}$ $\mathbf{H}$

Potentials

J

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$ **Vector Helmholtz equation**

A

$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

Potentials

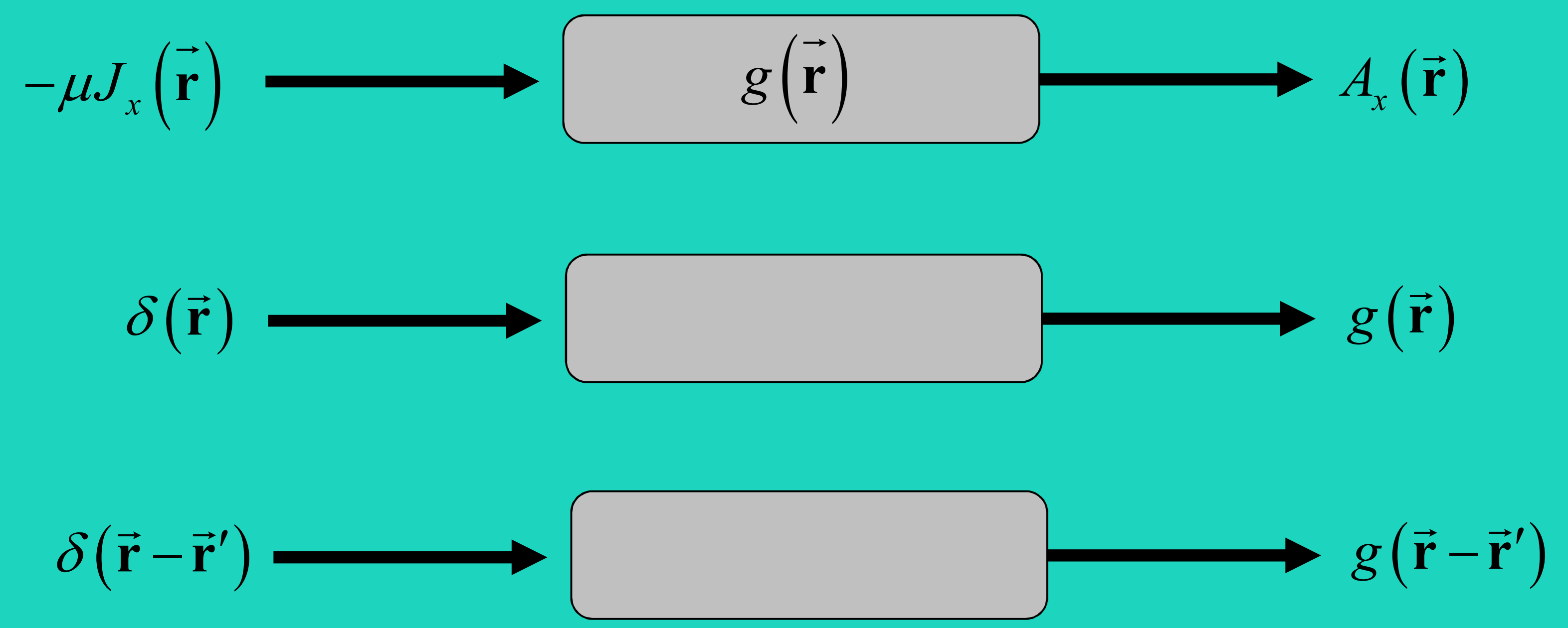
$$\begin{cases} \nabla^2 A_x + k^2 A_x = -\mu J_x \\ \nabla^2 A_y + k^2 A_y = -\mu J_y \\ \nabla^2 A_z + k^2 A_z = -\mu J_z \end{cases}$$

Let us address the solution of the following scalar Helmholtz equation

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$



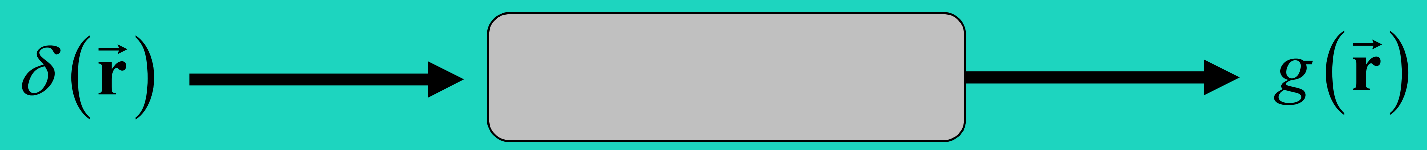
$$A_x(\vec{\mathbf{r}}) = \int -\mu J_x(\vec{\mathbf{r}}')\, g(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\, d\vec{\mathbf{r}}'$$

Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

Potentials

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$



$$\nabla^2 g(\vec{\mathbf{r}}) + k^2 g(\vec{\mathbf{r}}) = \delta(\vec{\mathbf{r}})$$