# Campi Elettromagnetici

## Corso di Laurea in Ingegneria Informatica, Biomedica e delle Telecomunicazioni

**a.a. 2020-2021 - Laurea “Triennale” – Secondo semestre - Secondo anno**


**Università degli Studi di Napoli “Parthenope”**

**Stefano Perna**

# Color legend

New formulas, important considerations,
important formulas, important concepts

Very important for the discussion

Memo

Mathematical tools to be exploited

Mathematics

# Mathematical tools that we will exploit today

$$\vec{\mathbf{A}}(\vec{\mathbf{r}})\cdot\left[\vec{\mathbf{B}}(\vec{\mathbf{r}})\times\vec{\mathbf{C}}(\vec{\mathbf{r}})\right]=\vec{\mathbf{C}}(\vec{\mathbf{r}})\cdot\left[\vec{\mathbf{A}}(\vec{\mathbf{r}})\times\vec{\mathbf{B}}(\vec{\mathbf{r}})\right]=\vec{\mathbf{B}}(\vec{\mathbf{r}})\cdot\left[\vec{\mathbf{C}}(\vec{\mathbf{r}})\times\vec{\mathbf{A}}(\vec{\mathbf{r}})\right]$$

# Fields at boundaries

Let us consider nonhomogeneous media, so that constitutive relations may change as a function of space.

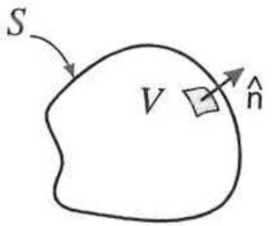
Let us suppose that this change is abrupt across space, that is, we have two media of different characteristics separated by a smooth surface.

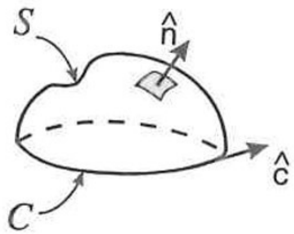
The solutions of Maxwell equations can be obtained in both the two regions: matching conditions are needed on the boundaries.

# Fields at boundaries

## Integral form

$$\begin{cases} \oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\frac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \frac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oiint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\ \oiint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0 \end{cases}$$

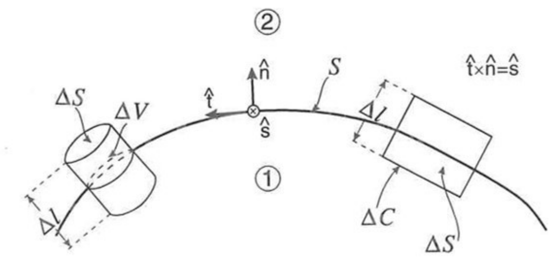


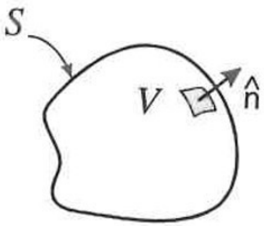



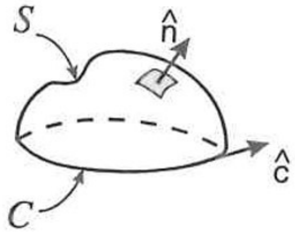
# Fields at boundaries

**Integral form**

$$
\begin{cases}
\oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\dfrac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \dfrac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oiint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\
\oiint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0
\end{cases}
$$



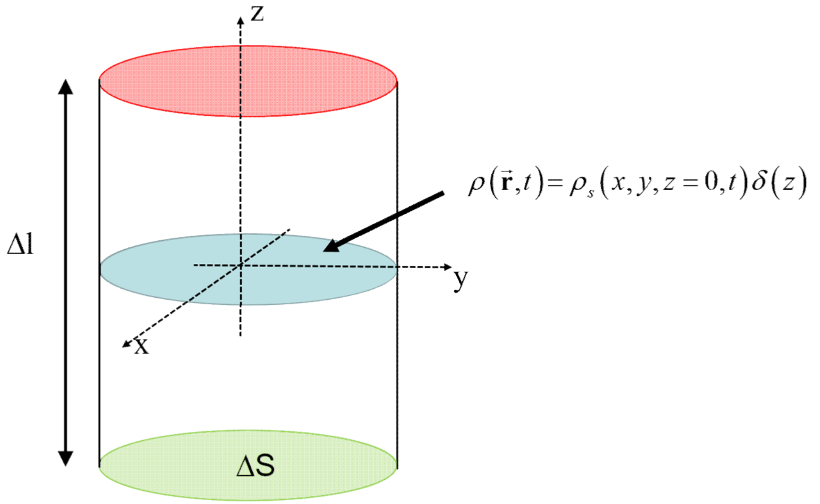



# Fields at boundaries

$$\rho(\vec{\mathbf{r}},t)=\rho_s(x,y,z=0,t)\delta(z)$$

$$\iiint_V dV\,\rho(x,y,z,t) = \iint_{\Delta S} dS \int_{\Delta l} dz\,\rho(x,y,z,t)$$

$$= \iint_{\Delta S} dS \int_{\Delta l} dz\,\rho_s(x,y,z=0,t)\delta(z) = \iint_{\Delta S} dS\,\rho_s(x,y,z=0,t)$$




$$\oint\!\!\!\oint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t)$$

$\Delta l \to 0$

$$\iint_{\Delta S} dS\vec{\mathbf{d}}_2(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} - \iint_{\Delta S} dS\vec{\mathbf{d}}_1(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t)$$

$$\iint_{\Delta S} dS\left[\vec{\mathbf{d}}_2(\vec{\mathbf{r}},t)-\vec{\mathbf{d}}_1(\vec{\mathbf{r}},t)\right]\cdot\hat{\mathbf{n}} = \iint_{\Delta S} dS\,\rho_s(\vec{\mathbf{r}},t)$$

$$\left(\vec{\mathbf{d}}_2-\vec{\mathbf{d}}_1\right)\cdot\hat{\mathbf{n}} = \rho_s$$

When a localized charge distribution is present over the boundary, so that $\rho$ is infinite there

$[\rho]: \dfrac{Coulomb}{m^3}$

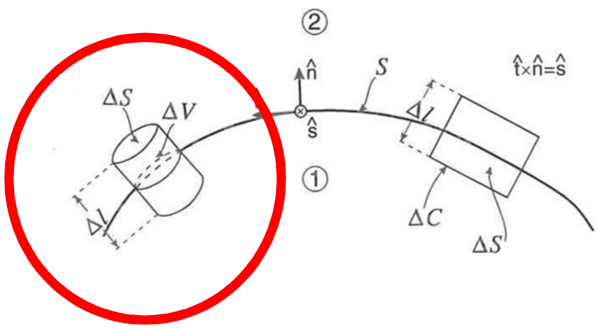
$[\delta(z)]: \dfrac{1}{m}$

$[\rho_s]: \dfrac{Coulomb}{m^2}$

# Fields at boundaries

**Integral form**

$$\begin{cases} \oint_C dc\, \vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\frac{d}{dt}\iint_S dS\, \vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oint_C dc\, \vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \frac{d}{dt}\iint_S dS\, \vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\, \vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oiint_S dS\, \vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\, \rho(\vec{\mathbf{r}},t) \\ \oiint_S dS\, \vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0 \end{cases}$$



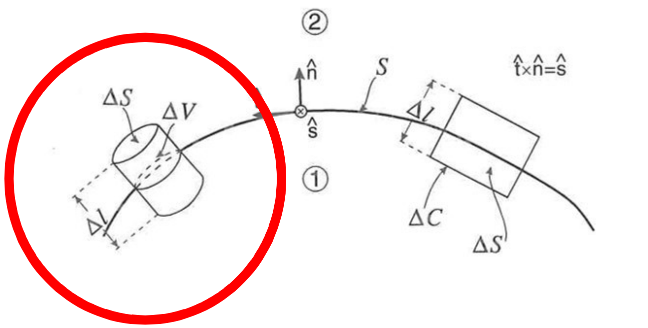

$$\oiint_S dS\, \vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\, \rho(\vec{\mathbf{r}},t) \quad \Rightarrow \quad \left(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1\right)\cdot\hat{\mathbf{n}} = \rho_s$$

$$\oiint_S dS\, \vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0 \quad \Rightarrow \quad \left(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1\right)\cdot\hat{\mathbf{n}} = 0$$

# Fields at boundaries

**Integral form**

$$
\begin{cases}
\oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\frac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \frac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oint\!\!\oint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\
\oint\!\!\oint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0
\end{cases}
$$




$$(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1)\cdot\hat{\mathbf{n}} = \rho_s$$

$$(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1)\cdot\hat{\mathbf{n}} = 0$$

# Fields at boundaries

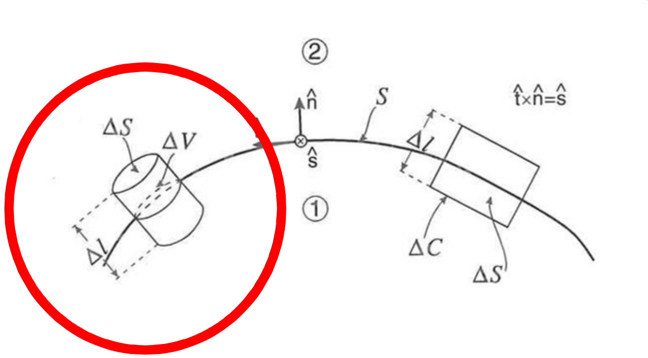
## Integral form

$$\begin{cases} \oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\frac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \frac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oint\!\!\oint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\ \oint\!\!\oint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0 \end{cases}$$

+

**Current density equation**

$$\oint\!\!\oint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \frac{dq(t)}{dt} = 0$$




$$\left(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1\right)\cdot\hat{\mathbf{n}} = \rho_s$$

$$\left(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1\right)\cdot\hat{\mathbf{n}} = 0$$

# Fields at boundaries

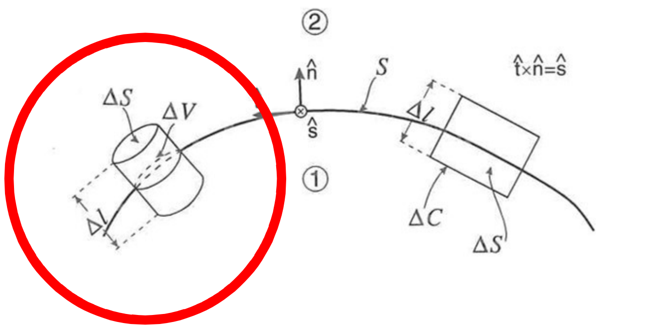
## Integral form

$$\begin{cases} \oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\frac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \frac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oiint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\ \oiint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0 \end{cases}$$

+

**Current density equation**

$$\oiint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \frac{d}{dt}\iiint_V dV\,\rho(\vec{\mathbf{r}},t) = 0$$




$$\left(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1\right)\cdot\hat{\mathbf{n}} = \rho_s$$

$$\left(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1\right)\cdot\hat{\mathbf{n}} = 0$$

# Fields at boundaries

## Integral form

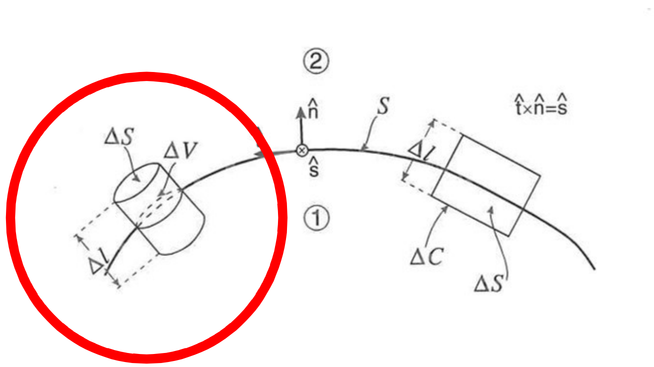
$$
\begin{cases}
\oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\dfrac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \dfrac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oiint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\
\oiint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0
\end{cases}
$$

+

**Current density equation**

$$\oiint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \frac{d}{dt}\iiint_V dV\,\rho(\vec{\mathbf{r}},t) = 0$$

$$\left(\vec{\mathbf{j}}_2 - \vec{\mathbf{j}}_1\right)\cdot\hat{\mathbf{n}} = -\frac{\partial \rho_s}{\partial t}$$




$$\left(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1\right)\cdot\hat{\mathbf{n}} = \rho_s$$

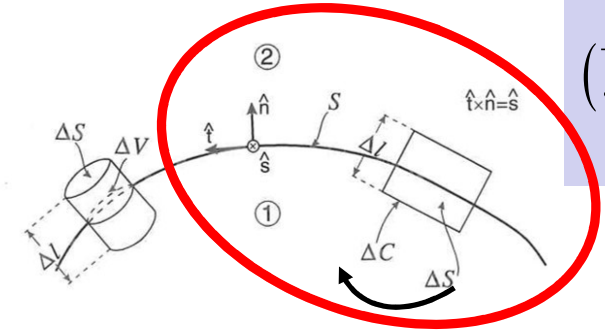
$$\left(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1\right)\cdot\hat{\mathbf{n}} = 0$$

# Fields at boundaries

## Integral form

$$\begin{cases} \oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\dfrac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \dfrac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oiint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\ \oiint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0 \end{cases}$$

$$\hat{\mathbf{t}} = \hat{\mathbf{n}}\times\hat{\mathbf{s}}$$




$$(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1)\cdot\hat{\mathbf{n}} = \rho_s$$

$$(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1)\cdot\hat{\mathbf{n}} = 0$$

$$(\vec{\mathbf{j}}_2 - \vec{\mathbf{j}}_1)\cdot\hat{\mathbf{n}} = -\frac{\partial \rho_s}{\partial t}$$

$$\vec{\mathbf{A}}\cdot[\vec{\mathbf{B}}\times\vec{\mathbf{C}}] = \vec{\mathbf{C}}\cdot[\vec{\mathbf{A}}\times\vec{\mathbf{B}}]$$

$\Delta l \to 0$

$$\oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \int_{\Delta C} dc\,\vec{\mathbf{h}}_1(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{t}} - \int_{\Delta C} dc\,\vec{\mathbf{h}}_2(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{t}} = \int_{\Delta C} dc\left[\vec{\mathbf{h}}_1(\vec{\mathbf{r}},t) - \vec{\mathbf{h}}_2(\vec{\mathbf{r}},t)\right]\cdot\hat{\mathbf{t}} = \int_{\Delta C} dc\left[\vec{\mathbf{h}}_1(\vec{\mathbf{r}},t) - \vec{\mathbf{h}}_2(\vec{\mathbf{r}},t)\right]\cdot(\hat{\mathbf{n}}\times\hat{\mathbf{s}})$$

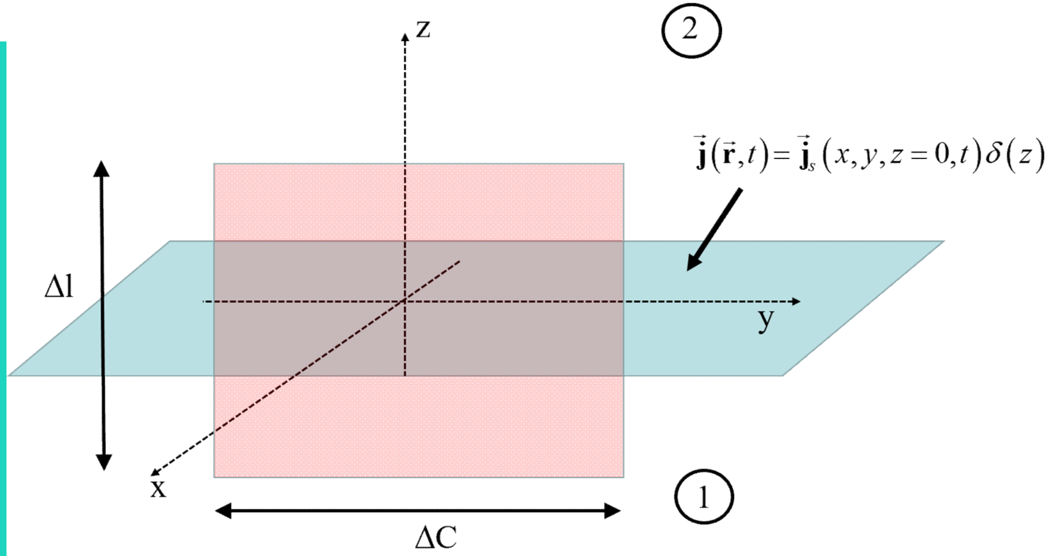
$$= \int_{\Delta C} dc\,\hat{\mathbf{s}}\cdot\left(\left[\vec{\mathbf{h}}_1(\vec{\mathbf{r}},t) - \vec{\mathbf{h}}_2(\vec{\mathbf{r}},t)\right]\times\hat{\mathbf{n}}\right) = \int_{\Delta C} dc\,\hat{\mathbf{s}}\cdot\left(\hat{\mathbf{n}}\times\left[\vec{\mathbf{h}}_2(\vec{\mathbf{r}},t) - \vec{\mathbf{h}}_1(\vec{\mathbf{r}},t)\right]\right) = \int_{\Delta C} dc\,\left(\hat{\mathbf{n}}\times\left[\vec{\mathbf{h}}_2(\vec{\mathbf{r}},t) - \vec{\mathbf{h}}_1(\vec{\mathbf{r}},t)\right]\right)\cdot\hat{\mathbf{s}}$$

# Fields at boundaries

$$\vec{\mathbf{j}}(x,y,z,t)=\vec{\mathbf{j}}_s(x,y,z=0,t)\delta(z)$$

$$\iint_{\Delta S} dS\,\vec{\mathbf{j}}(x,y,z,t)\cdot\hat{\mathbf{i}}_{\mathbf{x}}=\int_{\Delta C} dy\int_{\Delta l} dz\,\vec{\mathbf{j}}(x,y,z,t)\cdot\hat{\mathbf{i}}_{\mathbf{x}}$$

$$=\int_{\Delta C} dy\int_{\Delta l} dz\,\vec{\mathbf{j}}_s(x,y,z=0,t)\delta(z)\cdot\hat{\mathbf{i}}_{\mathbf{x}}\;=\int_{\Delta C} dy\vec{\mathbf{j}}_s(x,y,z=0,t)\cdot\hat{\mathbf{i}}_{\mathbf{x}}$$




$\Delta l \to 0$

$$\oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}}\;=\int_{\Delta C} dc\,\left(\hat{\mathbf{n}}\times\left[\vec{\mathbf{h}}_2(\vec{\mathbf{r}},t)-\vec{\mathbf{h}}_1(\vec{\mathbf{r}},t)\right]\right)\cdot\hat{\mathbf{s}}$$

$$\iint_{\Delta S} dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}}$$

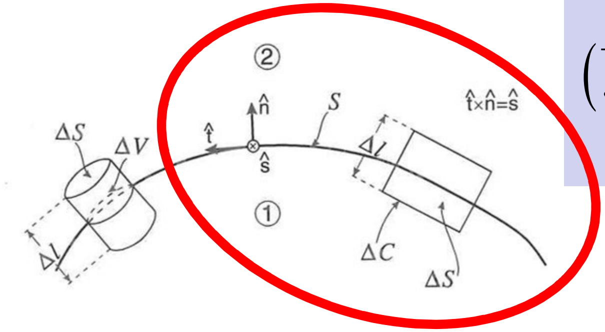
When a localized current is present over the boundary, so that ***j*** is infinite there

# Fields at boundaries

$$\vec{\mathbf{j}}(x,y,z,t)=\vec{\mathbf{j}}_s(x,y,z=0,t)\,\delta(z)$$

$$\iint_{\Delta S} dS\,\vec{\mathbf{j}}(x,y,z,t)\cdot\hat{\mathbf{i}}_{\mathbf{x}}=\int_{\Delta C} dy\int_{\Delta l} dz\,\vec{\mathbf{j}}(x,y,z,t)\cdot\hat{\mathbf{i}}_{\mathbf{x}}$$

$$=\int_{\Delta C} dy\int_{\Delta l} dz\,\vec{\mathbf{j}}_s(x,y,z=0,t)\,\delta(z)\cdot\hat{\mathbf{i}}_{\mathbf{x}}\;=\int_{\Delta C} dy\,\vec{\mathbf{j}}_s(x,y,z=0,t)\cdot\hat{\mathbf{i}}_{\mathbf{x}}$$




$$\left(\vec{\mathbf{d}}_2-\vec{\mathbf{d}}_1\right)\cdot\hat{\mathbf{n}}=\rho_s$$

$$\left(\vec{\mathbf{b}}_2-\vec{\mathbf{b}}_1\right)\cdot\hat{\mathbf{n}}=0$$

$$\left(\vec{\mathbf{j}}_2-\vec{\mathbf{j}}_1\right)\cdot\hat{\mathbf{n}}=-\frac{\partial\rho_s}{\partial t}$$

$\Delta l\rightarrow 0$

$$\oint_{C} dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}}\;=\int_{\Delta C} dc\,\left(\hat{\mathbf{n}}\times\left[\vec{\mathbf{h}}_2(\vec{\mathbf{r}},t)-\vec{\mathbf{h}}_1(\vec{\mathbf{r}},t)\right]\right)\cdot\hat{\mathbf{s}}$$

$$\iint_{\Delta S} dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}}\;=\int_{\Delta C} dc\,\vec{\mathbf{j}}_s(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}}$$

When a localized current is present over the boundary, so that ***j*** is infinite there

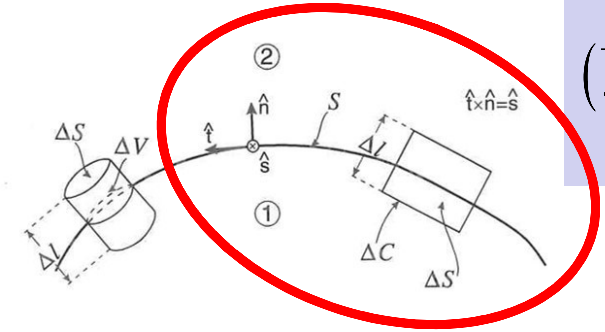
$$\left[\vec{\mathbf{j}}\right]:\frac{Ampere}{m^2}\qquad\left[\vec{\mathbf{j}}_s\right]:\frac{Ampere}{m}$$

$$\left[\delta(z)\right]:\frac{1}{m}$$

# Fields at boundaries

## Integral form

$$
\begin{cases}
\oint_C dc\, \vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\dfrac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oint_C dc\, \vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \dfrac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oiint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\
\oiint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0
\end{cases}
$$




$$(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1)\cdot\hat{\mathbf{n}} = \rho_s$$

$$(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1)\cdot\hat{\mathbf{n}} = 0$$

$$(\vec{\mathbf{j}}_2 - \vec{\mathbf{j}}_1)\cdot\hat{\mathbf{n}} = -\frac{\partial \rho_s}{\partial t}$$

$\Delta l \to 0$

$$\iint_{\Delta S} dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}} = 0$$

$$\oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \int_{\Delta C} dc\,\left(\hat{\mathbf{n}}\times\left[\vec{\mathbf{h}}_2(\vec{\mathbf{r}},t) - \vec{\mathbf{h}}_1(\vec{\mathbf{r}},t)\right]\right)\cdot\hat{\mathbf{s}}$$

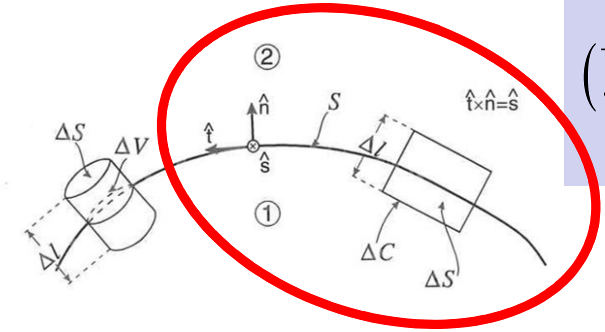
$$\iint_{\Delta S} dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}} = \int_{\Delta C} dc\,\vec{\mathbf{j}}_s(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}}$$

When a localized current is present over the boundary, so that ***j*** is infinite there

# Fields at boundaries

## Integral form

$$
\begin{cases}
\oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\frac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \frac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oiint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\
\oiint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0
\end{cases}
$$




$$(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1)\cdot\hat{\mathbf{n}} = \rho_s$$

$$(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1)\cdot\hat{\mathbf{n}} = 0$$

$$(\vec{\mathbf{j}}_2 - \vec{\mathbf{j}}_1)\cdot\hat{\mathbf{n}} = -\frac{\partial\rho_s}{\partial t}$$

$\Delta l \to 0$

$$\oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \int_{\Delta C} dc\,\left(\hat{\mathbf{n}}\times\left[\vec{\mathbf{h}}_2(\vec{\mathbf{r}},t)-\vec{\mathbf{h}}_1(\vec{\mathbf{r}},t)\right]\right)\cdot\hat{\mathbf{s}}$$

$$\iint_{\Delta S} dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}} = \int_{\Delta C} dc\,\vec{\mathbf{j}}_s(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}}$$

$$\iint_{\Delta S} dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}} = 0$$

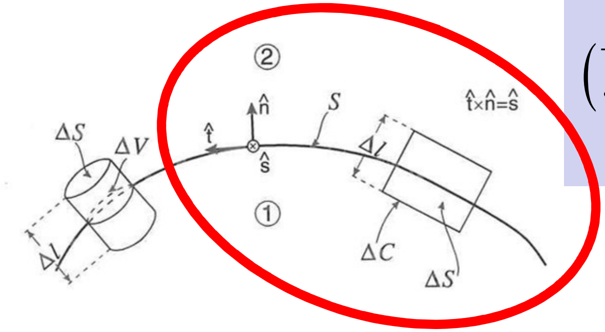
$$\int_{\Delta C} dc\,\left(\hat{\mathbf{n}}\times\left[\vec{\mathbf{h}}_2(\vec{\mathbf{r}},t)-\vec{\mathbf{h}}_1(\vec{\mathbf{r}},t)\right]\right)\cdot\hat{\mathbf{s}} = \int_{\Delta C} dc\,\vec{\mathbf{j}}_s(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}}$$

When a localized current is present over the boundary, so that ***j*** is infinite there

# Fields at boundaries

## Integral form

$$
\begin{cases}
\oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\frac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \frac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\
\oiint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\
\oiint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0
\end{cases}
$$




$$(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1)\cdot\hat{\mathbf{n}} = \rho_s$$

$$(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1)\cdot\hat{\mathbf{n}} = 0$$

$$(\vec{\mathbf{j}}_2 - \vec{\mathbf{j}}_1)\cdot\hat{\mathbf{n}} = -\frac{\partial\rho_s}{\partial t}$$

$\Delta l \to 0$

$$\int_{\Delta C} dc\,\left(\hat{\mathbf{n}}\times\left[\vec{\mathbf{h}}_2(\vec{\mathbf{r}},t) - \vec{\mathbf{h}}_1(\vec{\mathbf{r}},t)\right]\right)\cdot\hat{\mathbf{s}} = \int_{\Delta C} dc\,\vec{\mathbf{j}}_s(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{s}}$$

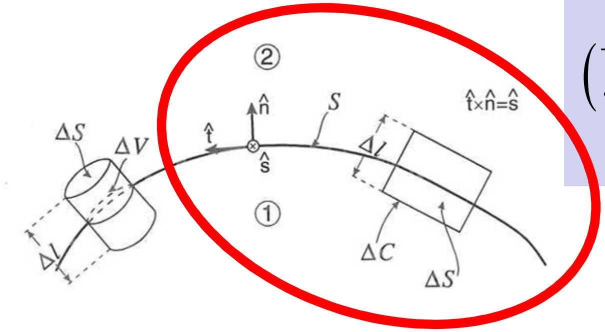
$$\hat{\mathbf{n}}\times(\vec{\mathbf{h}}_2 - \vec{\mathbf{h}}_1) = \vec{\mathbf{j}}_s$$

When a localized current is present over the boundary, so that ***j*** is infinite there

# Fields at boundaries

Integral form

$$\begin{cases}\oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\frac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \frac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \\ \oiint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = \iiint_V dV\,\rho(\vec{\mathbf{r}},t) \\ \oiint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} = 0\end{cases}$$




$$(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1)\cdot\hat{\mathbf{n}} = \rho_s$$

$$(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1)\cdot\hat{\mathbf{n}} = 0$$

$$(\vec{\mathbf{j}}_2 - \vec{\mathbf{j}}_1)\cdot\hat{\mathbf{n}} = -\frac{\partial\rho_s}{\partial t}$$

$$\oint_C dc\,\vec{\mathbf{h}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = \frac{d}{dt}\iint_S dS\,\vec{\mathbf{d}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} + \iint_S dS\,\vec{\mathbf{j}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \;\Rightarrow\; \hat{\mathbf{n}}\times(\vec{\mathbf{h}}_2 - \vec{\mathbf{h}}_1) = \vec{\mathbf{j}}_s$$

$$\oint_C dc\,\vec{\mathbf{e}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{c}} = -\frac{d}{dt}\iint_S dS\,\vec{\mathbf{b}}(\vec{\mathbf{r}},t)\cdot\hat{\mathbf{n}} \;\Rightarrow\; \hat{\mathbf{n}}\times(\vec{\mathbf{e}}_2 - \vec{\mathbf{e}}_1) = \mathbf{0}$$

# Fields at boundaries

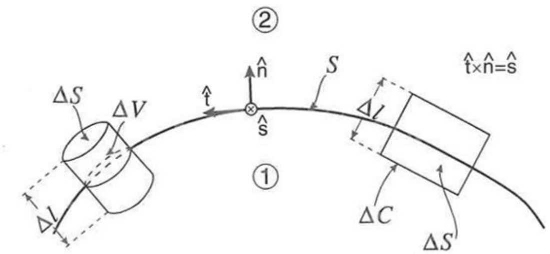
$$\hat{\mathbf{n}} \times \left( \vec{\mathbf{e}}_2 - \vec{\mathbf{e}}_1 \right) = \mathbf{0}$$

$$\hat{\mathbf{n}} \times \left( \vec{\mathbf{h}}_2 - \vec{\mathbf{h}}_1 \right) = \vec{\mathbf{j}}_s$$

$$\left( \vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1 \right) \cdot \hat{\mathbf{n}} = \rho_s$$

$$\left( \vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1 \right) \cdot \hat{\mathbf{n}} = 0$$

$$\left( \vec{\mathbf{j}}_2 - \vec{\mathbf{j}}_1 \right) \cdot \hat{\mathbf{n}} = -\frac{\partial \rho_s}{\partial t}$$



# Fields at boundaries

## One example: the medium 1 is a PEC

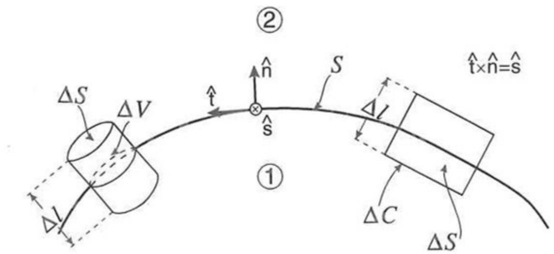
$$\hat{\mathbf{n}} \times \left( \vec{\mathbf{e}}_2 - \vec{\mathbf{e}}_1 \right) = \mathbf{0}$$

$$\hat{\mathbf{n}} \times \left( \vec{\mathbf{h}}_2 - \vec{\mathbf{h}}_1 \right) = \vec{\mathbf{j}}_s$$

$$\left( \vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1 \right) \cdot \hat{\mathbf{n}} = \rho_s$$

$$\left( \vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1 \right) \cdot \hat{\mathbf{n}} = 0$$

$$\left( \vec{\mathbf{j}}_2 - \vec{\mathbf{j}}_1 \right) \cdot \hat{\mathbf{n}} = -\frac{\partial \rho_s}{\partial t}$$




$$\vec{\mathbf{e}}_1 = \mathbf{0}$$

$$\vec{\mathbf{h}}_1 = \mathbf{0}$$

$$\vec{\mathbf{d}}_1 = \mathbf{0}$$

$$\vec{\mathbf{b}}_1 = \mathbf{0}$$

# Fields at boundaries

## One example: the medium 1 is a PEC

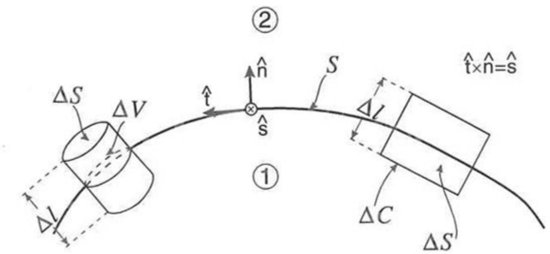
$$\hat{\mathbf{n}} \times (\vec{\mathbf{e}}_2 - \vec{\mathbf{e}}_1) = \mathbf{0}$$

$$\hat{\mathbf{n}} \times (\vec{\mathbf{h}}_2 - \vec{\mathbf{h}}_1) = \vec{\mathbf{j}}_s$$

$$(\vec{\mathbf{d}}_2 - \vec{\mathbf{d}}_1) \cdot \hat{\mathbf{n}} = \rho_s$$

$$(\vec{\mathbf{b}}_2 - \vec{\mathbf{b}}_1) \cdot \hat{\mathbf{n}} = 0$$

$$(\vec{\mathbf{j}}_2 - \vec{\mathbf{j}}_1) \cdot \hat{\mathbf{n}} = -\frac{\partial \rho_s}{\partial t}$$




$$\vec{\mathbf{e}}_1 = \mathbf{0} \qquad \hat{\mathbf{n}} \times \vec{\mathbf{e}}_2 = \mathbf{0}$$

$$\vec{\mathbf{h}}_1 = \mathbf{0} \qquad \hat{\mathbf{n}} \times \vec{\mathbf{h}}_2 = \vec{\mathbf{j}}_s$$

$$\vec{\mathbf{d}}_1 = \mathbf{0} \qquad \vec{\mathbf{d}}_2 \cdot \hat{\mathbf{n}} = \rho_s$$

$$\vec{\mathbf{b}}_1 = \mathbf{0} \qquad \vec{\mathbf{b}}_2 \cdot \hat{\mathbf{n}} = 0$$